PDE, HW 3, due Monday 10/17/05

1. Weierstrass' approximation theorem allows one to approximate a continuous function on the cube $[-1,1]^n$ by polynomials. Can one replace the word 'polynomials' by the phrase 'harmonic polynomials'?

2. Show that regularity of a boundary point is a local property: y is a regular boundary point for ∂U if and only if y is a regular boundary point for ∂U_r where $U_r = U \cap B(y,r)$ for some $r > 0$.

3. Let C be a right circular cone with angle $0 < \alpha < \pi$ and axis x_n based at the origin in $\mathbb{R}^n$. That is,

$$C = \{x \in \mathbb{R}^n | \cos\theta := \frac{x_n}{|x|} > \cos\alpha\}.$$

Construct an axisymmetric positive harmonic function in C of the form $u = |x|^\lambda f(\theta)$. Deduce that if U is an open bounded set and $y \in \partial U$ satisfies an exterior cone condition locally, then y is regular.

4. Problem 7 (c), p. 103 in John's book.

5. *Mollification.* (a) Construct a $\mathbb{C}^\infty$ function $\varphi \geq 0$ that has support in $B(0,1)$ and $\int_{\mathbb{R}^n} \varphi(x)\,dx = 1$. (b) Suppose U is a bounded, open set. Let $\mathbf{1}_U$ denote the indicator function for this set. For any $\delta > 0$ let

$$g_\delta(x) = \int_{\mathbb{R}^n} \delta^{-n}\varphi(y\delta^{-1})\mathbf{1}_U(x-y)\,dy.$$

Show that g_δ is C^∞ and $g_\delta(x) = \mathbf{1}_U(x)$ if $\mathrm{dist}(x, \partial U) > \delta$.

The following exercises deal with capacity of sets in $\mathbb{R}^n, n \geq 3$. Notation: $F \subset \mathbb{R}^n$ is compact; $U = \mathbb{R}^n \backslash F$ is unbounded; p_F is the potential of F.

6. Prove that p_F does not depend on the choice of approximating domains.

7. If $x \in U$, $r(x) = \min_{y\in F}|x-y|$, $R(x) = \max_{y \in F}|x-y|$ then

$$\mathrm{cap}(F)R^{2-n} \leq p_F(x) \leq \mathrm{cap}(F)r^{2-n}.$$

8. For fixed $\beta > 0$ let F_β denote the image of F under the dilation $x \mapsto \beta x$. Show that $\mathrm{cap}(F_\beta) = \beta^{n-2}\mathrm{cap}(F)$. Now use Wiener's criterion to provide another proof of the exterior cone condition.

9. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map such that $\sup|Df| \leq 1$. Show that $\mathrm{cap}(f(F)) \leq \mathrm{cap}(F)$ (a cat has reason to curl up before it sleeps...).