METAMORPHOSES THROUGH LIE GROUP ACTION

ALAIN TROUVÉ AND LAURENT YOUNES

Abstract. We formally analyze a computational problem which has important applications in image understanding and shape analysis. The problem can be summarized as follows. Starting from a group action on a Riemannian manifold M, we introduce a modification of the metric by partly expressing displacements on M as an effect of the action of some group element. The study of this new structure relates to evolutions on M under the combined effect of the action and of residual displacements, called metamorphoses. This can and have been applied to image processing problems, providing in particular diffeomorphic matching algorithms for pattern recognition.

AMS Subject Classification: 68T45, 53B21

Contents

1. Notation 1
2. A new metric on M 4
2.1. General form 4
2.2. Examples 5
3. Geodesic equations 7
3.1. General form 7
3.2. Examples 8
4. Evolution equations 11
4.1. Evolution in z 11
4.2. Evolution of the velocity 13
4.3. Examples 13
5. A last example: deforming geometric curves 14
Appendix A. Existence of solutions of $\frac{dg_t}{dt} = d_eR_{g_t}v_t$ 17
Appendix B. Proof of proposition 2 18
References 18

1. Notation

Many situations in image analysis model a set of *visual objets*, like images, shapes, patterns of points, which can be affected by "deformations". Following the seminal approach of Grenander's deformable template theory [6, 7], we have designed during the last decade, theoretical and numerical methods to analyze the action of diffeomorphisms on geometric structures, like landmarks, shapes and images [16, 21, 22, 17, 13, 2, 18, 12]. The basic assumption of the theory of deformable templates, observable objects are assumed to belong to the orbit of a fixed template, under the action of a group which would here be a group of diffeomorphisms. Here, following [13, 2, 18], the point of view is slightly different, because the basepoint of the deformation (the template) is allowed to vary during a process called a metamorphosis (this term is borrowed from computer graphics where it indicates a "morphing" process, generally on face images, but we use it here in fairly larger framework). Metamorphoses therefore are deformable templates with varying templates. In this paper, we study their metric and geometric properties from an abstract, general point of view, which has the advantage of embedding several apparently distinct situations into a unifying framework.

Here are the basic assumptions. The deformations belong to a Lie group G with Lie algebra denoted $\mathfrak{g}$, acting on a Riemannian manifold M which contains visual objects. We assume that $\mathfrak{g}$ is a Hilbert space with norm $|\,|_{\mathfrak{g}}$; the metric on M at a given point $m \in M$ is denoted $\langle . , . \rangle_m$ and the corresponding norm $|\,.\,|_m$.

Notation 1. *Assuming that action of $G \times M \to M$ is C^1, for any $g \in G$, the mapping $A(g) : m \to gm$ is a diffeomorphism on M. Then, $A : G \to Diff(M)$ is an homomorphism of groups, so that we can identify G with a subgroup of $Diff(M)$. We will use the notation $A(g) = \mathbf{g}$ so that $\mathbf{g}(m) = gm$.*

Moreover, since for m fixed, the mapping $R_m : g \to gm$ is differentiable with respect to g at the identity element e of G, then for any $v \in \mathfrak{g}$, $d_eR_m(v) \in T_mM$ and $m \to d_eR_m(v) \in \chi(M)$, the set of continuous vector fields on M. Hence, $\mathfrak{g}$ can be identified with a subspace of $\chi(M)$ equipped with a Hilbertian metric inherited from $\mathfrak{g}$. Using this identification, we will denote for any $v \in \mathfrak{g}$, $\mathbf{v}(m) \doteq d_eR_m(v)$. It will sometime be convenient to also use the notation δ_m instead of d_eR_m so that

$$\delta_m v = \mathbf{v}(m)\,.$$

If $(g_t, t \in [0,1])$ is a differentiable curve on G, and R_g is the right-multiplication in G ($R_g(g') = g'g$), we define its velocity v_t (which is a curve on $\mathfrak{g}$) by the relation:

$$\frac{dg_t}{dt} = d_eR_{g_t}v_t\,, \tag{1}$$

or using the previous identification

$$\frac{d\mathbf{g}_t}{dt} = \mathbf{v}_t \circ \mathbf{g}_t\,. \tag{2}$$

A *metamorphosis* on M is a pair of curves (g_t, μ_t) respectively on G and M, with $g_0 = e$. Its image is the curve m_t on M defined by $m_t = \mathbf{g}_t(\mu_t)$. We call g_t the deformation part of the metamorphosis, and μ_t the template evolution part. When μ_t is constant, we say that the metamorphosis is a pure deformation. In the rest of the paper, we will study how metamorphoses can be used to define a new metric on M, and obtain the corresponding geodesic equations. These will be essentially abstract developments, but they will be illustrated by the following examples, which correspond to useful situations in image analysis.

Example 1: Landmark matching with affine robustness. We here consider landmarks which are collections of labeled points in $\mathbb{R}^k$. Elements of M are N-tuples of points, so that M can be identified to $(\mathbb{R}^k)^N$. Let the Riemannian metric on M be the usual Euclidean metric, so that, if $m = (m^1, \ldots, m^N) \in M$ and $\eta = (\eta^1, \ldots, \eta^N) \in T_mM \sim (\mathbb{R}^k)^N$, we have

$$|\eta|_m^2 = \sum_{i=1}^N |\eta^i|^2_{\mathbb{R}^k}$$

The group G is the affine group on $\mathbb{R}^k$: elements of G are pairs (B,T) where B is an invertible $k \times k$ matrix and $T \in \mathbb{R}^k$, with the (semi-direct) product $(B,T)(B',T') = (BB', BT' + T)$. The action of G on M is defined for $g = (B,T)$ and $m = (m^1, \cdots, m^N)$ by

$$\mathbf{g}(m) = (Bm^1 + T, \ldots, Bm^N + T)$$

The Lie algebra on G is equal to $\mathcal{M}_k(\mathbb{R}) \times \mathbb{R}^k$. There are many possible definitions for the norm on $\mathfrak{g}$, and we shall choose the simplest one, for which, letting $v = (\beta, \tau) \in \mathfrak{g}$,

$$|v|^2_{\mathfrak{g}} = \mathrm{trace}({}^t\beta\beta) + \|\tau\|^2_{\mathbb{R}^k}$$

The velocity (β_t, τ_t) of a curve (B_t, T_t) on G is characterized by the system

$$\begin{cases} \dfrac{dB_t}{dt} = \beta_t B_t \\[2ex] \dfrac{dT_t}{dt} = \beta_t B_t + \tau_t \end{cases}$$

According to notation 1, G is identified with a subgroup of Diff(M) i.e. with a subgroup of diffeomorphisms on the space of N-uple of points in $\mathbb{R}^k$. More precisely, G is identified with a subgroup of Aff(M), the group of affine transformations on M.

Exemple 2: Deformable landmarks on a manifold [8, 2, 19]. We here consider general deformations acting on collections of points $(y^1, \ldots, y^N)$, each y^i being assumed to belong to an open and bounded subset Ω of a smooth Riemannian manifold M_0 (the preferred application being $M_0 = \mathbb{R}^k$, but the general case also has some interest [5]). We therefore set $M = \Omega^N$ and, like in example 1, the metric on M is the product metric, setting, for $m = (y^1, \ldots, y^N) \in M$ and $\eta = (\eta^1, \ldots, \eta^N) \in T_m M \sim \prod_{i=1}^N T_{y^i} M_0$:

$$|\eta|_m^2 = \sum_{i=1}^N |\eta^i|_{y^i}^2$$

The group G we consider here is a group of diffeomorphisms of Ω, with action: $\mathbf{g}(y^1, \ldots, y^N) = (g(y^1), \ldots, g(y^N))$. Considering such an infinite dimensional group brings additional difficulties, which will however be occulted in the forthcoming abstract derivations. The underlying construction we are assuming here has been proposed in [15] and [3]. The problem is that it is not possible to obtain all together the Lie group properties and the (Riemannian) metric property which are immediate in finite dimensions. Lie groups of diffeomorphisms on manifolds has been the subject of intensive studies in the framework of global analysis (eg. [14, 4, 9]). Strictly enforcing the Lie group properties comes at the cost of only considering smooth (C^∞) diffeomorphisms, and of the absence of any nice metric properties (like completeness and existence and geodesics) for the kind of Riemannian metric one is likely to consider. This metric aspect is more important for applications, since one ends up solving variational problems, for which it is important to know that solutions exists. Trying to build a group of diffeomorphisms which has most of the properties of a complete Riemannian manifold now implies relaxing some of the Lie group conditions. This is the construction we now summarize.

This starts with selecting $\mathfrak{g}$, as a Hilbert space (not necessarily a Lie algebra), assuming that it is continuously embedded into $\mathcal{X}^1(\Omega)$, the Banach space of continuously differentiable vector fields on Ω. We also assume that elements in $\mathfrak{g}$ have null boundary conditions, in the sense that $\mathcal{X}_c^\infty(\Omega)$ (the space of C^∞ vector fields with compact support on Ω) is dense in $\mathfrak{g}$. A generic way for building such a space is based on the Friedrich extension of an admissible operator [23]; a vector field on Ω is square integrable if

$$\int_\Omega \|v(y)\|_y^2 \, d\nu_0(y) < \infty$$

where ν_0 is the volume form on M_0. We denote $L^2(\Omega)$ the set of square integrable vector fields on Ω. We say that a symmetric operator $L : \mathcal{X}_c^\infty(\Omega) \to L^2(\Omega)$ is admissible if for all $y \in \Omega$, $Lv(y) \in T_y M$ and, for some constant K

$$(Lv, v) := \int_\Omega \langle Lv(y)\,,\, v(y)\rangle_y d\nu_0(y) \geq K \max\left\{\|v(y)\|_y^2 + \|\nabla_y v\|_y^2, x \in \Omega\right\}$$

Under this assumption, $\mathcal{X}_c^\infty(\Omega)$ and L can be completed into a Hilbert space $\mathfrak{g} \subset \mathcal{X}^1(\Omega)$ and an operator $L : \mathfrak{g} \to \mathfrak{g}^*$ such that

$$\|v\|_{\mathfrak{g}}^2 = (Lv, v)\,.$$

If $v_t \in L^1([0,1], \mathfrak{g})$, the differential equation $\frac{dy_t}{dt} = v_t(y_t)$, $y_0 = x$ has solutions over $[0,1]$. For $t \in [0,1]$, this solution is denoted $g_t^v(x)$, and the set $G = \left\{g_1^v, v \in L^1([0,1], \mathfrak{g})\right\}$ is a group (with product $gh = g \circ h$), which will be hereafter our group of diffeomorphisms of Ω. Note that, in this setting (1) precisely gives

$$\frac{dg_t}{dt} = v_t \circ g_t$$

Here, note that for any $g \in G \subset \text{Diff}(\Omega)$, $\mathbf{g}$ belongs to $\text{Diff}(\Omega^M)$.

Example 3: Deformable images [13, 18]. We consider, here again, a group G of diffeomorphisms of Ω, restricting, for simplicity, to the case $M_0 = I\!R^k$, and M is a set of square integrable functions $m : \Omega \to I\!R$. We let G act on M by $gm = m \circ g^{-1}$, and use the L^2 norm on the Hilbert space $L^2(\Omega, I\!R^k)$. Following Notation 1, $m \to gm$ defines an element $\mathbf{g} \in \mathcal{L}(L^2)$ i.e. the linear group on L^2. Note that in example 1, $\mathbf{g}$ also belongs to a linear group (but on a finite dimensional vector space), and we shall see that both examples share important structural properties.

2. A NEW METRIC ON M

2.1. **General form.** We return to the abstract setting to show how metamorphoses can be used to place a new Riemannian metric on M, which will take the action of G into account. A simple motivation for this can be taken from the following situation in example 1: if $m = (y^1, \dots, y^N)$ belongs to M, and $(T^1, \dots, T^N)$ are randomly chosen unit vectors in $I\!R^k$, m will be at equal Euclidean distance, N, from the configurations $(y^1 + T^1, \dots, y^N + T^1)$ and $(y^1 + T^1, \dots, y^N + T^N)$. However, in situations when m represents a shape, it is natural to consider that the first configuration, which is a translation of m, should be much closer to m than the second one. In other terms, we want to assign a different cost to the variation in cases when it can be, at least partially, explained by the action of G.

Metamorphoses, by the evolution of their image, provide a convenient representation of combinations of a group action and of a variation on M. Indeed, if $((g_t, \mu_t), t \in [0,1])$ is given (with $g_t \in G$ and $\mu_t \in M$) and $m_t = g_t(\mu_t)$ is its image, a straightforward computation yields, v_t being the velocity of g_t (defined in equation (1)):

$$\begin{aligned} \frac{dm_t}{dt} &= d_{\mu_t}\mathbf{g}_t(\frac{d\mu_t}{dt}) + \frac{d\mathbf{g}_t}{dt}(\mu_t) \\ &= d_{\mu_t}\mathbf{g}_t(\frac{d\mu_t}{dt}) + \mathbf{v}_t(m_t) \end{aligned} \tag{3}$$

In particular, when $t = 0$:

$$\frac{dm_t}{dt}_{|t=0} = \frac{d\mu_t}{dt}_{|t=0} + \mathbf{v}_0(m_0) \tag{4}$$

This expression provides the decomposition of a generic element $\eta \in T_m M$ in terms of an *infinitesimal metamorphosis*, represented by an element of $\mathfrak{g} \times T_m M$. Indeed, for $m \in M$, introduce the map

$$\begin{aligned} \Phi_m : \quad \mathfrak{g} \times T_m M &\to T_m M \\ (v, \delta) &\mapsto \mathbf{v}(m) + \delta \end{aligned}$$

so that (4) can be written

$$\frac{dm_t}{dt}_{|t=0} = \Phi_m\left(\frac{d\mu_t}{dt}_{|t=0}, v_0\right)$$

Fix $\sigma^2 > 0$. Since Φ_m is surjective ($\Phi_m(0, \delta) = \delta$), we can define a new metric on M by

$$\|\eta\|_m^2 = \inf\left\{|v|_{\mathfrak{g}}^2 + \frac{1}{\sigma^2}|\delta|_m^2 : \eta = \Phi_m(v, \delta)\right\}$$

(note that we are using double lines instead of simple to distinguish between the new metric and the original one).

Define $V_m = \Phi_m^{-1}(0)$. It is a linear subspace of $\mathfrak{g} \times T_m M$ and $\|\eta\|_m^2$ is the norm of the linear projection of $(0, \eta)$ on V_m, for the Hilbert structure on $\mathfrak{g} \times T_m M$ defined by

$$\|(v, \delta)\|_{e,m}^2 = |v|_{\mathfrak{g}}^2 + \frac{1}{\sigma^2}|\delta|_m^2$$

Thus $\|.\|_m = \|\pi_{V_m}(0, \eta)\|_{e,m}$ is associated to an inner product. Since it is a projection on a close subspace, the infimum is attained and by definition, cannot vanish unless $\eta = 0$. This therefore provides a new Riemannian metric on M.

With this metric, the energy of a curve is

$$E(m_t) = \int_0^1 \left\| \frac{dm_t}{dt} \right\|_{m_t}^2 dt = \inf \left(\int_0^1 |v_t|_{\mathfrak{g}}^2 \, dt + \frac{1}{\sigma^2} \int_0^1 \left| \frac{dm_t}{dt} - \mathbf{v}_t(m_t) \right|_{m_t}^2 dt \right) \tag{5}$$

the infimum being over all curves $t \mapsto v_t$ on $\mathfrak{g}$.

The distance between two elements m and m' in M can therefore be computed by minimizing

$$U(v_t, m_t) = \int_0^1 |v_t|_{\mathfrak{g}}^2 \, dt + \frac{1}{\sigma^2} \int_0^1 \left| \frac{dm_t}{dt} - \mathbf{v}_t(m_t) \right|_{m_t}^2$$

over all curves $((v_t, m_t), t \in [0,1])$ on $\mathfrak{g} \times M$, with boundary conditions $m_0 = m$ and $m_1 = m'$.

Introducing g_t, the solution[1] of (1), and the metamorphosis (g_t, μ_t), with $\mu_t = g_t^{-1}(m_t)$, equation (3) provides a second expression for the energy:

$$E(m_t) = \inf \left(\int_0^1 |v_t|_{\mathfrak{g}}^2 \, dt + \frac{1}{\sigma^2} \int_0^1 \left| d_{\mu_t} \mathbf{g}_t \left(\frac{d\mu_t}{dt} \right) \right|_{m_t}^2 dt \right) \tag{6}$$

2.2. **Examples.** We now specialize this computation to the 3 examples we are considering. This essentially means computing δ_m et $v \mapsto \mathbf{v}$ in these situations.

2.2.1. *Example 1.* If $m = (m^1, \dots, m^N)$, differentiating

$$(B, T) \mapsto (Bm^1 + T, \dots, Bm^N + T)$$

at $(B,T) = (\mathrm{Id}, 0)$ gives for $v = (\beta, \tau) \in \mathfrak{g}$

$$\mathbf{v}(m) = (\beta m^1 + \tau, \dots, \beta m^N + \tau) \tag{7}$$

so that, letting $m_t = (m_t^1, \dots, m_t^N)$,

$$U(v_t, m_t) = \int_0^1 (\mathrm{trace}\,({}^t\beta_t \beta_t) + |\tau_t|_{\mathbb{R}^k}^2) dt + \frac{1}{\sigma^2} \sum_{i=1}^N \int_0^1 \left| \frac{dm_t^i}{dt} - \beta m_t^i - \tau \right|_{\mathbb{R}^k}^2$$

Similarly, for $g = (B, T) \in G$, differentiating $\mu \mapsto \mathbf{g}(\mu)$ with respect to $\mu = (\mu^1, \dots, \mu^N)$ yields

$$d_\mu \mathbf{g}(\eta) = (B\eta^1, \dots, B\eta^N)$$

where $\eta = (\eta^1, \cdots, \eta^N)$. so that, the alternate form of the energy, (6), is

$$\tilde{U}(g_t, \mu_t) = \int_0^1 \left(\mathrm{trace}\,({}^t\beta_t \beta_t) + |\tau_t|_{\mathbb{R}^k}^2 \right) dt + \frac{1}{\sigma^2} \sum_{i=1}^N \int_0^1 \left| B_t^{-1} \frac{d\mu_t^i}{dt} \right|_{\mathbb{R}^k}^2 dt$$

2.2.2. *Example 2.* Since $\mathbf{g}(m) = (g(m^1), \dots, g(m^N))$, we have $\mathbf{v}(m) = (v(m^1), \dots, v(m^N))$ for $v \in \mathfrak{g}$, so that

$$U(v_t, m_t) = \int_0^1 |v_t|_{\mathfrak{g}}^2 \, dt + \frac{1}{\sigma^2} \sum_{i=1}^N \int_0^1 \left| \frac{dm_t^i}{dt} - v(m_t^i) \right|_{m^i}^2 dt$$

and (6) may also be computed from the expression $d_\mu \mathbf{g}(\eta) = (d_{\mu^1} g(\eta^1), \dots, d_{\mu^N} g(\eta^N))$ for $\eta = (\eta^1, \cdots, \eta^N)$. The result of minimizing the geodesic energy between two sets of 100 hundred 2D landmarks is provided in figure 1.

[1] The existence of this solution when $v \in L^2([0,1], \mathfrak{g})$ is validated in the appendix

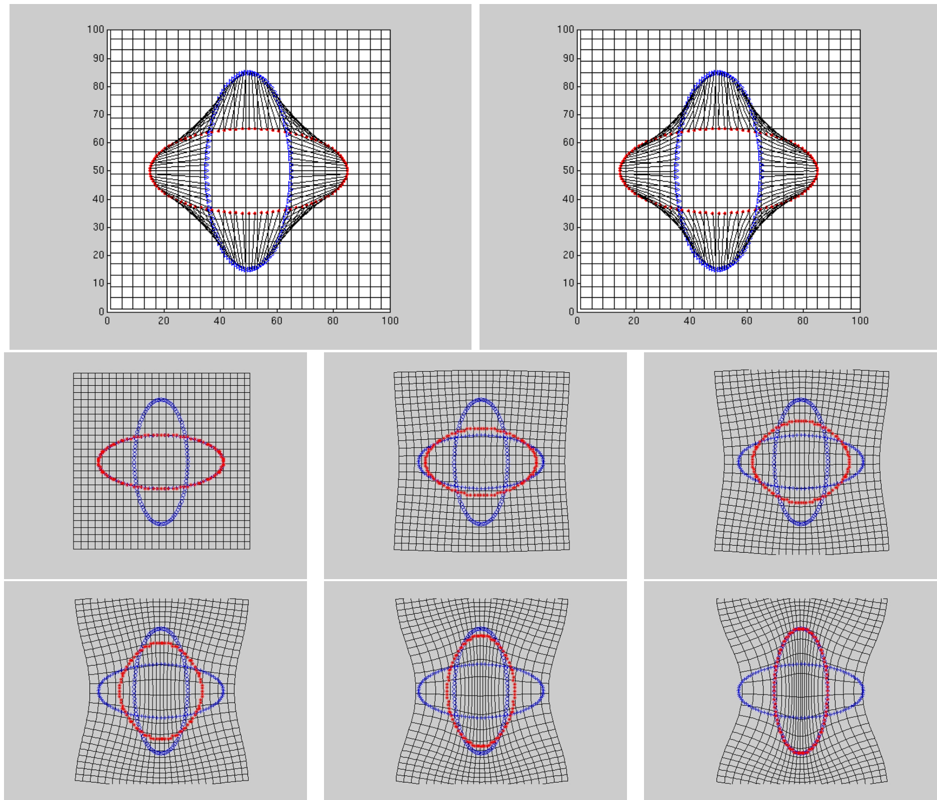

FIGURE 1. An example of geodesic between landmarks. The first row provides the initial requirements (100 landmarks placed on a horizontal ellipse to be displaced to a vertical ellipse), followed, in the second image by the geodesic trajectories. The next two rows provide the intermediate positions of the landmarks (red dots) and the effect of the underlying diffeomorphism on a grid.

2.2.3. *Example 3.* This example is more problematic: m being a function on Ω, we have for $g \in G$, $\mathbf{g}(m) = m \circ g^{-1}$; formal differentiation in the neighborhood of the identity would yield $\mathbf{v}(m) = -\langle \nabla m\,,\, v \rangle_{I\!R^k}$ for $v \in \mathfrak{g}$. For $m \in L^2(\Omega, I\!R)$, this is not necessarily defined in the strong sense, and can be given a generalized meaning, $\eta = \mathbf{v}(m)$ being identified to the linear form, defined, over all smooth functions φ on Ω with compact support, by

$$\varphi \mapsto \int_\Omega m(x) \mathrm{div}_x(\varphi v) dx\,.$$

This coincides with $-\int_\Omega \langle \nabla_x m\,,\, v(x) \rangle_{I\!R^k} \varphi(x) dx$ when m is C^1. Note that, for the energy $U(v_t, m_t)$ to be finite, we still need the sum $\frac{dm_t}{dt} - \mathbf{v}_t(m)$ to be square integrable as a function of two variables t and x.

The second form of the energy is simpler, since $d_\mu \mathbf{g}(\eta) = \eta \circ g^{-1}$, so that (6) is

$$\int_0^1 |v_t|^2_{\mathfrak{g}}\, dt + \frac{1}{\sigma^2} \int_0^1 \left| \frac{d\mu_t}{dt} \circ g_t^{-1} \right|^2 dx dt$$

which has a meaning whenever $\frac{d\mu_t}{dt}$ is square integrable. However, it is now the boundary condition $\mu_1 \circ g_1^{-1} = m'$ which may be hard to fulfill if m and m' are simply assumed to belong to L^2. This implies that m_t and v_t can no more be treated as independent variables in the variational analysis of this problem, which becomes, because of this, much more intricate.

To simplify computations, we will restrict, in the remaining of the paper, to the case when the compared images are smooth (at least C^1). Under these conditions, it may be assumed that m_t is

FIGURE 2. An example of geodesic between images (original images taken from the Olivetti face database). The three intermediate images are generated by the optimization algorithm.

C^1 at all times, because it can be shown to be so for the optimal path. More general situations are taken into account in [18]. An example of geodesic obtained by minimizing this energy between two given images is given in figure 2

3. Geodesic equations

3.1. General form. In this section, we compute the Euler-Lagrange equations for the energy

$$U(v_t, m_t) = \int_0^1 |v_t|_{\mathfrak{g}}^2 \, dt + \frac{1}{\sigma^2} \int_0^1 \left| \frac{dm_t}{dt} - \mathbf{v}_t(m_t) \right|_{m_t}^2 dt$$

These equations are important, because, on the first hand, they characterize geodesics on M for the new metric, and, on the second hand, they also provide gradient increments which can be used for the numerical computation of the geodesics.

To obtain the first equation, start with a variation $v_t \mapsto v_t + \varepsilon w_t$ in $\mathfrak{g}$. Let $f(\varepsilon) = U(v_t + \varepsilon w_t, m_t)$, so that

$$f'(0) = 2 \int_0^1 \langle v_t \, , \, w_t \rangle_{\mathfrak{g}} dt - 2 \int_0^1 \langle z_t \, , \, \mathbf{w}_t(m_t) \rangle_{m_t} dt \, ,$$

where we have introduced the notation

$$\sigma^2 z_t = \frac{dm_t}{dt} - \mathbf{v}_t(m_t) = d_{\mu_t} \mathbf{g}_t \Big(\frac{d\mu_t}{dt} \Big) \, . \tag{8}$$

Hence, we get for any $w \in \mathfrak{g}$,

$$\langle v_t \, , \, w \rangle_{\mathfrak{g}} = \langle z_t \, , \, \mathbf{w}(m_t) \rangle_{m_t} \, . \tag{9}$$

When A is a continuous operator between two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$, the adjoint of A, denoted $A^\dagger : \mathcal{H}' \to \mathcal{H}$ is uniquely defined by

$$\langle h \, , \, Av \rangle_{\mathcal{H}'} = \langle A^\dagger h \, , \, v \rangle_{\mathcal{H}}$$

Using this notation (with $\mathcal{H} = \mathfrak{g}$ and $\mathcal{H}' = T_m M$), and recalling the notation $\delta_m : \mathfrak{g} \to T_m M$ for the continous linear operator defined by $\delta_m(v) = \mathbf{v}(m)$, the identity $f'(0) = 0$ yields our first Euler-Lagrange equation:

$$v_t - \delta_{m_t}^\dagger (z_t) = 0 \tag{10}$$

which gives (from the expression of z_t)

$$\frac{dm_t}{dt} = \mathbf{v}_t(m_t) + \sigma^2 z_t = (\delta_m \delta_m^\dagger + \sigma^2 \mathrm{Id}) z_t \, . \tag{11}$$

Let $H_m^\sigma : T_m M \to T_m M$ be equal to $(\delta_m \delta_m^\dagger + \sigma^2 \mathrm{Id})$. It is a symmetric and positive operator. If $T_m M$ has finite dimensions, or if $\delta_m \delta_m^\dagger$ is compact, H_m^σ is invertible and equation (11) implies, as shown below,

$$E(m_t) = \int_0^1 \left\langle \frac{dm_t}{dt} \, , \, (H_m^\sigma)^{-1} \frac{dm_t}{dt} \right\rangle_{m_t} dt \, . \tag{12}$$

Indeed, since $\frac{1}{\sigma^2}\left|\frac{dm_t}{dt} - \mathbf{v}_t(m_t)\right|^2_{m_t} = \sigma^2\langle z_t\,,\,z_t\rangle_{m_t}$, we deduce from (9) and (11) that

$$|v|^2_{\mathfrak{g}}+\frac{1}{\sigma^2}\left|\frac{dm_t}{dt} - \mathbf{v}_t(m_t)\right|^2_{m_t} = \langle z_t\,,\,\mathbf{v}_t(m)+\sigma^2 z_t\rangle_{m_t} = \left\langle z_t\,,\,\frac{dm_t}{dt}\right\rangle_{m_t} = \left\langle \frac{dm_t}{dt}\,,\,(H^\sigma_m)^{-1}\frac{dm_t}{dt}\right\rangle_{m_t}.$$

Equality (12) provides an intrinsic expression for the new metric. In other terms

$$\|\eta\|^2_m = \langle \eta\,,\,(H^\sigma_m)^{-1}\eta\rangle_m\,.$$

We pass to the variation with respect to the curve m on M. For this, we consider a variation $m_{t,\varepsilon}$ with $m_{t,0}=m_t$, $m_{0,\varepsilon}=m_0$ and $m_{1,\varepsilon}=m_1$ and let $f(\varepsilon)=\frac{\sigma^2}{2}U(v,m_{t,\varepsilon})$. We have, letting $\eta_t = \frac{dm_{t,\varepsilon}}{d\varepsilon}$,

$$\begin{aligned} f'(0) &= \int_0^1 \left\langle z_t\,,\,\nabla_{\frac{\partial m}{\partial\varepsilon}}\left(\frac{\partial m}{\partial t}-\mathbf{v}_t\right)\right\rangle_{m_t} dt \\ &= \int_0^1 \left\langle z_t\,,\,\nabla_{\frac{\partial m}{\partial t}}\frac{\partial m}{\partial\varepsilon} - \nabla_{\frac{\partial m}{\partial\varepsilon}}\mathbf{v}_t\right\rangle_{m_t} dt \\ &= -\int_0^1 \left\langle \nabla_{\frac{\partial m}{\partial t}} z_t\,,\,\eta_t\right\rangle_{m_t} dt - \int_0^1 \langle z_t\,,\,\nabla_{\eta_t}\mathbf{v}_t\rangle_{m_t} dt \end{aligned}$$

This implies that, for all t and all $\eta\in T_{m_t}M$,

$$\left\langle \nabla_{\frac{\partial m}{\partial t}} z_t\,,\,\eta\right\rangle_{m_t} = -\langle z_t\,,\,\nabla_\eta \mathbf{v}_t\rangle_{m_t}\,. \tag{13}$$

If we define $\nabla^\dagger_\xi$ by $\left\langle \nabla^\dagger_\xi\chi\,,\,\eta\right\rangle_m = \langle\xi\,,\,\nabla_\eta\chi\rangle_m$ (ξ,η,χ being vector fileds on M) this may be written

$$\nabla_{\frac{\partial m}{\partial t}} z_t + \nabla^\dagger_{z_t}\mathbf{v}_t = 0 \tag{14}$$

The following proposition summarizes the previous results

Proposition 1. *The geodesic equations for the new Riemannian structure on M write*

$$\left\{\begin{array}{l} \frac{dm_t}{dt} = \mathbf{v}_t(m_t) = d_{\mu_t}\mathbf{g}_t\left(\frac{d\mu_t}{dt}\right)+\sigma^2 z_t \\ \nabla_{\frac{\partial m}{\partial t}} z_t + \nabla^\dagger_{z_t}\mathbf{v}_t = 0 \\ v_t = \delta^\dagger_{m_t}(z_t) \end{array}\right. \tag{15}$$

Note also that the previous computation provides the gradient of the energy U with respect to v and m, which is given by

$$\left\{\begin{array}{l} \mathrm{Grad}_v U = 2(v_t - \delta^\dagger_{m_t}(z_t)) \\ \mathrm{Grad}_m U = -2\left(\nabla_{\frac{\partial m}{\partial t}} z_t + \nabla^\dagger_{z_t}\mathbf{v}_t\right) \end{array}\right.$$

3.2. Examples.

3.2.1. *Example 1.* Letting $v_t=(\beta_t,\tau_t)$, $m_t=(m^1_t,\dots,m^N_t)$, we have $z_t=(z^1_t,\dots,z^N_t)$ with

$$\sigma^2 z^i_t = \frac{dm^i_t}{dt} - \beta m^i_t - \tau$$

To explicit (10), we compute, for $\xi=(\xi^1,\dots,\xi^N)$, and $m=(m^1,\dots,m^N)$, the element $\delta^\dagger_m\xi = (\beta,\tau)\in\mathfrak{g}$. It is characterized by the identity: for $(\beta',\tau')\in\mathfrak{g}$:

$$\mathrm{trace}({}^t\beta\beta') + \langle\tau\,,\,\tau'\rangle_{I\!R^k} = \sum_{i=1}^N \langle \xi^i\,,\,\beta' m^i+\tau'\rangle_{I\!R^k}$$

which immediately provides

$$\beta = \sum_{i=1}^N \xi^{i\,t} m^i, \tau = \sum_{i=1}^N \xi^i$$

so that equation (10) is

$$\begin{cases} \beta_t = \sum_{i=1}^N z_t^{i\,t} m_t^i \\ \tau_t = \sum_{i=1}^N z_t^i \end{cases}$$

Moreover, we have

$$\delta_m \delta_m^\dagger \xi = (\beta m^1 + \tau, \dots, \beta m^N + \tau)$$

and

$$\beta m^i + \tau = \sum_{j=1}^N (1 + \langle m^i \,,\, m^j \rangle_{I\!R^k}) \xi^j$$

so that the matrix H_m^σ can be identified to the $Nk \times Nk$ block matrix, for which the (i,j) $k \times k$ block is $(1 + \langle m^i \,,\, m^j \rangle_{I\!R^k})\mathrm{Id}$ if $i \neq j$ and $(1 + \sigma^2 + |m^i|^2_{I\!R^k})\mathrm{Id}$ if $i = j$.

We now compute equation (14). Since M is Euclidean, we have $\nabla_{\frac{\partial m}{\partial t}} z_t = \frac{dZ}{dt}$ and $\nabla_\eta \chi = d\chi\eta$. For $v = (\beta, \tau)$, and $\chi = \mathbf{v}$, as given by equation (7), this yields

$$\nabla_\eta \mathbf{v} = (\beta\eta^1, \dots, \beta\eta^n)$$

so that

$$\langle z \,,\, \nabla_\eta \mathbf{v} \rangle_m = \sum_{i=1}^N \langle z_i \,,\, \beta\eta^i \rangle_{I\!R^k} = \sum_{i=1}^N \langle {}^t\beta z_i \,,\, \eta^i \rangle_{I\!R^k}$$

Thus, equation (14) gives, in this case:

$$\frac{dz_t^i}{dt} + {}^t\beta_t z_t^i = 0$$

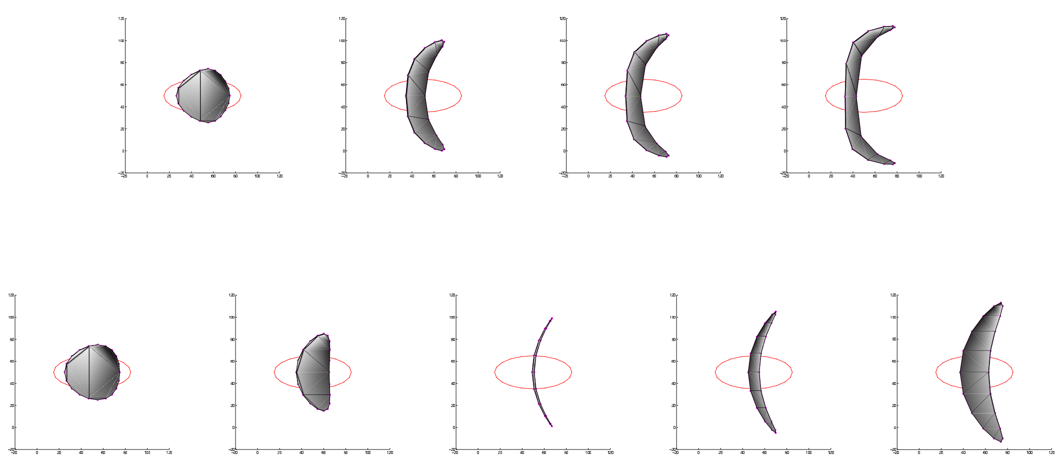

Figure 3. Comparison of geodesics under the new metric and the Euclidean one for landmarks in $I\!R^2$ under diffeomorphic action. The template is the central ellipse and both evolutions are designed to coincide at time $t = 1$. Images are given at times $t = 0.5, 1, 1.7, 2, 2.4$. The geodesics for the new metric progressively deform the ellipse in a way which is consistent with the notion of deformation (first column). Euclidean geodesics (second column) squeeze the initial ellipse, and then invert it to start expanding again.

3.2.2. *Example 2.* G now is a group of diffeomorphisms on $\Omega \subset M_0$. We have

$$\sigma^2 z_t^i = \frac{dm_t^i}{dt} - v_t(m_t^i)$$

To compute $\delta_{m_t}^\dagger(z_t)$, which is characterized by

$$\langle z\,,\,\delta_m(v)\rangle_m = \left\langle v\,,\,\delta_m^\dagger(z)\right\rangle_{\mathfrak{g}}$$

we need to introduce the reproducing kernel, K, of $\mathfrak{g}$. It associates to $(x,y)\in\Omega^2$ a linear operator, denoted $K(x,y)$, from T_yM_0 to T_xM_0 which satisfies

- for all $x\in\Omega$ and for $\eta\in T_xM_0$, the vector field $y\mapsto K(y,x)\eta$, which is hereafter denoted $K_x\eta$, belongs to $\mathfrak{g}$
- for all $x\in\Omega$ and $\eta\in T_x\Omega$

$$\langle K_x\eta\,,\,v\rangle_{\mathfrak{g}} = \langle \eta\,,\,v(x)\rangle_x$$

Note that the kernel has the property $K(x,y)^\dagger = K(y,x)$, ie

$$\langle K(x,y)\eta\,,\,\eta'\rangle_x = \left\langle \eta\,,\,K(y,x)\eta'\right\rangle_y$$

for $\eta\in T_yM_0$ and $\eta'\in T_xM_0$.

Given this kernel, the product $\langle \xi\,,\,\delta_m(v)\rangle_m$ may be written

$$\langle \xi\,,\,\delta_m(v)\rangle_m = \sum_{i=1}^N \left\langle \xi^i\,,\,v(m^i)\right\rangle_{m^i} = \sum_{i=1}^N \left\langle K_{m^i}\xi^i\,,\,v\right\rangle_{\mathfrak{g}}$$

so that $\delta_m^\dagger(\xi) = \sum_{i=1}^N K_{m^i}\xi^i$ and (10) is

$$v_t = \sum_{i=1}^N K_{m_t^i} z_t^i$$

We have also, $\delta_m \circ \delta_m^\dagger(\xi) = (\tilde\xi^1,\ldots,\tilde\xi^N)$ with

$$\tilde\xi^j = \sum_{i=1}^N K_{m^i}(m^j)\xi^i = \sum_{i=1}^N K(m^j,m^i)\xi^i$$

Thus, $H_m^\sigma : \prod_{i=1}^N T_{m_i}M_0 \to \prod_{i=1}^N T_{m_i}M_0$ is defined by

$$(H_m^\sigma \xi)^i = \sigma^2\xi^i + \sum_{j=1}^N K(m^i,m^j)\xi^j$$

It is a $Nk\times Nk$ block matrix, for which the (i,j) $k\times k$ block is $K(m^i,m^j)$ if $i\neq j$ and $\sigma^2\mathrm{Id} + K(m^i,m^j)$ if $i=j$. Note that the previous example appears as a particular case, with $K(x,y) = (1+\langle x\,,\,y\rangle)\mathrm{Id}$.

We now explicit equation (14). We have for $\eta = (\eta^1,\ldots,\eta^N)\in T_mM$:

$$\nabla_\eta \mathbf{v} = (\nabla_{\eta^1}v,\ldots,\nabla_{\eta^N}v)$$

so that

$$\nabla_z^\dagger \mathbf{v} = (\nabla_{z^1}^\dagger v,\ldots,\nabla_{z^N}^\dagger v)$$

and equation (14) becomes (for $i=1,\ldots,N$)

$$\nabla_{\frac{dm^i}{ddt}} z_t^i + \nabla_{z_t^i}^\dagger v_t = 0$$

where the covariant derivative now is the one on M_0. When $M_0 = \mathbb{R}^k$, we have $\nabla_\eta v = dv(\eta)$ and this equation writes

$$\frac{dz_t^i}{dt} + {}^t d_{m_t^i} v_t(z_t^i) = 0$$

Figure 3 provides an example of geodesic for landmarks in $\mathbb{R}^2$ (obtained by solving the evolution equation), with a comparison with the corresponding geodesic in the Euclidean space.

3.2.3. *Example 3.* Here, $\delta_m(v) = -\langle \nabla m\,,\, v\rangle$ (recall that we restrict to the case when m is C^1), and, using the reproducing kernel of $\mathfrak{g}$,

$$\begin{aligned}\langle z\,,\,\delta_m(v)\rangle_{L^2} &= -\int_\Omega z(x)\langle \nabla_x m\,,\, v(x)\rangle_{I\!R^k} dx\\ &= -\int_\Omega \langle K_x(z(x)\nabla_x m)\,,\, v\rangle_{\mathfrak{g}} dx\\ &= -\langle K(z\nabla m)\,,\, v\rangle_{\mathfrak{g}}\end{aligned}$$

with the notation $Kf = \int_\Omega K_x f(x)dx$. Equation (10) therefore writes

$$v_t = -K(z_t\nabla m_t)$$

We have $\delta_m \circ \delta_m^\dagger(\eta) = \langle \nabla m\,,\, K(\eta\nabla m)\rangle_{I\!R^k}$, i.e.

$$(\delta_m\circ\delta_m^\dagger(\eta))(y) = \int_\Omega \langle \nabla_y m\,,\, K(y,x)\nabla_x m)\rangle_{I\!R^k}\eta(x)dx$$

so that $\delta_m \circ \delta_m^\dagger$ is a kernel operator, therefore compact, which implies that H_m^σ is invertible and expression (12) is well-defined.

Consider now equation (14). We have $\mathbf{v}(m) = \delta_m(v) = -\langle \nabla m\,,\, v\rangle$, which is linear in m, so that $\nabla_\eta \mathbf{v} = -\langle \nabla\eta\,,\, v\rangle$ and

$$\langle z\,,\,\nabla_\eta\mathbf{v}\rangle_m = -\int_\Omega \langle \nabla_x\eta\,,\, v(x)\rangle z(x)dx = \int_\Omega \eta(x)\mathrm{div}_x(zv)dx$$

which implies

$$\frac{dz_t}{dt} + \mathrm{div}(z_t v_t) = 0$$

4. Evolution equations

4.1. **Evolution in z.** Equation (14) already provides an evolution equation for z. We now proceed to an interpretation of it in terms of conservation of a certain quantity. For any vector field η on M, we have, by equation (13), at all times t

$$\left\langle \nabla_{\frac{\partial m}{\partial t}} z_t\,,\,\eta\right\rangle_{m_t} = -\langle z_t\,,\,\nabla_\eta \mathbf{v}_t\rangle_{m_t}$$

Let $\mu_{t,\varepsilon}$ be a perturbation of μ_t along η i.e. $\mu_{t,0} = \mu_t$ and $\frac{\partial\mu_{t,\varepsilon}}{\partial\varepsilon}{}_{|\varepsilon=0} = \eta$. Now, define, α and β being positive numbers,

$$m_{\alpha,\beta,\varepsilon} = \mathbf{g}_\alpha(\mu_{\beta,\varepsilon})$$

so that $m_{t,t,0} = m_t$. For $\frac{d}{dt} = \frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\beta}$, we have around $(\alpha,\beta,\varepsilon) = (t,t,0)$

$$\frac{d}{dt}\left\langle z_t\,,\,\frac{\partial m}{\partial\varepsilon}\right\rangle_{m_t} = \left\langle \nabla_{\frac{dm}{dt}} z_t\,,\,\frac{\partial m}{\partial\varepsilon}\right\rangle_{m_t} + \left\langle z_t\,,\,\nabla_{\frac{dm}{dt}}\frac{\partial m}{\partial\varepsilon}\right\rangle_{m_t}$$

Since $\frac{dm}{dt} = \frac{\partial m}{\partial\alpha} + \frac{\partial m}{\partial\beta}$, and $\nabla_{\frac{\partial m}{\partial\alpha}}\frac{\partial m}{\partial\varepsilon} = \nabla_{\frac{\partial m}{\partial\varepsilon}}\frac{\partial m}{\partial\alpha}$, we deduce that

$$\frac{d}{dt}\left\langle z_t\,,\,\frac{\partial m}{\partial\varepsilon}\right\rangle_{m_t} = \left\langle \nabla_{\frac{dm}{dt}} z_t\,,\,\frac{\partial m}{\partial\varepsilon}\right\rangle_{m_t} + \left\langle z_t\,,\,\nabla_{\frac{\partial m}{\partial\varepsilon}}\frac{\partial m}{\partial\alpha}\right\rangle_{m_t} + \left\langle z_t\,,\,\nabla_{\frac{\partial m}{\partial\beta}}\frac{\partial m}{\partial\varepsilon}\right\rangle_{m_t}$$

so that using equality (13) and the fact that $\mathbf{v}_t(m_t) = \frac{\partial m}{\partial\alpha}{}_{|\alpha=t,\beta=t,\varepsilon=0}$, we get finally

$$\frac{d}{dt}\left\langle z_t\,,\,\frac{\partial m}{\partial\varepsilon}\right\rangle_{m_t} = \left\langle z_t\,,\,\nabla_{\frac{\partial m}{\partial\beta}}\frac{\partial m}{\partial\varepsilon}\right\rangle_{m_t}. \tag{16}$$

Since $\frac{\partial m}{\partial\epsilon}{}_{|\alpha=t,\beta,\epsilon=0} = d_{\mu_\beta}\mathbf{g}_t(\eta(\mu_\beta))$ and $\frac{\partial m}{\partial\beta}{}_{|\alpha=\beta=t,\epsilon=0} = \sigma^2 z_t$, we obtain

$$\frac{d}{dt}\langle z_t\,,\,\mathbf{Ad}_{\mathbf{g}_t}(\eta)\rangle_{m_t} = \langle z_t\,,\,\nabla_{\sigma^2 z_t}\mathbf{Ad}_{\mathbf{g}_t}(\eta)\rangle_{m_t} \tag{17}$$

where $\mathrm{Ad} : G\times\mathfrak{g} \to \mathfrak{g}$ is the adjoint action defined by $\mathrm{Ad}_g.w = d_e\Phi_g w$ with $\Phi_g(h) = ghg^{-1}$, $h \in G$, and $\mathbf{Ad}_\mathbf{g}$ the induced action on $\chi(M)$ by the identification provided in Notation 1 ($\mathbf{Ad}_\mathbf{g}(\mathbf{v}) = d_{\mathbf{g}^{-1}}\mathbf{v}\mathbf{v} \circ \mathbf{g}^{-1}$. This equation receives an interesting interpretation if we consider the pull-back of the metric by $\mathbf{g}_t$. Indeed, for any $g \in G$, let $\langle\, , \rangle^g$ denote the pull-back of the metric on M by $\mathbf{g}$ defined for any $u, u' \in T_\mu M$ by

$$\langle u\, ,\, u'\rangle^g_\mu = \langle d_\mu\mathbf{g}(u)\, ,\, d_\mu\mathbf{g}(u')\rangle_{\mathbf{g}(\mu)}$$

and let ∇^g denote the associated Levi-Civita connection defined for any $u \in T_\mu M$ and $\eta \in \mathfrak{g}$ by

$$(d_\mu\mathbf{g})(\nabla^g_u\eta) = \nabla_{d_\mu\mathbf{g}(u)}\mathbf{Ad}_\mathbf{g}(\eta)\,. \tag{18}$$

Since $\sigma^2 z_t = d_{\mu_t}\mathbf{g}_t(\frac{d\mu}{dt})$, we get from (17) that

$$\frac{d}{dt}\left\langle \frac{d\mu}{dt}\, ,\, \eta\right\rangle^{g_t}_{\mu_t} = \left\langle \frac{d\mu}{dt}\, ,\, \nabla^{g_t}_{\frac{d\mu}{dt}}\eta\right\rangle^{g_t}_{\mu_t}. \tag{19}$$

We now introduce a new notion related to metamorphoses. We say that a vector field η on M such that $\nabla^{g_t}_{\frac{d\mu_t}{dt}}\eta = 0$, is *morphoparallel* along the metamorphosis. We therefore have obtained the fact that

$$\frac{d}{dt}\left\langle \frac{d\mu}{dt}\, ,\, \eta\right\rangle^{g_t}_{\mu_t} = 0$$

whenever η is morphoparallel.

It can easily be checked from the chart representation that morphoparallelism only involves the values of η along the curve μ_t. Moreover, as a first order linear differential equation on TM, this has a unique solution starting from a tangent vector η_0 at $T_{\mu_0}M$. The solution at time t of this ODE with value $\eta_s \in T_{\mu_s}M$ at time s can be called morphoparallel transport along the metamorphosis and will be denoted $\theta_{st}\eta_s$. Thus, if η is morphoparallel along (g_t, μ_t), we have

$$\langle \dot\mu_t\, ,\, \theta_{0t}(\eta_0)\rangle^{g_t}_{\mu_t} = \langle \dot\mu_0\, ,\, \eta_0\rangle_{\mu_0} \tag{20}$$

where $\dot\mu_s = \frac{d\mu}{dt}_{|t=s}$. This implies that $\langle z_t\, ,\, d_{\mu_t}\mathbf{g}_t(\theta_{0t}(\eta_0))\rangle_{m_t} = \langle \dot\mu_0\, ,\, \eta_0\rangle_{\mu_0}$. Hence, since $\dot\mu_0 = z_0$ and $\mu_0 = m_0$ we get finally

$$z_t = (d_{m_t}\mathbf{g}_t^{-1})^\dagger \circ \theta^\dagger_{t0}(z_0)\,. \tag{21}$$

4.1.1. *The affine case.* In general, the morphoparallel transport equation cannot be solved analytically, so that little is gained, from a practical point of view, compared to the initial equation (14). This will be illustrated in example 2 below. But, as shown in example 1 and 3, morphoparallel translation may sometimes coincide with parallel transport along μ_t, in which case equation (21) becomes quite useful. This arises when the metric ∇^g coincides with ∇ for all $g \in G$ as defined more formally below.

Definition 1. *Let* $\mathbf{g} : M \to M$ *be a* C^2 *invertible mapping. We say that* $\mathbf{g}$ *let invariant the Levi-Civita connexion, if for any* $X, Y \in \chi^1(M)$*, we have*

$$\mathbf{Ad}_\mathbf{g}(\nabla_X Y) = \nabla_{\mathbf{Ad}_\mathbf{g}(X)}\mathbf{Ad}_\mathbf{g}(Y)\,.$$

When M is a flat space, it is a known fact that $\mathbf{g}$ let the connexion invariant iff $\mathbf{g}$ is affine as stated in the following result (the proof is given for completeness in appendix B):

Proposition 2. *Assume that* M *is an Hilbert space considered as a Riemannian manifold with metric induced by the dot product. Let* $\mathbf{g} : M \to M$ *be a* C^2 *invertible mapping. Then* $\mathbf{g}$ *let invariant the Levi-Civita connexion iff i.e. for any* $u \in M$*, we have* $\mathbf{g}(u) = \mathbf{g}(0) + d_0\mathbf{g}(u)$*.*

Thus, in this case, the morphoparallel transport is the parallel transport which is simply a translation. This implies that $\theta_{0t} = \mathrm{Id}$ and (21) becomes

$$z_t = (d_{m_t}\mathbf{g}_t^{-1})^\dagger(z_0)\,. \tag{22}$$

As we have seen, this is the situation of examples 1 and 3 which can be equivalently derived as arising from a semi-direct product $G \ltimes M$ with $G \subset \mathcal{L}(M)$ [11]. So, the affine case, or semi-direct product case, corresponds to some important situations as metamorphosis between fonctionnal data like grey-level images. However, example 2 cannot be handled in this framework and more generally, when M is finite dimensional and the group acting on M is infinite dimensional, we leave the affine framework and the morphoparallel transport is different from the parallel transport.

By extension, when M is non flat, the set of diffeomorphisms of M which leave the metric invariant is still called the affine group of M, and denoted Aff(M). Thus, when $G \subset \text{Aff}(M)$, equation (21) becomes computationally trivialized.

When M is a compact manifold, it is known [20] that the connected components of the identity in the affine group and in the group of isometries of M coincide. Note that, by definition of the metamorphoses, we are only interested in diffeomorphisms $\mathbf{g}$ which belong to this connected component. The group of isometries of a compact finite dimensional manifold is itself a compact finite dimensional manifold. Thus, if M is compact, morphoparallel transport coincides with parallel transport only in specific situations.

4.2. **Evolution of the velocity.** From (10) and (21), we get immediately

$$v_t = \delta_m^\dagger \circ (d_{m_t} g_t^{-1})^\dagger \circ \theta_{t0}^\dagger(z_0)\,. \tag{23}$$

From (9), we get, for $w \in \mathfrak{g}$ $\frac{d}{dt}\langle z_t\,,\, \mathbf{Ad}_{\mathbf{g}_t}(\mathbf{w})\rangle_{m_t} = \frac{d}{dt}\langle v_t\,,\, \text{Ad}_{g_t}(w)\rangle_{\mathfrak{g}}$. Thus we deduce from (17) that

$$\frac{d}{dt}\langle v_t\,,\, \text{Ad}_{g_t}(w)\rangle_{\mathfrak{g}} = \sigma^2\langle z_t\,,\, \nabla_{z_t}\mathbf{Ad}_{g_t}\mathbf{w}\rangle_{m_t} \tag{24}$$

This equation without second term only depends on the Lie group structure. It has the general form of a geodesic equation on a Lie group with a right-invariant metric, as derived in [1]. The non-vanishing left-hand term modifies this conservative evolution to account for the template evolution part of the optimal metamorphosis. This part cannot vanish unless the the metamorphosis is constant, since (9) and $z_0 = 0$ implies $v_0 = 0$ as initial condition. Thus, a pure metamorphosis can only be obtained a the limit process for which $\sigma \to 0$, while z_0 remains non zero.

If we let $S_z(w) = \nabla_z\mathbf{w}$, for $w \in \mathfrak{g}$, we may finally write

$$\frac{d}{dt}\big\langle (Ad_{g_t})^\dagger(v_t)\,,\, w\big\rangle_{\mathfrak{g}} = \big\langle (Ad_{g_t})^\dagger \circ S_{z_t}^\dagger(z_t)\,,\, w\big\rangle_{\mathfrak{g}}$$

or

$$\frac{d}{dt}(Ad_{g_t})^\dagger v_t - (Ad_{g_t})^\dagger S_{z_t}^\dagger z_t = 0 \tag{25}$$

Since, as we have seen $\langle z_t\,,\, \nabla_{\sigma^2 z_t}\mathbf{Ad}_{\mathbf{g}_t}\mathbf{w}\rangle_{m_t} = \left\langle \dot\mu_t\,,\, \nabla_{\dot\mu_t}^{g_t}\eta\right\rangle_{\mu_t}^{g_t}$, we get from (20) that

$$\frac{d}{dt}\langle v_t\,,\, Ad_{g_t}(\eta)\rangle_{\mathfrak{g}} = \left\langle z_0\,,\, \theta_{t0}(\nabla_{\dot\mu_t}^{g_t}\mathbf{w})\right\rangle_{m_0}$$

which yields,

$$\langle v_t\,,\, Ad_{g_t}w\rangle_V = \langle v_0\,,\, w\rangle_{\mathfrak{g}} + \left\langle z_0\,,\, \int_0^t \theta_{s0}(\nabla_{\dot\mu_s}^{g_s}\mathbf{w})ds\right\rangle_{m_0} \tag{26}$$

Finally, we can reintroduce the expression of $\dot\mu_s$ in function of $\dot\mu_0 = z_0$, yielding $\dot\mu_s = \theta_{t0}^\dagger(z_0)$ to explicit the dependency of v_t as a function of v_0 and z_0.

4.3. Examples.

4.3.1. *Example 1.* This first example enters exactly in the setting of affine action on M. Here $M = (\mathbb{R}^k)^N$ with the usual dot product. If $g = (B, T) \in G$, g can be represented as an affine transformation on M whose differential is given by

$$d_m g((\eta^1, \cdots, \eta^N) = (B\eta^1, \cdots, B\eta^N)$$

so that (25) gives

$$z_t^i = {}^tB_t^{-1} z_0^i$$

and (23) gives

$$\begin{aligned}\beta_t &= {}^tB_t^{-1}\sum_{i=1}^N z_0^i\, {}^tm_t^i \\ \tau_t &= {}^tB_t^{-1}\sum_{i=1}^N z_0^i\end{aligned}$$

4.3.2. *Example 2.* Here, the action of G is not affine. Let us verify this, assuming that $M_0 = I\!R^k$ to simplify. In this case, letting $m = (m^1, \ldots, m^N)$ and $\mu^i = g^{-1}(m^i)$:

$$\mathbf{Ad_g}(\eta)(m) = (d_{\mu^1}g(\eta^1(\mu^1)), \cdots, d_{\mu^N}g(\eta^N(\mu^N)))\,.$$

As a consequence, working on the ith component, we have for $\xi^i \in T_{\mu^i}M$

$$\begin{aligned}(\nabla^g_{\xi^i}\eta^i) &= d_{m^i}g^{-1}\left(d_{m^i}(d_{\mu^i}g\circ\eta^i\circ g^{-1})(d_{\mu^i}g(\xi^i))\right) \\ &= d_{m^i}g^{-1}\left(d^2_{\mu^i}g(\eta^i(\mu^i),\xi^i) + d_{\mu^i}g\circ d_{\mu^i}\eta^i(\xi^i)\right) \\ &= d_{\mu^i}\eta^i(\xi^i) + d_{m^i}g^{-1}\left(d^2_{\mu^i}g(\eta^i(\mu^i),\xi^i)\right)\,.\end{aligned}$$

Since $\nabla_{\xi^i}\eta^i = d_{\mu^i}\eta^i(\xi^i)$, the matric is not conserved. Morphoparallel transport along (g_t, μ_t) is the solution of the equation

$$\frac{d\eta_t^i}{dt} + (d_{\mu_t^i}g_t)^{-1}d^2_{\mu_t^i}g_t(\eta^i, \frac{d\mu^i}{dt}) = 0\,.$$

4.3.3. *Example 3.* This is again (at least formally) an affine action: M is a linear space and $m \to gm = m\circ g^{-1}$ is linear in m. To explicit equation (21), we need to compute, according to equation (22) $z_t = (d_{m_t}\mathbf{g}_t^{-1})^\dagger z_0$, which is characterized by

$$\int_\Omega z_t(y)\xi(y)dy = \int_\Omega z_0(x)\xi\circ g_t(x)dx = \int_\Omega z_0\circ g_t^{-1}(y)\xi(y)\left|d_yg_t^{-1}\right|dy$$

so that equation (21) is $z_t(y) = \left|d_yg_t^{-1}\right| z_0\circ g_t^{-1}(y)$.

Figure 4 provides examples of solutions of this system with a fixed initial image (template). The first two simply are reconstructions of precomputed geodesics, with two different target images. The third one provides two ways for averaging the previous two: first by averaging the target images, which clearly provides unsatisfactory results, then by by averaging the initial conditions z_0 before solving the evolution equations, with a much more consistent result.

5. A last example: deforming geometric curves

We conclude this paper with the presentation of a fourth example of metamorphosis, in which M is, as a set, a Hilbert space, but with a non-trivial metric. Indeed, we consider regular plane curves in parametric form, that is functions $m : [0,1] \to \Omega$, with Ω an open bounded subset of $I\!R^2$. We assume enough derivatives for the following formal computations, and we restrict to closed curves, $m(0) = m(1)$, together with all required derivatives.

On this "manifold" M, we will use the metric (with $\xi, \eta : [0,1] \to I\!R^2$ having the same regularity and periodicity condition)

$$\langle \xi\,,\,\eta\rangle_m = \int_0^1 {}^t\xi(x)\eta(x)\left|\frac{dm}{dx}\right|dx$$

ξ and η can in fact be considered as vector fields supported by the image of m in $I\!R^2$ and the corresponding metric is with respect to arc-length integration on m. In the following, we will denote $q_m(x) = |dm/dx| = ds/dx$, s being the arc-length on m. This is a natural metric if one is interested in the geometric aspects of the deformation. Recall that we restrict to regular curves so that q_m is non-vanishing on $[0,1]$.

We now consider the action of a group G of diffeomorphisms of Ω on M, letting $(gm)(x) = g\circ m(x)$. When G is defined like in examples 2 and 3, on the basis of a Hilbert space V of vector

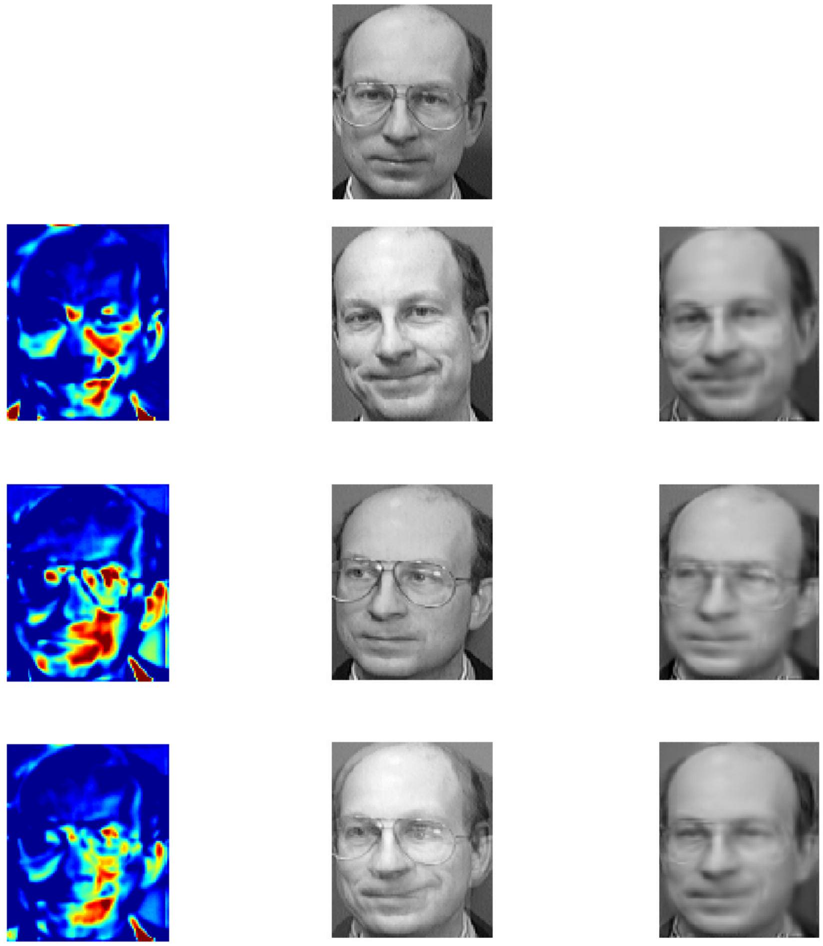

FIGURE 4. Solving the evolution equations with different initial conditions allows to represent a large range of image varieties. The first line, with a single image, is the template. The second line provides the initial z_0, the true target (from which z_0 is computed), and the reconstruction of the target by geodesic evolution. The third line provides the same, with a different target. Finally, the last line compares the naive averaging of the targets (second image) to the output of averaging the initial z_0 before solving the geodesic equations.

fields on Ω, the infinitesimal action, defined for $v \in V$ is $\mathbf{v}(m)(x) = v \circ m(x)$ and the energy of a metamorphosis (g_t, μ_t) is

$$\int_0^1 |v_t|_V \, dt + \frac{1}{\sigma^2} \int_0^1 \left| \frac{dm_t}{dt} - v_t \circ m_t \right|_{m_t}^2 dt$$

The geodesic equations (10) and (13) require the computation of the dual evaluation function, $\delta_m^\dagger$, and of the Levi-Civita connnection on M. Starting with $\delta_m^\dagger$, we have (K being the reproducible kernel on V)

$$\begin{aligned} \langle \xi \,,\, \delta_m(v) \rangle_m &= \int_0^1 {}^t\xi(x) v(m(x)) q_m(x) dx \\ &= \int_0^1 \left\langle K_{m(x)} \xi(x) \,,\, v \right\rangle_V q_m(x) dx \end{aligned}$$

so that

$$\delta^\dagger_m(x) = \int_0^1 q_m(x) K_{m(x)} \xi(x) dx$$

and equation (10) writes

$$v_t(y) = \int_0^1 K(y, m_t(x)) z_t(x) q_{m_t}(x) dx \tag{27}$$

with $\sigma^2 z_t = dm_t/dt - v_t \circ m_t$.

The covariant derivative can be computed from the standard formula

$$2\langle \zeta\,,\, \nabla_\xi \eta\rangle = \xi\langle \eta\,,\, \zeta\rangle + \eta\langle \xi\,,\, \eta\rangle - \zeta\langle \xi\,,\, \eta\rangle - \langle [\xi,\zeta]\,,\, \eta\rangle - \langle [\eta,\zeta]\,,\, \xi\rangle + \langle [\xi,\eta]\,,\, \zeta\rangle$$

which yields, after a simple computation,

$$\nabla_\xi \eta = \xi\eta + \frac{1}{2q}\left(\eta(\xi q) + \xi(\eta q) - \rho^{{}^t\eta\xi}\right)$$

where ρ^f is defined by

$$\int_0^1 {}^t\rho^f_m(x)\zeta_m(x)dx = \int_0^1 f_m(x)(\zeta q)_m(x)dx$$

Using the fact that $q_m(x) = |dm/dx|$, we have $(\zeta q)_m(x) = {}^t\frac{d\zeta_m}{dx}\tau_m(x)$ with $\tau_m(x) = (dm/dx)/\,|dm/dx|$ (the unit tangent) so that, using integration by parts,

$$\rho^f_m = -\frac{d}{dx}\left(f_m \tau_m\right)$$

and

$$\nabla_\xi \eta = \xi\eta + \frac{1}{2q}\left(\eta({}^t\frac{d\xi}{dx}\tau) + \xi({}^t\frac{d\eta}{dx}\tau) + \frac{d}{dx}\left(({}^t\xi\eta)\tau\right)\right)$$

Introducing the arc-length derivative $d/ds = (1/q)d/dx$, this writes

$$\nabla_\xi \eta = \xi\eta + \frac{1}{2}\left(\eta({}^t\frac{d\xi}{ds}\tau) + \xi({}^t\frac{d\eta}{ds}\tau) + \frac{d}{ds}\left(({}^t\xi\eta)\tau\right)\right)$$

In particular, adopting, for short, the notation $\dot u = \frac{\partial u}{\partial t}$ and $u' = \frac{\partial u}{\partial s}$, and using the fact that $(\dot m)' = \sigma^2 z' + d_m v\tau$

$$\begin{aligned}\nabla_{\dot m} z &= \dot z + \frac{1}{2}\left(z({}^t(dv\tau)\tau) + \sigma^2 z({}^t z'\tau) + \dot m({}^t z'\tau) + \left(({}^t z\dot m)\tau\right)'\right)\\ &= \dot z + \frac{1}{2}\left(z({}^t(dv\tau)\tau) + 2\sigma^2 z({}^t z'\tau) + v({}^t z'\tau) + \left(({}^t z\dot m)\tau\right)'\right)\end{aligned}$$

(v and dv being evaluated at m).

Since $\langle \zeta\,,\, \nabla_\xi\eta\rangle = \left\langle \nabla^\dagger_\zeta \eta\,,\, \xi\right\rangle$, the same analysis provides

$$\nabla^\dagger_\zeta \eta = \eta^\dagger\zeta + \frac{1}{2q}\left(\zeta(\eta q) - \eta(\zeta q) + \rho^{{}^t\eta\zeta}\right)$$

with $\left\langle \eta^\dagger\zeta\,,\, \xi\right\rangle = \langle \zeta\,,\, \xi\eta\rangle$, so that

$$\nabla^\dagger_\zeta \eta = \eta^\dagger\zeta + \frac{1}{2}\left(\zeta({}^t\frac{d\eta}{ds}\tau) - \eta({}^t\frac{d\zeta}{ds}\tau) - \frac{d}{ds}\left(({}^t\eta\zeta)\tau\right)\right)$$

Applying this to $\zeta = z$ and $\eta_m = \mathbf{v}(m) = v \circ m$ yields $z^\dagger \mathbf{v} = {}^t dvz$ and

$$\nabla^\dagger_z \mathbf{v} = {}^t dvz + \frac{1}{2}\left(z({}^t(dv\tau)\tau) - v(z'\tau) - \left(({}^t zv)\tau\right)'\right)$$

This yields the equation for z_t

$$\dot z + {}^t dvz + ({}^t\tau dv\tau)z + \sigma^2 z({}^t z'\tau) = 0 \tag{28}$$

We therefore an summarize the geodesic equation for the studied action by

$$\begin{cases} (\dot{m})' = \sigma^2 z' + d_m v\tau \\ \dot{z} + {}^t dvz + ({}^t\tau dv\tau)z + \sigma^2 z({}^t z'\tau) = 0 \\ v(y) = \int_0^1 K(y,m) z ds \end{cases}$$

Appendix A. Existence of solutions of $\frac{dg_t}{dt} = d_e R_{g_t} v_t$

In this section, we consider, on G, the right invariant Riemannian structure associated to the norm on $\mathfrak{g}$, and we assume that G is a complete Riemannian manifold. We will also assume that there exists a constant C such that $|[v,w]|_{\mathfrak{g}} \leq C\,|v|_{\mathfrak{g}}\,|w|_{\mathfrak{g}}$, for $v, w \in \mathfrak{g}$.

Our goal is to define solutions of equation (1), when v_t satisfies

$$\|v\|_{1,\mathfrak{g}} := \int_0^1 |v_t|_{\mathfrak{g}}\, dt < \infty .$$

First note that, if v_t is continuous in time, a solution exists at least in small time, and we now show that it can be extended to $[0,1]$. Along such a solution, we have

$$d_G(g_s, g_t) \leq \int_s^t |v_s|_{\mathfrak{g}}\, ds$$

which implies that if $[0,T[$ is a maximal interval for the solution, a limit can be found for g_t when t tends to T (because G is complete), and the solution can be further extended beyond T, unless of course $T = 1$.

We now show that, for any $h \in G$ and $u \in \mathfrak{g}$:

$$|Ad_{w_t} u|_{\mathfrak{g}} \leq \exp\left(C d_G(e, w_t)\right) |u|_{\mathfrak{g}}$$

Indeed let h_t be a geodesic between e and h (it exists because G is complete), and let $\tilde{u}_t = (d_e R_{h_t})^{-1}\frac{dh_t}{dt}$ so that

$$\left|\frac{d}{dt} Ad_{h_t} u\right|_{\mathfrak{g}} = |[\tilde{u}_t, Ad_{h_t} u]| \leq C\,|\tilde{u}_t|_{\mathfrak{g}}\,|Ad_{h_t} u|_{\mathfrak{g}}$$

and

$$\frac{d}{dt}\,|Ad_{h_t} u|^2_{\mathfrak{g}} \leq 2C\,|\tilde{u}_t|_{\mathfrak{g}}\,|Ad_{h_t} u|^2_{\mathfrak{g}} .$$

By Gronwall's lemma, this implies

$$|Ad_{h_t} u|_{\mathfrak{g}} \leq |u|_{\mathfrak{g}} \exp\left(C \int_0^t |\tilde{u}_s|_{\mathfrak{g}}\, ds\right) = |u|_{\mathfrak{g}} \exp\left(C d_G(e, h_t)\right) .$$

Now, consider the map $v \mapsto g_t^v$ which associates to $(v : s \mapsto v_s)$, assumed to be continuous and belonging to $L^1([0,1],\mathfrak{g})$, the solution of (1) at time t. If $v, w \in L^1([0,1],\mathfrak{g})$, we have, letting $q_t = g_t^v (g_t^w)^{-1}$

$$d_G(g_t^v, g_t^w) = d_G(e, q_t) \leq \int_0^t \left|(d_e R_{q_t})^{-1}\frac{dq_t}{dt}\right|_{\mathfrak{g}} dt$$

A straightforward computation shows that

$$\begin{aligned} \rho_s := (d_e R_{q_s})^{-1}\frac{dq_s}{ds} &= v_s - Ad_{q_s^{-1}} w_s \\ &= v_s - w_s - \int_0^s \frac{d}{du} Ad_{q_u^{-1}} w_s du \\ &= v_s - w_s - \int_0^s [\rho_u, Ad_{q_u^{-1}} w_s] du \end{aligned}$$

Since $d_G(e, q_u^{-1}) = d_G(g_u^v, g_u^w) \leq d_G(e, g_u^v) + d_G(e, g_u^w) \leq \|v\|_{1,\mathfrak{g}} + \|w\|_{1,\mathfrak{g}}$, we have

$$|\rho_s|_{\mathfrak{g}} \leq |w_s - v_s|_{\mathfrak{g}} + C \exp\left(C(\|v\|_{1,\mathfrak{g}} + \|w\|_{1,\mathfrak{g}})\right) \int_0^s |\rho_u|_{\mathfrak{g}}\,|w_u|_{\mathfrak{g}}\, du$$

and by Gronwall's lemma again, there exists a continuous function $K(\|v\|_{1,\mathfrak{g}}, \|w\|_{1,\mathfrak{g}})$ such that

$$|\rho_s|_{\mathfrak{g}} \leq K(\|v\|_{1,\mathfrak{g}}, \|w\|_{1,\mathfrak{g}}) \|w - v\|_{1,\mathfrak{g}}$$

so that

$$d_G(g_t^v, g_t^w) \leq K(\|v\|_{1,\mathfrak{g}}, \|w\|_{1,\mathfrak{g}}) \|w - v\|_{1,\mathfrak{g}}$$

This implies that g_t^v is uniformly continuous over bounded subsets of $L^1([0,1],\mathfrak{g}) \cap C^0([0,1],\mathfrak{g})$, and can therefore be extended by continuity to $L^1([0,1],\mathfrak{g})$.

Appendix B. Proof of proposition 2

Proof. ($\Rightarrow$) Indeed, for any $u \in M$, let $\mathbf{u} : M \to M$, be the constant vector field defined by $\mathbf{u}(m) = u$ for any $m \in M$. Let $\mathbf{v} = \mathrm{Ad}_g(\mathbf{u})$. If $x_t = tu$, then $\frac{dx}{dt} = u$ and if $y_t = g(x_t)$, we have

$$\frac{dy}{dt} = d_{x_t} g(\frac{dx}{dt}) = d_{x_t} g(u) = \mathrm{Ad}_g(\mathbf{u})(y_t) = \mathbf{v}(y_t)\,.$$

Then

$$\nabla_{\frac{dy}{dt}} \frac{dy}{dt} = \nabla_{\mathbf{v}} \mathbf{v}(y_t) \overset{(a)}{=} d_{x_t} g(\nabla_{\mathbf{u}} \mathbf{u}(x_t)) = d_{x_t} g(\nabla_{\frac{dx}{dt}} \frac{dx}{dt}) = 0\,,$$

where (a) comes from the invariance property of g. Thus, $\frac{dy}{dt}$ is constant and $g(u) = y_0 + \int_0^1 \frac{dy}{dt} dt = g(0) + d_0 g(u)$.

($\Leftarrow$): Assume that $g : M \to M$ is defined by $g(u) = g(0) + B(u)$ where B is a continuous invertible linear operateur on M. For any X and $Y \in \chi^1(M)$, since in this flat case, $\nabla_X Y = dY(X)$, we get

$$\mathrm{Ad}_g(\nabla_X Y)(m) = B \circ d_{g^{-1}(m)} Y(X(g^{-1}(m)))\,.$$

Now, $d_m(\mathrm{Ad}_g(Y)) = B \circ d_{g^{-1}(m)} Y \circ B^{-1}$, so that

$$\nabla_{\mathrm{Ad}_g(X)} \mathrm{Ad}_g(Y) = B \circ d_{g^{-1}(m)} Y(X(g^{-1}(m))) = \mathrm{Ad}_g(\nabla_X Y)(m)\,,$$

and the result is proved. □

References

[1] I. Arnold, V, *Mathematical methods of Classical Mechanics*, Springer, 1978. Second Edition: 1989.

[2] V. Camion and L. Younes, *Geodesic interpolating splines*, in EMMCVPR 2001, M. Figueiredo, J. Zerubia, and K. Jain, A, eds., vol. 2134 of Lecture notes in computer sciences, Springer, 2001.

[3] P. Dupuis, U. Grenander, and M. Miller, *Variational problems on flows of diffeomorphisms for image matching*, Quaterly of Applied Math., (1998).

[4] G. Ebin, D and E. Marsden, J, *Groups of diffeomorphisms and the motion of an incompressible fluid*, Ann. of Math, 92 (1970), pp. 102–163.

[5] J. Glaunès, M. Vaillant, and I. Miller, M, *Landmark matching via large deformation diffeomorphisms on the sphere*, Journal of Mathematical Imaging and Vision, MIA 2002 special issue, (2003).

[6] U. Grenander, *General Pattern Theory*, Oxford Science Publications, 1993.

[7] U. Grenander and I. Miller, M, *Representation of knowledge in complex systems (with discussion section)*, J. Royal Stat. Soc., 56 (1994), pp. 569–603.

[8] S. Joshi and M. Miller, *Landmark matching via large deformation diffeomorphisms*, IEEE transactions in image processing, 9 (2000), pp. 1357–1370.

[9] A. Kriegl and W. Michor, P, *Regular infinite dimensional lie groups*, J. of Lie Theory, 7 (1997), pp. 61–99.

[10] E. Marsden, J and S. Ratiu, T, *Introduction to Mechanics and Symmetry*, Springer, 1999.

[11] E. Marsden, J, T. Ratiu, and A. Wenstein, *Semidirect products and reduction in mechanics*, Trans. Am. Math. Soc., 281 (1984).

[12] I. Miller, M, A. Trouvé, and L. Younes, *On the metrics and euler-lagrange equations of computational anatomy*, Annual Review of biomedical Engineering, 4 (2002), pp. 375–405.

[13] I. Miller, M and L. Younes, *Group action, diffeomorphism and matching: a general framework*, Int. J. Comp. Vis, 41 (2001), pp. 61–84. (*Originally published in electronic form in: Proceeding of SCTV 99, http://www.cis.ohio-state.edu/ szhu/SCTV99.html*).

[14] R. Palais, *Foundations of Glonal Non-linear Geometry*, Benjamin, 1968.

[15] A. Trouvé, *Infinite dimensional group action and pattern recognition*, tech. rep., DMI, Ecole Normale Supérieure, 1995.

[16] ———, *Diffeomorphism groups and pattern matching in image analysis*, Int. J. of Comp. Vis., 28 (1998), pp. 213–221.

[17] A. Trouvé and L. Younes, *On a class of optimal matching problems in 1 dimension*, Siam J. Control Opt. (to appear), (2000).
[18] A. Trouvé and L. Younes, *Local geometry of deformable templates*, tech. rep., Université Paris 13, 2002.
[19] C. Twinings, S. Marsland, and C. Taylor, *Measuring geodesic distances on the space of bounded diffeomorphisms*, in British Macine Vision Conference, 2002.
[20] K. Yano, *On harmonic and killing vector fields*, Ann. Math., (1952), pp. 38–45.
[21] L. Younes, *Computable elastic distances between shapes*, SIAM J. Appl. Math, 58 (1998), pp. 565–586.
[22] L. Younes, *Optimal matching between shapes via elastic deformations*, Image and Vision Computing, (1999).
[23] E. Zeidler, *Applied Functional Analysis. Applications to mathematical physics*, Springer, 1995.

Alain Trouvé, CMLA, ENS de Cachan, 61, avenue du Président Wilson, 94235 Cachan CEDEX, France. Email: trouve@cmla.ens-cachan.fr

Laurent Younes, Department of Applied Mathematics and Statistics and Center for Imaging Science, The Johns Hopkins University, 3400 N-Charles Street Baltimore MD 21218-2686. Email: younes@cis.jhu.edu. (Corresponding author)