Linear Algebra

MA 242 (Spring 2013)

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Inner product

Given two vectors u and v in $\mathbb{R}^n$ the product

$$u \cdot v = u^T v = [u_1 \; u_2 \; \cdots u_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + \ldots + u_n v_n$$

is called **inner product** of u and v (also called **dot product** or **scalar product** in the literature).

** Properties*

- For $u, v \in \mathbb{R}^n$ it holds that
$$u \cdot v = v \cdot u$$
- For $u_1, \ldots, u_p, w \in \mathbb{R}^n$ and scalars $c_1, \ldots, c_p$ it holds that
$$(c_1 u_1 + c_2 u_2 + \ldots + c_p u_p) \cdot w = c_1 (u_1 \cdot w) + c_2 (u_2 \cdot w) + \ldots + c_p (u_p \cdot w)$$
- For $u \in \mathbb{R}^n$ it holds that
$$u \cdot u \geq 0$$
and $u \cdot u = 0$ if and only if $u = 0$.

** Inner product and length (or norm) of a vector*

For $u \in \mathbb{R}^n$ it holds that

$$\sqrt{u \cdot u} = \sqrt{u_1^2 + \ldots + u_n^2} = \|u\|$$

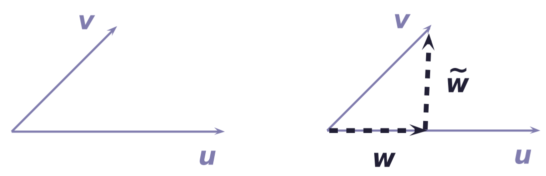
** Inner product and angle between two vectors*

For $u, v \in \mathbb{R}^n$ it holds that

$$u \cdot v = \|u\| \, \|v\| \, \cos(\phi)$$

where ϕ is the angle between the lines Span$\{u\}$ and Span$\{v\}$. (Recall: Law of cosines)

** Inner product and projections*



Write

$$v = w + \widetilde{w},$$

where

$$w \in \text{Span}\{u\}, \quad w \cdot \widetilde{w} = 0,$$

so $w = \alpha u$ for some (unknown) scalar α, that can be computed by

$$\alpha = \frac{v \cdot u}{u \cdot u}.$$