Linear Algebra

MA 242 (Spring 2013)
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EIGENVALUES AND EIGENVECTORS

Definition

Let A be an $n \times n$ matrix. Then a *nonzero* vector $v \in \mathbb{R}^n$ is called an **eigenvector** of A if

$$Av = \lambda v$$

for some scalar λ. This scalar λ is then called **eigenvalue** of A and v is said to be the corresponding eigenvector. The set of all solutions to

$$(A - \lambda I)v = 0$$

is called the **eigenspace** corresponding to λ.

Computing eigenvalues

A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if the equation

$$(A - \lambda I)v = 0$$

has nontrivial solutions, which is only true if $(A - \lambda I)$ is not invertible (otherwise it has the unique solution $v = 0$), which is only true if

$$\det(A - \lambda I) = 0.$$

Example in $\mathbb{R}^2$: Compute eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 8 \\ 0 & 2 \end{bmatrix}$

1. $A - \lambda I =$
2. $\det(A - \lambda I) =$
3. $\det(A - \lambda I) = 0$ has solutions $\lambda_1 =$, $\lambda_2 =$.
4. <u>eigenvector for λ_1</u>:

5. <u>eigenvector for λ_2</u>:

Example in $\mathbb{R}^3$: Compute eigenvalues and eigenvectors of $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$.

1. $A - \lambda I = \begin{bmatrix} 4-\lambda & -1 & 6 \\ 2 & 1-\lambda & 6 \\ 2 & -1 & 8-\lambda \end{bmatrix}$

2. $\det(A - \lambda I) = -(\lambda - 9)(\lambda - 2)^2$. <u>(characteristic equation)</u>

3. $\lambda_1 = 9$ (with multiplicity one), $\lambda_2 = 2$ (with multiplicity two)

4. <u>eigenvector for $\lambda_1 = 9$:</u>

$$\left[\begin{array}{ccc|c} 4-9 & -1 & 6 & 0 \\ 2 & 1-9 & 6 & 0 \\ 2 & -1 & 8-9 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad \text{so } v = s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, s \in \mathbb{R}$$

   Choose, for instance, $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Note: Eigenvectors are only unique up to scaling.

   Check: $Av_1 = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} =$

   $\rightsquigarrow$ Eigenspace for $\lambda_1 = 9$ is one dimensional.

5. <u>eigenvector for $\lambda_2 = 2$:</u>

$$\left[\begin{array}{ccc|c} 4-2 & -1 & 6 & 0 \\ 2 & 1-2 & 6 & 0 \\ 2 & -1 & 8-2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad \text{so } v = s \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, s, t \in \mathbb{R}$$

   Any choice of s, t will give an eigenvector (for instance, $s = 1, t = 0$). $\rightsquigarrow$ Eigenspace for $\lambda_2 = 2$ is two dimensional.