Chapter 2 SUPPLEMENTARY EXERCISES

1. a. True. If A and B are $m \times n$ matrices, then B^T has as many rows as A has columns, so AB^T is defined. Also, A^TB is defined because A^T has m columns and B has m rows.

   b. False. B must have 2 columns. A has as many columns as B has rows.

   c. True. The ith row of A has the form $(0, \ldots, d_i, \ldots, 0)$. So the ith row of AB is $(0, \ldots, d_i, \ldots, 0)B$, which is d_i times the ith row of B.

   d. False. Take the zero matrix for B. Or, construct a matrix B such that the equation $B\mathbf{x} = \mathbf{0}$ has nontrivial solutions, and construct C and D so that $C \neq D$ and the columns of $C - D$ satisfy the equation $B\mathbf{x} = \mathbf{0}$. Then $B(C - D) = 0$ and $BC = BD$.

   e. False. Counterexample: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

   f. False. $(A + B)(A - B) = A^2 - AB + BA - B^2$. This equals $A^2 - B^2$ if and only if A commutes with B.

17. Let A be a 6x4 matrix and B a 4x6 matrix. Since B has more columns than rows, its six columns are linearly dependent and there is a nonzero $\mathbf{x}$ such that $B\mathbf{x} = \mathbf{0}$. Thus $AB\mathbf{x} = A\mathbf{0} = \mathbf{0}$. This shows that the matrix AB is not invertible, by the IMT. (Basically the same argument was used to solve Exercise 22 in Section 2.1.)

Note: (In the *Study Guide*) It is possible that BA is invertible. For example, let C be an invertible 4x4 matrix and construct $A = \begin{bmatrix} C \\ 0 \end{bmatrix}$ and $B = [C^{-1} \;\; 0]$. Then $BA = I_4$, which is invertible.

18. By hypothesis, A is 5x3, C is 3x5, and $CA = I_3$. Suppose $\mathbf{x}$ satisfies $A\mathbf{x} = \mathbf{b}$. Then $CA\mathbf{x} = C\mathbf{b}$. Since $CA = I$, $\mathbf{x}$ must be $C\mathbf{b}$. This shows that $C\mathbf{b}$ is the only solution of $A\mathbf{x} = \mathbf{b}$.

Section 3.2, page 175

2. A constant may be factored out of one row.

4. A row replacement operation does not change the determinant.

24. Linearly independent

26. Linearly dependent

Ch.3 Suppl. 6. 12

Section 3.3, page 184

20. 7   22. 21   24. 15

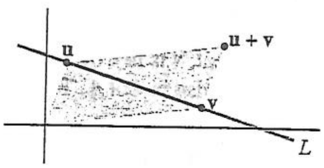
Chapter 4

Section 4.1, page 195

2. a. Given $\begin{bmatrix} x \\ y \end{bmatrix}$ in W and any scalar c, the vector $c\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$ is in W because $(cx)(cy) = c^2(xy) \geq 0$, since $xy \geq 0$.

   b. *Example:* If $\mathbf{u} = \begin{bmatrix} -1 \\ -7 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, then $\mathbf{u}$ and $\mathbf{v}$ are in W, but $\mathbf{u}+\mathbf{v}$ is not in W.

4.



u and v are on the line, but u + v is not.

6. No, the zero polynomial is not in the set.

8. Yes. The zero vector is in the set, H. If $\mathbf{p}$ and $\mathbf{q}$ are in H, then $(\mathbf{p}+\mathbf{q})(0) = \mathbf{p}(0) + \mathbf{q}(0) = 0$, so $\mathbf{p}+\mathbf{q}$ is in H. Also, for any scalar c, $(c\mathbf{p})(0) = c \cdot \mathbf{p}(0) = c \cdot 0 = 0$, so $c\mathbf{p}$ is in H.

10. $H = \text{Span}\,\{\mathbf{v}\}$, where $\mathbf{v} = \begin{bmatrix} 3 \\ 0 \\ -7 \end{bmatrix}$. By Theorem 1, H is a subspace of $\mathbb{R}^3$.

12. $W = \text{Span}\,\{\mathbf{u}, \mathbf{v}\}$, where $\mathbf{u} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 4 \\ 0 \\ -3 \\ 5 \end{bmatrix}$. By Theorem 1, W is a subspace of $\mathbb{R}^4$.

20. a. The constant function $\mathbf{f}(t) = 0$ is continuous. The sum of two continuous functions is continuous. A constant multiple of a continuous function is continuous.

   b. Let $H = \{\mathbf{f} \text{ in } C[a,b] : \mathbf{f}(a) = \mathbf{f}(b)\}$. Take $\mathbf{f}$ and $\mathbf{g}$ in H and let c be a real number. Then, the function $\mathbf{f}+\mathbf{g}$ is in $C(a,b)$, because the sum of two continuous functions is continuous. Also,

$$(\mathbf{f}+\mathbf{g})(a) = \mathbf{f}(a) + \mathbf{g}(a) = \mathbf{f}(b) + \mathbf{g}(b) = (\mathbf{f}+\mathbf{g})(b),$$

   which shows that $\mathbf{f}+\mathbf{g}$ is in H. Next, using the definition of $c\mathbf{f}$,

$$(c\mathbf{f})(a) = c(\mathbf{f}(a)) = c(\mathbf{f}(b)),$$

   because $\mathbf{f}$ is in H. Also, $c(\mathbf{f}(b)) = c\mathbf{f}(b) = (c\mathbf{f})(b)$. This shows that the scalar multiple, $c\mathbf{f}$, is in H. Thus H is closed under sums and scalar multiples, so H is a subspace.

22. Yes. See the proof of Theorem 12 in Section 2.8 for a proof that is similar to the one needed here.

Section 4.2, page 205

2. $\begin{bmatrix} 2 & 6 & 4 \\ -3 & 2 & 5 \\ -5 & -4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, so $\mathbf{w}$ is in Nul A.

8. W is not a subspace because $\mathbf{0}$ is not in W. The vector $(0, 0, 0)$ does not satisfy the condition $3r - 2 = 3s + t$.

10. W is a subspace of $\mathbb{R}^4$ by Theorem 2, because W is the set of solutions of the homogeneous system

32. $\mathbf{p}_1(t) = t$, $\mathbf{p}_2(t) = t^2$. The range of T is $\left\{\begin{bmatrix} a \\ a \end{bmatrix} : a \text{ real}\right\}$.

Section 4.3, page 213

2. This set does not form a basis for $\mathbb{R}^3$. The set is linearly dependent because the zero vector is in the set. The columns of $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ do not span $\mathbb{R}^3$, by the Invertible Matrix Theorem.

4. These vectors form a basis for $\mathbb{R}^3$. See Example 5 for an example of a justification.

10. $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ 12. $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$

20. The three simplest answers are $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_3\}$, and $\{\mathbf{v}_2, \mathbf{v}_3\}$. Other answers are possible.

24. Let $A = [\,\mathbf{v}_1 \ \cdots \ \mathbf{v}_n\,]$. Since A is square and its columns are linearly independent, its columns also span $\mathbb{R}^n$, by the Invertible Matrix Theorem. So $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$.

Section 4.4, page 222

2. $\begin{bmatrix} -26 \\ 1 \end{bmatrix}$ 4. $\begin{bmatrix} 8 \\ -5 \\ 1 \end{bmatrix}$ 6. $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ 8. $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Section 4.5, page 229

6. $\begin{bmatrix} 3 \\ 0 \\ -7 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 5 \\ 1 \end{bmatrix}$; dim is 3

8. $\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$; dim is 3

10. 1   12. 3   14. 3, 4   16. 0, 2   18. 1, 2

Chapter 5

Section 5.1, page 271

2. Yes   4. Yes, $\lambda = 3$ 6. No

8. Yes, $\begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$ 12. $\lambda = 3$: $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$; $\lambda = 7$: $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ 14. $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

18. 5, 0, 3

Section 5.2, page 279

2. $\lambda^2 + 3\lambda + 2$; $-2, -1$

4. $\lambda^2 - 11\lambda + 18$; 2, 9

6. $\lambda^2 - 14\lambda + 49$; 7, 7

8. $\lambda^2 + 3\lambda - 10$; $-5, 2$

10. $-\lambda^3 + 15\lambda^2 - 73\lambda + 115$

16. 3, 2, 6, -5

18. $h = 3$

Section 5.3, page 286

2. $\begin{bmatrix} 321 & -160 \\ 480 & -239 \end{bmatrix}$

4. $\begin{bmatrix} -3\cdot(-3)^k + 4\cdot(-2)^k & 6\cdot(-3)^k - 6\cdot(-2)^k \\ -2\cdot(-3)^k + 2\cdot(-2)^k & 4\cdot(-3)^k - 3\cdot(-2)^k \end{bmatrix}$

6. $\lambda = 3$: $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix}$; $\lambda = 4$: $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

When an answer involves a diagonalization, $A = PDP^{-1}$, the factors P and D are not unique, so your answer may differ from that given here.

8. Not diagonalizable. The eigenvalue 3 has multiplicity two, but the associated eigenspace is only one-dimensional.

12. $P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

16. Not diagonalizable. The only real eigenvalue is 0 and its eigenspace is only one-dimensional.

18. $P = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$, $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Section 5.4, page 293

10. a. For any $\mathbf{p}, \mathbf{q}$ in $\mathbb{P}_3$ and any scalar c,

$$T(\mathbf{p}+\mathbf{q}) = \begin{bmatrix} (\mathbf{p}+\mathbf{q})(-2) \\ (\mathbf{p}+\mathbf{q})(3) \\ (\mathbf{p}+\mathbf{q})(1) \\ (\mathbf{p}+\mathbf{q})(0) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(-2) \\ \mathbf{p}(3) \\ \mathbf{p}(1) \\ \mathbf{p}(0) \end{bmatrix} + \begin{bmatrix} \mathbf{q}(-2) \\ \mathbf{q}(3) \\ \mathbf{q}(1) \\ \mathbf{q}(0) \end{bmatrix} = T(\mathbf{p}) + T(\mathbf{q})$$

$$T(c\cdot\mathbf{p}) = \begin{bmatrix} (c\cdot\mathbf{p})(-2) \\ (c\cdot\mathbf{p})(3) \\ (c\cdot\mathbf{p})(1) \\ (c\cdot\mathbf{p})(0) \end{bmatrix} = c\cdot\begin{bmatrix} \mathbf{p}(-2) \\ \mathbf{p}(3) \\ \mathbf{p}(1) \\ \mathbf{p}(0) \end{bmatrix} = c\cdot T(\mathbf{p})$$

b. $\begin{bmatrix} 1 & -2 & 4 & -8 \\ 1 & 3 & 9 & 27 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

12. $\begin{bmatrix} -4 & 0 \\ 2 & -2 \end{bmatrix}$

16. $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

18. If there is a basis $\mathcal{B}$ such that $[T]_\mathcal{B}$ is diagonal, then A is similar to a diagonal matrix, by the second paragraph following Example 3. If A has a set of three eigenvectors that is linearly independent, then there will be a choice of $\mathcal{B}$ such that $[T]_\mathcal{B}$ is diagonal. However, since A has only two distinct eigenvalues, it may be the case that a set of linearly independent eigenvectors contains at most two vectors.

5-6 (last page)

Chapter 5 Supplementary Exercises, page 326

1. a. True. If A is invertible and if $A\mathbf{x} = 1\cdot\mathbf{x}$ for some nonzero $\mathbf{x}$, then left-multiply by A^{-1} to obtain $\mathbf{x} = A^{-1}\mathbf{x}$, which may be rewritten as $A^{-1}\mathbf{x} = 1\cdot\mathbf{x}$. Since $\mathbf{x}$ is nonzero, this shows that 1 is an eigenvalue of A^{-1}.

b. False. If A is row equivalent to the identity matrix, then A is invertible. The matrix in Example 4 in Section 5.3 shows that an invertible matrix need not be diagonalizable. Also, see Exercise 31 in Section 5.3.

c. True. If A contains a row or column of zeros, then A is not row equivalent to the identity matrix and thus is not invertible. By the Invertible Matrix Theorem (as stated in Section 5.2), 0 is an eigenvalue of A.

d. False. Consider a diagonal matrix D whose eigenvalues are 1 and 3; that is, its diagonal entries are 1 and 3. Then D^2 is a diagonal matrix whose eigenvalues (diagonal entries) are 1 and 9. In general, the eigenvalues of A^2 are the *squares* of the eigenvalues of A.

e. True. Suppose a nonzero vector $\mathbf{x}$ satisfies $A\mathbf{x} = \lambda\mathbf{x}$, then

$$A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda A\mathbf{x} = \lambda^2\mathbf{x}$$

This shows that $\mathbf{x}$ is also an eigenvector of A^2.

f. True. Suppose a nonzero vector $\mathbf{x}$ satisfies $A\mathbf{x} = \lambda\mathbf{x}$, then left-multiply by A^{-1} to obtain $\mathbf{x} = A^{-1}(\lambda\mathbf{x}) = \lambda A^{-1}\mathbf{x}$. Since A is invertible, the eigenvalue λ is not zero. So $\lambda^{-1}\mathbf{x} = A^{-1}\mathbf{x}$, which shows that $\mathbf{x}$ is also an eigenvector of A^{-1}.

g. False. Zero is an eigenvalue of each singular square matrix.

h. True. By definition, an eigenvector must be nonzero.

i. False. If the dimension of the eigenspace is at least 2, then there are at least two linearly independent eigenvectors in the same subspace.

j. True. This follows from Theorem 4 in Section 5.2.

k. False. Let A be the 3×3 matrix in Example 3 in Section 5.3. Then A is similar to a diagonal matrix D. The eigenvectors of D are the columns of I_3, but the eigenvectors of A are entirely different.

l. False. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Then $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are eigenvectors of A, but $\mathbf{e}_1 + \mathbf{e}_2$ is not. (Actually, it can be shown that if two eigenvectors of A correspond to distinct eigenvalues, then their sum cannot be an eigenvector.)

m. False. *All* the diagonal entries of an upper triangular matrix are the eigenvalues of the matrix (Theorem 1 in Section 5.1). A diagonal entry may be zero.

n. True. Matrices A and A^T have the same characteristic polynomial, because $\det(A^T - \lambda I) = \det(A - \lambda I)^T = \det(A - \lambda I)$, by the determinant transpose property.

o. False. Counterexample: Let A be the 5×5 identity matrix.

p. True. For example, let A be the matrix that rotates vectors through $\pi/2$ radians about the origin. Then $A\mathbf{x}$ is not a multiple of $\mathbf{x}$ when $\mathbf{x}$ is nonzero.

q. False. If A is a diagonal matrix with a zero on the diagonal, then the columns of A are not linearly independent.

r. True. If $A\mathbf{x} = \lambda_1\mathbf{x}$ and $A\mathbf{x} = \lambda_2\mathbf{x}$, then $\lambda_1\mathbf{x} = \lambda_2\mathbf{x}$ and $(\lambda_1 - \lambda_2)\mathbf{x} = \mathbf{0}$. If $\mathbf{x} \neq \mathbf{0}$, then λ_1 must equal λ_2.

s. False. Let A be a singular matrix that is diagonalizable. (For instance, let A be a diagonal matrix with a zero on the diagonal.) Then, by Theorem 8 in Section 5.4, the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is represented by a diagonal matrix relative to a coordinate system determined by eigenvectors of A.

t. True. By definition of matrix multiplication,

$$A = AI = A[\,\mathbf{e}_1 \quad \mathbf{e}_2 \quad \cdots \quad \mathbf{e}_n\,] = [\,A\mathbf{e}_1 \quad A\mathbf{e}_2 \quad \cdots \quad A\mathbf{e}_n\,]$$

If $A\mathbf{e}_j = d_j\mathbf{e}_j$ for $j = 1, \ldots, n$, then A is a diagonal matrix with diagonal entries $d_1, \ldots, d_n$.

u. True. If $B = PDP^{-1}$, where D is a diagonal matrix, and if $A = QBQ^{-1}$, then $A = Q(PDP^{-1})Q^{-1} = (QP)D(QP)^{-1}$, which shows that A is diagonalizable.

v. True. Since B is invertible, AB is similar to $B(AB)B^{-1}$, which equals BA.

w. False. Having n linearly independent eigenvectors makes an $n \times n$ matrix diagonalizable (by the Diagonalization Theorem in Section 5.3), but not necessarily invertible. One of the eigenvalues of the matrix could be zero.

x. True. If A is diagonalizable, then by the Diagonalization Theorem, A has n linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in $\mathbb{R}^n$. By the Basis Theorem, $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ spans $\mathbb{R}^n$. This means that each vector in $\mathbb{R}^n$ can be written as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

Chapter 6

Section 6.1, page 336

2. $35, 5, \frac{1}{7}$ **4.** $\begin{bmatrix} -1/5 \\ 2/5 \end{bmatrix}$ **6.** $\begin{bmatrix} 30/49 \\ -10/49 \\ 15/49 \end{bmatrix}$

8. 7 **10.** $\begin{bmatrix} -6/\sqrt{61} \\ 4/\sqrt{61} \\ -3/\sqrt{61} \end{bmatrix}$

16. Orthogonal **18.** Not orthogonal

22. $\mathbf{u}\cdot\mathbf{u} \geq 0$ because $\mathbf{u}\cdot\mathbf{u}$ is a sum of squares of the entries in $\mathbf{u}$. The sum of squares of numbers is zero if and only if all the numbers are themselves zero.

24.
$$\begin{aligned} \|\mathbf{u}+\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} \\ &= \|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2 \\ \|\mathbf{u}-\mathbf{v}\|^2 &= (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) \\ &= \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot(-\mathbf{v}) - \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} \\ &= \|\mathbf{u}\|^2 - 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2 \end{aligned}$$

When $\|\mathbf{u}+\mathbf{v}\|^2$ and $\|\mathbf{u}-\mathbf{v}\|^2$ are added, the $\mathbf{u}\cdot\mathbf{v}$ terms cancel, and the result is $2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$.

26. Theorem 2 in Chapter 4, because W is the null space of the $1 \times n$ matrix $\mathbf{u}^T$. W is a plane through the origin of $\mathbb{R}^3$.

Section 6.2, page 344

12. $\begin{bmatrix} .4 \\ -1.2 \end{bmatrix}$ **14.** $\mathbf{y} = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} + \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix}$

Section 6.3, page 352

2. $\mathbf{v} = 2\mathbf{u}_1 + \frac{3}{7}\mathbf{u}_2 + \frac{12}{7}\mathbf{u}_3 - \frac{8}{7}\mathbf{u}_4$; $\mathbf{v} = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ -5 \\ 1 \end{bmatrix}$

Section 6.4, page 358

2. $\begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix}$

6. $\begin{bmatrix} 3 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ -3 \\ 0 \end{bmatrix}$

1. The exercise does not specify the matrix A, but only lists the eigenvalues 3 and 1/3, and the corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Also, $\mathbf{x}_0 = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$.

 a. To find the action of A on $\mathbf{x}_0$, express $\mathbf{x}_0$ in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$. That is, find c_1 and c_2 such that $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. This is certainly possible because the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent (by inspection and also because they correspond to distinct eigenvalues) and hence form a basis for $\mathbb{R}^2$. (Two linearly independent vectors in $\mathbb{R}^2$ automatically span $\mathbb{R}^2$.) The row reduction $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{x}_0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 9 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -4 \end{bmatrix}$ shows that $\mathbf{x}_0 = 5\mathbf{v}_1 - 4\mathbf{v}_2$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors (for the eigenvalues 3 and 1/3):

$$\mathbf{x}_1 = A\mathbf{x}_0 = 5A\mathbf{v}_1 - 4A\mathbf{v}_2 = 5\cdot 3\mathbf{v}_1 - 4\cdot(1/3)\mathbf{v}_2 = \begin{bmatrix} 15 \\ 15 \end{bmatrix} - \begin{bmatrix} -4/3 \\ 4/3 \end{bmatrix} = \begin{bmatrix} 49/3 \\ 41/3 \end{bmatrix}$$

 b. Each time A acts on a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, the $\mathbf{v}_1$ term is multiplied by the eigenvalue 3 and the $\mathbf{v}_2$ term is multiplied by the eigenvalue 1/3:

$$\mathbf{x}_2 = A\mathbf{x}_1 = A[5\cdot 3\mathbf{v}_1 - 4(1/3)\mathbf{v}_2] = 5(3)^2\mathbf{v}_1 - 4(1/3)^2\mathbf{v}_2$$

 In general, $\mathbf{x}_k = 5(3)^k\mathbf{v}_1 - 4(1/3)^k\mathbf{v}_2$, for $k \ge 0$.

2. The vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ -3 \\ 7 \end{bmatrix}$ are eigenvectors of a 3×3 matrix A, corresponding to eigenvalues 3, 4/5, and 3/5, respectively. Also, $\mathbf{x}_0 = \begin{bmatrix} -2 \\ -5 \\ 3 \end{bmatrix}$. To describe the solution of the equation $\mathbf{x}_{k+1} = A\mathbf{x}_k\ (k = 1, 2, \ldots)$, first write $\mathbf{x}_0$ in terms of the eigenvectors.

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{x}_0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 & -2 \\ 0 & 1 & -3 & -5 \\ -3 & -5 & 7 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \Rightarrow \mathbf{x}_0 = 2\mathbf{v}_1 + \mathbf{v}_2 + 2\mathbf{v}_3$$

Then, $\mathbf{x}_1 = A(2\mathbf{v}_1 + \mathbf{v}_2 + 2\mathbf{v}_3) = 2A\mathbf{v}_1 + A\mathbf{v}_2 + 2A\mathbf{v}_3 = 2\cdot 3\mathbf{v}_1 + (4/5)\mathbf{v}_2 + 2\cdot(3/5)\mathbf{v}_3$. In general, $\mathbf{x}_k = 2\cdot 3^k\mathbf{v}_1 + (4/5)^k\mathbf{v}_2 + 2\cdot(3/5)^k\mathbf{v}_3$. For all k sufficiently large,

$$\mathbf{x}_k \approx 2\cdot 3^k\mathbf{v}_1 = 2\cdot 3^k\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$$