Linear Algebra

MA 242 (Spring 2013)
Instructor: M. Chirilus-Bruckner

Matrix Algebra

– sum, scalar multiple, product, powers, inverse, transpose –

- **Properties of matrix addition and scalings**

  Let A, B and C be matrices of the same size, and let r and s be scalars.

  a. $A + B = B + A$
  b. $(A + B) + C = A + (B + C)$
  c. $A + 0 = A$
  d. $r(A + B) = rA + rB$
  e. $(r + s)A = rA + sA$
  f. $r(sA) = s(rA)$

- **Matrix multiplication**

  If A is a $m \times n$ matrix and B is a $n \times p$ matrix and $b_1, \ldots, b_p$ are the columns of B, then the product AB is

$$AB = A\,[b_1 \cdots b_p] = [Ab_1 \cdots Ab_p]$$

$$\begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ldots & \vdots & \ldots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \ldots & \vdots & \ldots & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

- **Row-Column rule for matrix multiplication**

  $(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$

- $\boxed{\textbf{Properties of matrix multiplication}}$

  Let A be an $m \times n$ matrix and B, C such that all the sums and products are defined.

  a. $A(BC) = (AB)C$ (associative law)
  b. $A(B + C) = AB + AC$ (left distributive law)
  c. $(B + C)A = BA + CA$ (right distributive law)
  d. $r(AB) = (rA)B = A(rB)$ (scaling products)
  e. $I_m A = A = AI_n$ (identity for matrix multiplication)

- $\boxed{\textbf{Peculiarities of matrix multiplication}}$

  1. In general, $AB \neq BA$.
  2. If $AB = AC$ it is in general not true that $B = C$.
  3. If $AB = 0$, then you cannot conclude that either $A = 0$ or $B = 0$.

- $\boxed{\textbf{Powers of a matrix } A^k}$

  If A is a $n \times n$ matrix, then $A^k = \underbrace{A \cdots A}_{k \ times}$. If $k = 0$, then $A^k = A^0 = I$.

- $\boxed{\textbf{Transpose of a matrix } A^T}$

  For a given $m \times n$ matrix A, its transpose A^T has as columns the rows of A, so A^T is a $n \times m$ matrix.

- $\boxed{\textbf{Inverse of a matrix } A^{-1}}$

  A matrix A of size $n \times n$ is said to be invertible if there is an $n \times n$ matrix X such that

  $$XA = I \qquad AX = I$$

  where $I = I_n$ is the identity matrix. This matrix X is called the inverse of A and is usually denoted by A^{-1}. In other words,

  $$A^{-1}A = I \qquad AA^{-1} = I$$

  Analogy with numbers: $5^{-1}5 = 1, 55^{-1} = 1$ *Note:* The inverse might not always exist!

- $\boxed{\textbf{Algorithm for finding } A^{-1}}$

  Row reduce the augmented matrix for $AX = I$:

  $$[A \,|\, I] \sim \left[I \,|\, A^{-1}\right]$$

.

- $\boxed{\textbf{Formula for finding } A^{-1} \textbf{ for in the } 2\times 2 \textbf{ case and the determinant of a matrix}}$

  If for a matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

  the **determinant** does not vanish, so

$$\det(A) = ad - bc \neq 0,$$

  then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

" Switch diagonally, negate the wings and scale with a cross."