# PDE, Final exam. Due by 5 pm, Wednesday, 14 Dec. 05

## Instructions

1. You may use any reference sources (books, notes, internet), provided you cite these sources appropriately.
2. Resist the temptation to discuss the exam with fellow students.
3. If stuck, try to formulate a partial answer for special cases. Do not be discouraged if you cannot solve all problems.
4. Please inform me of any errors you find in the exam, I will post them immediately.

## Problems

1. (a) Suppose $f \in \mathcal{D}'(\mathbb{R}^n)$ . Is it always possible to solve $\triangle L = f$ in the sense of distributions? (b) A distribution $f$ has compact support if there is a compact set $F \subset \mathbb{R}^n$, such that $< f, \varphi >= 0$ for all test functions $\varphi$ with $\text{supp}(\varphi) \cap F = \phi$. Can one solve $\triangle L = f$ for every $f$ with compact support?

2. *Weyl's lemma:* Suppose $L$ is a harmonic distribution ($\triangle L = 0$ in $\mathcal{D}'(\mathbb{R}^n)$). Then $L$ is a harmonic *function.*

3. Let $F \subset \mathbb{R}^n$ be compact, and suppose $U = \mathbb{R}^n \backslash F$ has a $C^2$ boundary. Consider the positive solution to $u_t = \triangle u$, $x \in U$, $t > 0$ with initial data $u(x, 0) = 0$, $x \in U$ and boundary data $u(x, t) = 1$, $x \in \partial U$, $t > 0$.

(a) Show that $\lim_{t\to\infty} u(x, t) = p_F(x)$, $x \in U$, where $p_F$ denotes the potential of the set $F$.

(b) Use radial symmetry to find $u$ when $F$ is the closed ball $\overline{B(0, R)}$.

(c) Physically, this is a model for heat flow from a body held at unit temperature. The energy lost upto time $t$ is

$$E(t) = \int_U u(x, t)\, dx.$$

It turns out that $\lim_{t\to\infty} t^{-1}E(t) = \text{cap}(F)$. This requires some work, so try and prove the simpler estimate $\limsup_{t\to\infty} t^{-1}E(t) < \infty$ (do not assume $F$ is a ball here, but it may help to use $F \subset B(0, R)$ for $R$ large enough).

4. Problem 4, John p. 162. To simplify matters suppose $g$ in $\mathcal{S}(\mathbb{R}^n)$.

5. *A discrete vortex.* A piecewise constant vector field $(u_1, u_2)$ takes the values $(0, 1)$, $(1, 0)$, $(0, -1)$, and $(-1, 0)$ in the four symmetric parts of the

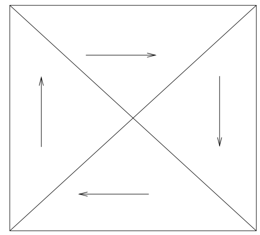

Figure 0.1: Divergence free vector field

square shown in Figure 0.1. Show that $\partial_{x_1} u_1 + \partial_{x_2} u_2 = 0$ in the sense of distributions.

6. Consider the solution operator for the wave equation $u_{tt} - \triangle u = 0$, $x \in \mathbb{R}^n, t > 0$ when $n = 2k+1, k \geq 1$ with initial data $u(x,0) = 0$, $u_t(x,0) = g$, $g \in \mathcal{S}(\mathbb{R}^n)$:

$$u(x,t) = \gamma_n^{-1} \left(t^{-1}\partial_t\right)^{\frac{n-3}{2}} \left( t^{n-2} \fint_{S(x,t)} g(y) dS_y \right),$$

where $\gamma_n = (n-2)(n-4)\ldots 5 \cdot 3$. Show that this solution formula agrees with that obtained by Fourier analysis

$$\hat{u}(\xi,t) = \frac{\sin|\xi|t}{|\xi|}\hat{g}(\xi).$$

7. *Oseen tensor.* The Stokes system is the following set of equations

$$\triangle u - Dp = f, \quad D \cdot u = 0.$$

Here $u : \mathbb{R}^n \to \mathbb{R}^n$ and $p : \mathbb{R}^n \to \mathbb{R}$ are unknown. $f = (f_1, \ldots, f_n)$ is a given forcing, $Dp$ denotes the gradient of $p$, and $D \cdot u$ denotes the divergence of $u$.

Suppose $n \geq 3$. Solve this problem for $f_i \in \mathcal{S}(\mathbb{R}^n)$, $i = 1, \ldots, n$ as follows. (i) To obtain a fundamental solution, fix a unit vector $\omega$, and consider the Stokes system with $f = \delta_0 \omega$. Take the Fourier transform of both sides and eliminate $\hat{p}$ to obtain a linear equation $\hat{u} = A(\xi)\omega$ with a matrix multiplier $A(\xi)$; (ii) Find the inverse Fourier transform of $A(\xi)$; (iii) Write the solution for $f$ using the fundamental solution.

8. We have shown that harmonic functions are characterized by the mean value property on balls. Prove the following characterization of balls by

harmonic functions. Suppose $U \subset \mathbb{R}^n$ is open, convex, bounded, has a $C^2$ boundary and contains the origin. If

$$\frac{1}{|U|} \int_U f(x)\, dx = f(0),$$

for every $f$ harmonic in $U$, then $U$ is a ball centered at the origin. This is also true without assumptions of convexity, but you do not need to prove this.