PDE, HW 1, due Friday 9/16/05

1. The mean value inequality can be used to *define* subharmonic functions. Henceforth, we will use the following definition.

Definition 0.1. A function $u \in C(U)$ is subharmonic if for every $x \in U$ there exists $\delta(x) > 0$ such that $S(x,\delta) \subset U$ and

$$u(x) \leq \fint_{S(x,r)} u(y) dS_y, \quad 0 < r \leq \delta.$$

Yet another definition is the following.

Definition 0.2. A function $u \in C(U)$ is subharmonic if for every $B(x,r) \subset\subset U$ and every harmonic function v defined on a domain containing $B(x,r)$ such that $u \leq v$ on $S(x,r)$, we have $u \leq v$ in $B(x,r)$.

Show that these definitions are equivalent; that is, each implies the other. You may assume the existence and uniqueness of solutions for the Dirichlet problem in the ball.

2. Let $f : \mathbb{R} \to R$ be convex; that is $f(ax + (1-a)y) \leq af(x) + (1-a)f(y)$ for every $a \in [0,1]$. Convex functions are continuous and satisfy Jensen's inequality (look this up, if you haven't seen it before). Let $u : U \to \mathbb{R}$ be harmonic. Show that $f \circ u$ is subharmonic. Deduce that the functions $|u|^p, p \geq 1$ and $|Du|^2$ are subharmonic.

3. Use Harnack's inequality to prove Liouville's theorem: a harmonic function on $\mathbb{R}^n$ that is bounded below is constant.

In the following questions, you will need the following definitions.

Definition 0.3. An $n \times n$ matrix F is *conformal* if $F^t F = \lambda I$ for a scalar $\lambda > 0$.

Geometrically, this means that F takes a circle about the origin to a circle about the origin. Maps whose gradients possess this property are of interest in connection with Laplace's equation.

Definition 0.4. A C^1 map $f : U \to \mathbb{R}^n$ is *conformal* if the gradient Df is conformal at every $x \in U$.

4. Inversion in the unit sphere is the map $f : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}^n$ defined by

$$f(x) = \frac{x}{|x|^2}.$$

(The origin is mapped to the point at infinity). Show that this map is conformal. Interpret the gradient Df geometrically.

5. Suppose $n = 2$. Let $f = (f_1(x), f_2(x))$ be a C^2 conformal map from an open subset $\mathbb{R}^2 \supset U \to \mathbb{R}^2$. Show that f_1 and f_2 are harmonic, that is $\triangle f_1 = \triangle f_2 = 0$. Thus, show that if $u : U \to \mathbb{R}$ is harmonic, so is $v = u \circ f$.

6. Show that inversion is not harmonic for $n \geq 3$ (a vector field is harmonic if each of its components is harmonic). The fix for this is in the following problem.

7. *Kelvin's transformation.* Let u be a harmonic function on $\mathbb{R}^n$. Show that $v(x) = |x|^{2-n} u(x/|x|^2)$ is a harmonic function for $x \neq 0$.

8. *Nonuniqueness for the exterior problem.* Let $U = \{x||x| > 1\}$ be the exterior of the unit ball in $\mathbb{R}^n, n \geq 3$. Consider the Dirichlet problem

$$\triangle u = 0, \quad x \in U, \qquad u = 1, \quad |x| = 1.$$

Show that there are infinitely many solutions to this problem. Which, if any, is the most appropriate solution?