PDE, Final exam solutions and scores

The exam scores in increasing order were: 37, 40, 47, 48, 53, 51, 55, 57, 57, 59, 71, 76.

Problem 2, Weyl's lemma. 1. We must establish that L is equivalent to a harmonic function. We will use mollification, and the fact that a uniform limit of harmonic functions is harmonic. Fix $\eta \in \mathcal{D}$ with $\int_{\mathbb{R}^n} \eta(x)\,dx = 1$ and $\eta(x) = \eta(-x)$ (for convenience). Let $\eta_m(x) = m^n \eta(mx)$. We know that $L \star \eta_m \to L$ in $\mathcal{D}'$. We will show that $L \star \eta_m$ converges uniformly on compact sets to a harmonic function f.

2. Fix m. Note that $\triangle(L \star \eta_m) = (\triangle L) \star \eta_m = 0$. Thus, we have the mean value property

$$L \star \eta_m(x) = \langle L, \tau_x \eta_m \rangle = ⨍_{S(x,r)} \langle L, \tau_y \eta_m \rangle \, dS_y = \left\langle L, ⨍_{S(x,r)} \tau_y \eta_m dS_y \right\rangle. \tag{0.1}$$

The test function in the last equality is defined by

$$\left(⨍_{S(x,r)} \tau_y \eta_m dS_y \right)(z) = ⨍_{S(x,r)} \eta_m(z-y) dS_y.$$

The last equality in (0.1) involves an interchange of limits justified by using Riemman sums that approximate the integral and converge in $\mathcal{D}$.

3. We cannot pass to the limit $m \to \infty$ yet. The key point is to smear the integral over the shell $S(x,r)$ into one over an annulus, so that we can pass to the limit. This is done as follows. Let ψ be a radial test function such that $\psi = 0$ for $r < 1$ and $r > 2$ and $\int_0^\infty \psi(r)dr = 1$. Since (0.1) holds for every $r > 0$, we integrate in r to obtain

$$L \star \eta_m(x) = \left\langle L, \int_0^r \psi(r) ⨍_{S(x,r)} \tau_y \eta_m dS_y dr \right\rangle.$$

The test function on the right hand side is defined by

$$\left(\int_0^r \psi(r) ⨍_{S(x,r)} \tau_y \eta_m dS_y dr \right)(z) = \int_0^\infty \psi(r) ⨍_{S(x,r)} \eta_m(z-y) dS_y dr$$
$$= \frac{1}{\omega_n} \int_0^\infty \int_{|\omega|=1} \eta_m(z - r\omega)\psi(r) d\omega dr = \frac{1}{\omega_n} \int_{\mathbb{R}^n} \eta_m(z-y) \frac{\psi(|y|)}{|y|^{n-1}} dy.$$

This is simply the convolution of η_m with the test function $\tilde{\psi}(y) := \psi(|y|)|y|^{1-n}$. As $m \to \infty$ we have $\eta_m \star \tilde{\psi} \to \tilde{\psi}$ in the space of test functions. Thus,

$$\lim_{m\to\infty} L \star \eta_m(x) = \left\langle L, \tau_x \tilde{\psi} \right\rangle := f(x),$$

with uniform convergence on compact sets. Since $L \star \eta_m$ is harmonic, so is f. On the other hand, $L \star \eta_m \to L$ in $\mathcal{D}'$. If $\varphi \in \mathcal{D}$ we now obtain

$$\langle L, \varphi \rangle = \int_{\mathbb{R}^n} f(x)\varphi(x)\, dx.$$

Thus, f defines the same distribution as L. □

Problem 3. (a). *1.* Here is a probabilistic proof for $u_t = \frac{1}{2}\triangle u$. Fix $x \in U$, $t > 0$. If W_t is a Brownian motion, let $T_x = \inf_{t>0}\{x + W_t \in \partial U\}$. We then have the probabilistic representation

$$u(x,t) = \mathbb{E}(\mathbf{1}_{T_x<t}) = P(T_x < t) \le 1.$$

Suppose $0 < t_1 < t_2$. Since $\{T_x < t_1\} \subset \{T_x < t_2\}$, we have $P(T_x < t_1) \le P(T_x < t_2)$. Thus, $u(x,t)$ is an increasing function and $\lim_{t\to\infty} u(x,t) := v(x)$ exists. Observe that $v(x) = P(T_x < \infty)$. We must show that $v = p_F(x)$.

2. Consider a sequence of times $0 < T_k \to \infty$ and the shifted solutions $u_k(x,t) = u(x, t + T_k)$, $t \ge -T_k$. Fix $0 < r < T_1$ and a heat ball $E(x,0;r)$ in $U \times (-\infty, 0)$. We then have the mean value property

$$u(x, T_k) = u_k(x,0) = \fint_{E(x,0;r)} u_k(y,s) \frac{|x-y|^2}{(t-s)^2} dy\, ds.$$

Let $k \to \infty$ and use the monotone convergence theorem to find

$$v(x) = \fint_{E(x,0;r)} v(y) \frac{|x-y|^2}{(t-s)^2} dy\, ds.$$

Thus, v is a solution to the heat equation that does not depend on t. That is, $\triangle v = 0$. We also have $v = 1$ on ∂U and $v \to 0$ as $x \to \infty$ since $v = P(T_x < \infty)$. By the uniqueness of the potential, we must have $v = p_F$.

(b). First solve the problem for the unit ball

$$u_t = u_{rr} + \frac{(n-1)}{r} u_r, \quad r > 1$$

subject to $u(1,t) = 1$, $t > 0$, and $u(r,0) = 0$. In such problems, one may reduce to homogeneous boundary conditions by subtracting the steady state solution $u_* = r^{2-n}$. Let $v(r,t) = u_*(r) - u(r,t)$. We then have

$$v_t = v_{rr} + \frac{(n-1)}{r} v_r, \quad v(1,t) = 0, \quad v(r,0) = r^{2-n}.$$

The solution is simplest when $n = 3$. In this case, we set $V = rv$ and obtain

$$V_{rr} = rv_{rr} + 2v_r = rv_t = V_t.$$

Thus, we obtain the 1-D heat equation. This is, of course, the method used for the wave equation. The boundary condition is $V(1,t) = 0, t > 0$ and the initial condition is $V(r,0) = 1$, $r > 0$. For convenience, let $s = r - 1$ so that we have

$$V_t = V_{ss},\ s,t > 0, \quad V(0,t) = 0,\ t > 0 \quad V(s,0) = 1,\ s > 0. \tag{0.2}$$

The Green's function for the heat equation on the half line with zero boundary condition is obtained by reflection. Let $k(s,y,t)$ denote the usual fundamental solution for the heat equation

$$k(s,y,t) = \frac{1}{\sqrt{4\pi t}} \exp(-\frac{(s-y)^2}{4t}).$$

Then the fundamental solution with absorbing boundary conditions satisfies

$$g_t = g_{ss}, \quad g(s,y,0) = \delta_y(s), \quad g(0,y,t) = 0,$$

and is given by reflection

$$g(s,y,t) = k(s,y,t) - k(s,-y,t), \quad s,y,t > 0.$$

The solution to (0.2) is given by

$$V(s,t) = \int_0^\infty g(s,y,t)\,dy = \int_{-s}^{s} k(s,y,t)\,dy = \frac{1}{\sqrt{4\pi t}} \int_{-s}^{s} e^{-y^2/4t}\,dy.$$

Since $\int_{\mathbb{R}} k(s,y,t)dy = 1$ we have

$$1 - V(s,t) = \frac{2}{\sqrt{4\pi t}} \int_s^\infty e^{-y^2/4t}\,dy.$$

Therefore, the solution to the problem is

$$u(r,t) = \frac{1}{r}(1 - V(r-1,t)) = \frac{2}{r\sqrt{4\pi t}} \int_{r-1}^\infty e^{-y^2/4t}\,dy.$$

Finally, the heat flow upto time t is

$$E(t) = \int_{\mathbb{R}^3\setminus B(0,1)} u(|x|,t)\,dx = 4\pi \int_1^\infty r^2 u(r,t)\,dr = 4\pi \int_0^\infty (s{+}1)(1{-}V(s,t))\,ds.$$

We use the solution for $V(s,t)$ and integrate by parts to obtain

$$\int_0^\infty s(1-V(s,t))\,ds = \frac{2}{\sqrt{4\pi t}} \int_0^\infty \frac{s^2}{2} e^{-s^2/4t}\,ds = t,$$

similarly, $\int_0^\infty (1-V(s,t))\,ds = 1$. To summarize, we have

$$E(t) = 4\pi(t+1).$$

If the radius of the ball is R, after a change of scale we have

$$E(t) = 4\pi R(t+1).$$

(c). Given a general compact set F, we enclose it within a ball $B(0,R)$. Let u_F and u_B denote the solutions to the heat equations on the respective domains with initial and boundary conditions of the kind we have considered. Since $u_F(x,t) < u_B(x,t) = 1$ for $x \in S(0,R), t>0$, $u_F(x,0) = u_B(x,0) = 0$ for $|x| > R$ and we have the uniform bound $0 \le u_F \le 1$, $0 \le u_B \le 1$ we apply the maximum principle to conclude $u_F(x,t) < u_B(x,t)$, $|x| > R$. It is then clear that

$$\begin{aligned} E_F(t) &= \int_{\mathbb{R}^n\setminus F} u_F(x,t)\,dx = \int_{B(0,R)\setminus F} u_F(x,t)dx + \int_{|x|>R} u_F(x,t)\,dx \\ &\le |B(0,R)| + E_B(t) = |B(0,R)| + 4\pi R(t+1). \end{aligned}$$

Thus, $\limsup_{t\to\infty} t^{-1} E_F(t) \le 4\pi R < \infty$. □

Problem 4, Radon transform. (a). I will use the notation $G(\omega,p)$ for the Radon transform, reserving the letter ξ for the Fourier transform. In all that follows $g \in \mathcal{S}(\mathbb{R}^n)$, $|\omega| = 1$, $d\omega$ is the surface measure on the unit sphere, and $d^{n-1}y$ denotes the $n-1$ dimensional Lebesgue measure on hyperplanes. To prove the first identity, after a suitable translation and rotation we may suppose that $x = 0$ and $\omega = (1,0,\ldots,0)$. We then have

$$\begin{aligned} \int_{\mathbb{R}^n} |y_1| \triangle_y g(y)\ ,dy &= \int_{\mathbb{R}^n} |y_1| \left(\partial_{y_1}^2 g + \sum_{k=2}^n \partial_{y_k}^2 \right) g(y)\,dy \\ &= -\int_{\mathbb{R}^n} \operatorname{sgn}(y_1) \partial_{y_1} g\,dy = \int_{y_1<0} \partial_{y_1} g\,dy - \int_{y_1>0} \partial_{y_1} g\,dy \\ &= 2\int_{y_1=0} g(0,y_2,\ldots,y_n)\,d^{n-1}y = 2G(\omega, x\cdot\omega). \end{aligned} \tag{0.3}$$

(b). We use (2.82) and integrate by parts to obtain

$$\begin{aligned} g(x) &= \int_{\mathbb{R}^n} k(x,y)\triangle_y^{\frac{n+1}{2}} g(y)\,dy = \int_{\mathbb{R}^n} \triangle_y^{\frac{n-1}{2}} k(x,y)\triangle_y g(y)\,dy \\ &= \int_{\mathbb{R}^n} \triangle_x^{\frac{n-1}{2}} k(x,y)\triangle_y g(y)\,dy = \triangle_x^{\frac{n-1}{2}} \int_{\mathbb{R}^n} k(x,y)\triangle_y g(y)\,dy. \end{aligned} \tag{0.4}$$

We then combine (2.81), (0.3) and (0.4) to obtain

$$\begin{aligned} g(x) &= \frac{d_n}{c_n}\triangle_x^{\frac{n-1}{2}} \int_{|\omega|=1}\int_{\mathbb{R}^n} |(x-y)\cdot\omega|\,\triangle_y g(y)\,dy \\ &= \frac{2d_n}{c_n}\triangle_x^{\frac{n-1}{2}} \int_{|\omega|=1} G(\omega, x\cdot\omega)\,d\omega. \end{aligned}$$

(c) Let $\hat{g}(\xi), \xi \in \mathbb{R}^n$ denote the Fourier transform of g as usual. Let $\xi = \omega q$.

$$\begin{aligned} &\hat{g}\,(\omega q) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-ix\cdot\omega q} g(x)dx = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}} e^{-isq}\int_{x\cdot\omega=s} g(x)d^{n-1}x\,ds \\ &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}} e^{-isq} G(\omega,s)\,ds = \frac{1}{(2\pi)^{\frac{n-1}{2}}}\hat{G}(\omega,q). \end{aligned}$$

Therefore, using the Fourier inversion formula

$$\begin{aligned} &g(x) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\hat{g}(\xi)d\xi = (2\pi)^{-n/2}\int_0^\infty q^{n-1}\,dq\int_{|\omega|=1} e^{iqx\cdot\omega}\hat{g}(q\omega)d\omega \\ &= (2\pi)^{\frac{1-2n}{2}}\int_0^\infty q^{n-1}\,dq\int_{|\omega|=1} e^{iqx\cdot\omega}\hat{G}(\omega,q)\,d\omega. \end{aligned}$$

(d) To obtain (2.90) from (2.89) we substitute the inversion formula

$$G(\omega, x\cdot\omega) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ix\cdot\omega q}\,\hat{G}(\omega,q)\,dq$$

in (2.89) to find

$$\begin{aligned} g(x) &= \frac{2d_n}{c_n\sqrt{2\pi}}\int_{\mathbb{R}}\int_{|\omega|=1}\triangle_x^{\frac{n-1}{2}} e^{ix\cdot\omega q}\,\hat{G}(\omega,q)\,dq\,d\omega \\ &= \frac{2d_n}{c_n\sqrt{2\pi}}\int_0^\infty\int_{|\omega|=1} (-|q|)^{n-1}\,e^{ix\cdot\omega q}\,\hat{G}(\omega,q)\,d\omega\,dq, \end{aligned}$$

using the symmetry of (2.92). Finally, plug in constants as in (2.91) to obtain (2.90).

(e) We switch to polar coordinates $x = r\cos\varphi, y = r\sin\varphi$. Then $p^2 + t^2 = r^2$ and for fixed p, $t = \pm\sqrt{r^2 - p^2}$. Therefore, taking into account both branches we have

$$\begin{aligned} G(\theta, p) &= \int_p^\infty g(p\cos\theta + \sqrt{r^2-p^2}\sin\theta, p\sin\theta - \sqrt{r^2-p^2}\cos\theta)\,\frac{dt}{\sqrt{r^2-p^2}} \\ &+ \int_p^\infty g(p\cos\theta - \sqrt{r^2-p^2}\sin\theta, p\sin\theta + \sqrt{r^2-p^2}\cos\theta)\,\frac{dt}{\sqrt{r^2-p^2}}, \end{aligned}$$

which may be integrated in θ to yield

$$\int_0^{2\pi} G(\theta, p)\, d\theta = 4\pi \int_p^\infty \frac{r}{\sqrt{r^2-p^2}} I(r)\, dr,$$

where $I(r) = (2\pi r)^{-1} \int_{S(0,r)} g dS$. If we multiply this equation by $p/\sqrt{p^2-s^2}$ and integrate from s to ∞ we have the integral

$$\int_s^\infty rI(r)\, dr \left(\int_s^r \frac{p\, dp}{\sqrt{p^2-s^2}\sqrt{r^2-p^2}} \right).$$

Observe that the inner integral is simply a constant. Indeed, we have

$$\int_s^r \frac{p\, dp}{\sqrt{p^2-s^2}\sqrt{r^2-p^2}} = \frac{1}{2}\int_{s^2}^{r^2} \frac{dx}{\sqrt{x-s^2}\sqrt{r^2-x}} = \frac{1}{2}\int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \frac{\pi}{2}.$$

Thus, we have the expression

$$\int_s^\infty rI(r)\, dr = \frac{1}{2\pi^2}\int_s^\infty \int_0^{2\pi} \frac{p}{\sqrt{p^2-s^2}} G(\theta, p)\, d\theta\, dp.$$

□

Problem 5, Discrete vortex. Since u is piecewise constant, it will suffice to study the jumps. The crux of the problem is that the normal component of u is continuous. For the geometry at hand, a simple proof goes as follows. Let φ be a test function with compact support in the first quadrant $(0,1)^2$. Let $V_\pm$ be as shown in Figure 0.1. By definition,

$$\langle \partial_{x_1} u_1 + \partial_{x_2} u_2, \varphi \rangle = -\langle u_1, \varphi_{x_1} \rangle - \langle u_2, \varphi_{x_2} \rangle = -\int_{V_-} \varphi_{x_1}\, dx + \int_{V_+} \varphi_{x_2}\, dx,$$

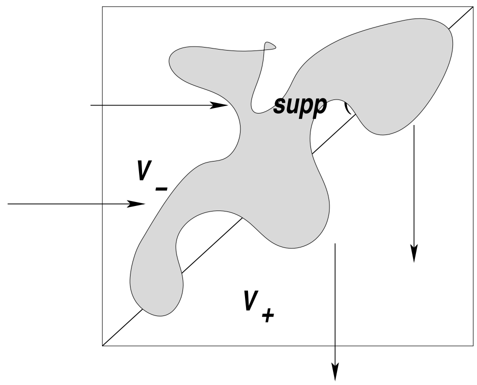

Figure 0.1: Discrete vortex

because $u = (1,0)$ in V_- and $u = (0,-1)$ in V_+. We now compute

$$\begin{aligned}
\int_{V_-} \varphi_{x_1}\,dx &= \int_0^1 \int_0^{x_2} \varphi_{x_1}\,dx_1\,dx_2 = \int_0^1 \varphi(x_2,x_2)\,dx_2,\\
\int_{V_+} \varphi_{x_2}\,dx &= \int_0^1 \int_0^{x_1} \varphi_{x_2}\,dx_2\,dx_1 = \int_0^1 \varphi(x_1,x_1)\,dx_1.
\end{aligned}$$

Thus, $\langle \partial_{x_1} u_1 + \partial_{x_2} u_2, \varphi\rangle = 0$ for φ with support in the first quadrant. A similar argument works for each quadrant, and also for a test function with support in a neighborhood of the origin. An arbitrary test function can be separated into such pieces by a partition of unity. $\square$

Problem 6. 1. Suppose $g \in \mathcal{S}(\mathbb{R}^n)$. The solution formula may be written as an integral over the unit sphere as

$$u(x,t) = \gamma_n^{-1}\left(t^{-1}\partial_t\right)^{\frac{n-3}{2}}\left(t^{n-2} ⨍_{S(0,1)} g(x+t\omega)d\omega\right), \tag{0.5}$$

with $\gamma_n = (n-2)(n-4)\ldots 5\cdot 3$. Take the Fourier transform of both sides, and switch the order of integrals using Fubini's theorem to find

$$\hat{u}(\xi,t) = \hat{g}(\xi)\left(\gamma_n^{-1}\left(t^{-1}\partial_t\right)^{\frac{n-3}{2}}\left(t^{n-2} ⨍_{S(0,1)} e^{i\xi\cdot t\omega}\,d\omega\right)\right).$$

The task is to show that the term in brackets is the multiplier $|\xi|^{-1}\sin|\xi|t$.

2. First compute the integral. Let $\xi = |\xi|\hat{\xi}$ and let θ be the polar angle from the unit vector $\hat{\xi}$. Then $\xi \cdot t\omega = |\xi| t\hat{\xi} \cdot \omega$ and we have

$$\fint_{S(0,1)} e^{i\xi\cdot t\omega}\, d\omega = \frac{\omega_{n-1}}{\omega_n} \int_0^\pi e^{-i|\xi|t\cos\theta} \sin^{n-1}\theta\, d\theta$$
$$= \frac{\omega_{n-1}}{\omega_n} \int_{-1}^1 \cos(p|\xi|t)(1-p^2)^{k-1}\, dp. \tag{0.6}$$

In the last step we have substituted $p = \cos\theta$, and $n = 2k+1$. If $n = 3$ (or $k = 1$) this is simply

$$\frac{2\pi}{4\pi}\frac{2\sin|\xi|t}{|\xi t} = \frac{\sin|\xi|t}{|\xi|t}.$$

We substitute in (0.5) and use $\gamma_3 = 1$ to obtain the desired multiplier.

3. For general n, one could use a table of mathematical functions. For example, the integral can be expressed in terms of Bessel functions as [1]

$$\int_{-1}^1 \cos(p|\xi|t)(1-p^2)^{k-1}\, dp = \frac{\pi^{1/2}\Gamma(k)}{\left(\frac{1}{2}|\xi|t\right)^{k-1/2}} J_{k-1/2}\left(|\xi|t\right).$$

After collecting constants (using for example, $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$) and a change of scale $z = |\xi|t$ it turns out that one has to verify the identity

$$c_n \left(z^{-1}\partial_z\right)^l \left[z^{l+1/2}J_{l+1/2}(z)\right] = \sin z,$$

where

$$n = 2l+3, \quad c_n = \frac{2^{n/2-1}\Gamma\left(\frac{n}{2}\right)}{\gamma_n}.$$

I had assumed that this would be easy to find in a table of special functions, but surprisingly didn't find it. This may be restated using 'spherical Bessel functions' [2]

$$j_l(z) = \sqrt{\frac{\frac{\pi}{2}}{z}} J_{l+1/2}(z),$$

as the identity

$$c_n\sqrt{\frac{\pi}{2}} \left(z^{-1}\partial_z\right)^l \left[z^{l+1}j_l(z)\right] = \sin z.$$

By a direct calculation I worked this out for

$$j_0(z) = \frac{\sin z}{z}, \quad j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z}.$$

[1] *M. Abramowitz, I. Stegun*, Handbook of Mathematical Functions, p. 360

[2] *op. cit.* , p. 437

There is an identity for j_l that goes the wrong way:

$$j_l(z) = z^l \left(-z^{-1}\partial_z\right)^l \frac{\sin z}{z}.$$

At the end of the day we must then verify that

$$c_n\sqrt{\frac{\pi}{2}}\left(z^{-1}\partial_z\right)^l \left[z^{2l+1}\left(-z^{-1}\partial_z\right)^l \frac{\sin z}{z}\right] = \sin z,$$

which seems surprising. $\square$

Problem 7, Oseen tensor. (a) Fix a unit vector ω, and consider the equation

$$\triangle u - Dp = \omega\delta_0, \quad D\cdot u = 0.$$

Take the Fourier transform on both sides to obtain

$$-|\xi|^2\hat{u} - i\xi\hat{p} = \frac{1}{(2\pi)^{n/2}}\omega, \quad \xi\cdot\hat{u} = 0.$$

Take the dot product of the first equation with ξ to eliminate $\hat{u}$, and find $-i|\xi|^2\hat{p} = (2\pi)^{-n/2}\omega\cdot\xi$. We then have

$$\hat{u} = \hat{A}(\xi)\omega, \quad \hat{A}(\xi) = \frac{|\xi|^{-2}}{(2\pi)^{n/2}}\left(Id - \frac{\xi\otimes\xi}{|\xi|^2}\right).$$

(Here $\xi\otimes\xi$ is the rank one matrix with components $\xi_i\xi_j$.
(b) $|\xi|^{-2}$ is a tempered distribution. For any $0<\alpha<n$ let

$$c_{n-\alpha} = \frac{2^{-\alpha/2}}{\Gamma\left(\frac{\alpha}{2}\right)}.$$

We then have the symmetric identity (I am not sure the constants were correct when I stated this in lecture)

$$\mathcal{F}\left(c_\alpha|x|^{-\alpha}\right) = c_{n-\alpha}|\xi|^{\alpha-n}, \quad 0<\alpha<n. \tag{0.7}$$

Of particular importance is the case when we wish to invert $(2\pi)^{-n/2}|\xi|^{-2}$. This yields the fundamental solution of the Laplacian, and so we have

$$\mathcal{F}\left(\frac{|x|^{2-n}}{\omega_n(n-2)}\right) = \frac{|\xi|^{-2}}{(2\pi)^{n/2}}, \quad \text{and} \quad \frac{c_{n-2}}{c_2} = \frac{(2\pi)^{n/2}}{\omega_n(n-2)}. \tag{0.8}$$

The second term in $\hat{A}(\xi)$ is a little more tricky. One approach is to guess

$$\mathcal{F}^{-1}\left(\hat{A}(\xi)\right) = A(x) = |x|^{2-n}\left(a_1 Id + a_2 \frac{x \otimes x}{|x|^2}\right),$$

for suitable constants a_i based on the symmetry of $\hat{A}(\xi)$. Here is another proof based on a calculation of interest in itself.

$$\mathcal{F}\left(\log|x|\right) = -(n-2)\frac{c_{n-2}}{c_2}|\xi|^{-n}. \tag{0.9}$$

This is an example of an 'endpoint' calculation (compare with (0.7)) We will use this calculation with the roles of x and ξ interchanged, but for future reference it seems better to state it in this form. There is a symmetry between x and ξ in transforms of power laws and one may use the notation interchangeably.

Proof. We have $\mathcal{F}(1) = (2\pi)^{n/2}\delta_0$. Therefore, $\mathcal{F}x_i = -i(2\pi)^{n/2}\partial_{\xi_i}\delta_0$ and by convolution (see eg. Rauch, p.83)

$$\begin{aligned}\mathcal{F}\left(\frac{x_i}{|x|^2}\right) &= -i\partial_{\xi_i}\delta_0 \star \frac{c_{n-2}}{c_2|\xi|^{n-2}} \\ &= -i\delta_0 \star \partial_{\xi_i}\frac{c_{n-2}}{c_2|\xi|^{n-2}} = i(n-2)\frac{c_{n-2}}{c_2}\frac{\xi_i}{|\xi|^n}.\end{aligned}$$

(A factor of $(2\pi)^{n/2}$ is absorbed in the convolution). On the other hand, we also have $\partial_{x_i}\log(|x|) = x_i|x|^{-2}$ so that $-i\xi_i\mathcal{F}\log|x| = \mathcal{F}x_i|x|^{-2}$. Now compare terms to obtain (0.9). $\square$

We use (0.9) as follows. We differentiate twice to find

$$\partial_{\xi_i}\partial_{\xi_j}\log(|\xi|) = -2\frac{\xi_i\xi_j}{|\xi|^4} + \frac{\delta_{ij}}{|\xi|^2}.$$

Now combine the various calculations so far to find

$$\mathcal{F}^{-1}\left(\frac{\xi_i\xi_j}{|\xi|^4}\right) = \frac{1}{2}\frac{c_{n-2}}{c_2|x|^{n-2}}\left(\delta_{ij} - (n-2)\frac{x_i x_j}{|x|^2}\right).$$

This may be combined with (0.8) to yield the fundamental matrix solution

$$A(x) = \frac{|x|^{2-n}}{2\omega_n(n-2)}\left(Id + (n-2)\frac{x \otimes x}{|x|^2}\right).$$

(c) A is called the Oseen tensor. If $f_i \in \mathcal{S}(\mathbb{R}^n)$ we have $u(x) = (A \star f)(x)$. $\square$

Problem 8, Kuran's theorem. The following elegant proof is due to Kuran (*Bull. London. Math. Soc., 4, p.311-312, 1972*).

Let $B := B(0,r) \subset U$ be the largest ball contained in U. Then there exists a point $x_0 \in U \backslash \overline{B(0,r)}$ such that $|x - x_0| = r$. Consider the function

$$h(x) = \frac{|x|^2 - r^2}{|x - x_0|^n} + r^{2-n}.$$

h is harmonic in $\mathbb{R}^n \backslash \{x_0\}$ because the first term is a constant multiple of the Poisson kernel (see Thm. 1.13 in the notes). By assumption,

$$0 = h(0) = \frac{1}{|U|} \int_U h(y)\, dy,$$

and since h is harmonic, we also have the mean value property

$$0 = h(0) = \frac{1}{|B|} \int_B h(y)\, dy.$$

We combine these equalities with $h \geq r^{2-n}$ on $U \backslash B$ to obtain

$$0 = \int_{U \backslash B} h(y)\, dy \geq r^{2-n} \int_{U \backslash B} dy = r^{2-n} \, |U \backslash B| \geq r^{2-n} \left| U \backslash \overline{B} \right|.$$

Since U is open, this implies $U = B$. Observe that the assumptions of convexity and smoothness of the boundary are not needed. $\square$

Remark 0.1. I apologize for the hint which can be described as misleading (if one is charitable), or wrong (if one is accurate). What I had in mind was an argument of the following kind. Suppose $n = 2$. Let U be a domain containing the origin with the mean value property. We then have the identity

$$|U| = \int_U e^{\xi z}\, dxdy, \quad \xi \in \mathbb{C}.$$

This is equivalent to an infinite set of 'moment equations'

$$|U| = \int_U dxdy, \quad 0 = \int_U z^k\, dxdy, \quad k \geq 1.$$

obtained by differentiation with respect to ξ, and evaluating at $\xi = 0$. The idea roughly is the following: if the domain has the property that it is annihilated by a rich enough class of functions (eg. exponentials, polynomials), is this enough to determine it is a ball? We used such a uniqueness principle to prove the potential of a measure is unique. If the domain is convex and we write its boundary as two graphs $a_\pm(y), y \in [y_{\min}, y_{\max}]$ by testing against $e^{\xi x} \sin \xi y$ with $\xi \in \mathbb{R}, \xi \neq 0$ we have

$$0 = \int_{y_{\min}}^{y_{\max}} \frac{\sin \xi y}{\xi} \left(e^{\xi a_+(y)} - e^{\xi a_-(y)} \right).$$

The flawed argument was to conclude that $e^{\xi a_+(y)} - e^{\xi a_-(y)}$ is even, and thus deduce some symmetry of U. But this is not true.