Linear Algebra

MA 242 (Spring 2013)
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EIGENVALUES, EIGENVECTORS, DIAGONALIZATION

– 2×2 example —

Given $A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$ compute eigenvalues, eigenspaces and its diagonalization.

"characteristic equation":

$$\det(A - \lambda I) = (-1-\lambda)^2 - 4 = \lambda^2 + 2\lambda - 3 = 0$$

"eigenvalues": characteristic equation is solved by $\lambda_{1/2} = \frac{-2\pm\sqrt{4+12}}{2}$, eigenvalues are

$$\lambda_1 = 1, \qquad \lambda_2 = -3$$

"eigenspace for $\lambda_1 = 1$" $=$ Nul$(A - \lambda_1 I)$:

$$\left[\begin{array}{cc|c} -1-1 & 2 & 0 \\ 2 & -1-1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 2 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \mathrm{Nul}(A-I) = \mathrm{Span}\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$$

Check: $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ✓ *Summary*: $Av_1 = \lambda_1 v_1$ for $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
In fact, $A(cv_1) = \lambda_1(cv_1)$ for any $c \in \mathbb{R}$ and $\dim(\mathrm{Nul}(A - \lambda_1 I)) = 1$.

"eigenspace for $\lambda_2 = -3$" $=$ Nul$(A - \lambda_2 I)$:

$$\left[\begin{array}{cc|c} -1-(-3) & 2 & 0 \\ 2 & -1-(-3) & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \mathrm{Nul}(A-(-3)I) = \mathrm{Span}\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\}$$

Check: $A\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} = (-3) \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ ✓ *Summary*: $Av_2 = \lambda_2 v_2$ for $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.
In fact, $A(cv_2) = \lambda_2(cv_2)$ for any $c \in \mathbb{R}$ and $\dim(\mathrm{Nul}(A - \lambda_2 I)) = 1$.

"diagonalization": Compile the matrix of eigenvectors $V = [v_1\ v_2] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Since v_1 and v_2 are linearly independent we can invert V. We have $V^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

The matrix A can be transformed into a diagonal matrix by

$$V^{-1}AV = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} = D.$$

When executing these multiplications note the following structure:

$$V^{-1}AV = V^{-1}[Av_1\ Av_2] = V^{-1}[\lambda_1 v_1\ \lambda_2 v_2] = V^{-1}V\mathrm{diag}[\lambda_1, \lambda_2] = \mathrm{diag}[\lambda_1, \lambda_2] = D$$