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Technical Report

State Estimation in Stochastic Hybrid Systems with Sparse Observations

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Abstract

In this paper we study the problem of state estimation for a class of sampled-measurement stochastic hybrid systems. In the model we consider, the *continuous* state x satisfies a linear stochastic differential equation, and noisy measurements y are taken at assigned discrete-time instants. The parameters of both the state and measurement equation depend on the *discrete* state q of a continuous-time finite Markov chain. Thus, switches may occur between two subsequent measurements. This makes the model well suited to systems subject to sudden changes occurring at a rate comparable to the sampling rate. On the other hand, it also makes state estimation an extremely hard task. We then focus on a typical fault detection setting, in which at most one switching takes place. Based on a convenient parametrization of the system, we derive on-line algorithms for the estimation of both the continuous and the discrete state as well as of the discrete-state switching time and final value. Numerical simulations are reported and illustrated.

Contents

1 Introduction

Over the last three decades, a substantial effort has been dedicated to the study of the so-called *Jump Markov Linear Systems* (JMLS). These are linear Gaussian systems switching among a finite number of linear modes indexed by a *discrete state* q. In general, measurements y of the *continuous state* x also depend on q. Switching follows the laws of a Markov chain independent of the initial state $x(0)$ and of the system inputs. Models as such are a natural description of systems subject to sudden changes and find applications in disparate fields including computer vision, robotics, and econometrics, to name a few. Typical linear system estimation problems carry over to JMLS, such as continuous state filtering and prediction. In addition, being the trajectory of q the outcome of a stochastic process, estimation of q from the available measurements is also a concern. However, for any fixed trajectory of q, the system is linear time-variant. It follows that the statistics of x and y are mixtures of Gaussian distributions, and their optimal (Bayesian) estimates are found by *averaging of conditional Kalman filters*. This also leads to an optimal estimation of q; however, the complexity of the solution is at any time equal to the number of possible discrete trajectories of the discrete state.

Most of the literature deals with discrete-time JMLS. For this class of systems, the complexity of the optimal Bayesian estimates of x and q is exponential in time. Thus, a big effort has been devoted to derive effective *suboptimal* algorithms of reasonable (finite) complexity. The most common approach in literature is to elicit a fixed number of "most probable" discrete state trajectories, and to obtain approximate estimates by suitable averaging. In the Generalized Pseudo Bayes approach of Ackerson and Fu [1], approximation is achieved by fitting a Gaussian distribution to the actual distribution of the state, by way of "ensemble" mean and covariance parameters. Tugnait [25, 26] uses a Detection-Estimation strategy: at each step, the most probable mode $q(t+1)$ is detected out of the possible transitions of $q(t)$ according to a new measurement of the system's state. Generalizations of these algorithms are also available. Blom & Bar-Shalom [3] keep track of a fixed number of trajectories $q(t)$ by "merging" those that prove "undistinguishable", and discarding the unlikely ones. Alternative directions of research include polynomial filtering [12], joint state-mode estimation [6, 7], and sequential hypothesis testing [24]. A randomized approach is offered by sequential Monte Carlo methods [8, 9, 18, 19, 21], which *explore* the space of all possible discrete trajectories based on a convenient generating distribution. By their nature, a large number of trajectory samples are needed for these methods to be effective. Further tecniques are illustrated in the works by Chen & Liu [4], Elliott [10], Hofbaur & Williams [14]. Lerner *et al.* [20] and Murphy [23]. Less attention has been dedicated to the continuous-time counterpart. The interested reader may consult the work by Hibey & Charalambous [13], Hu *et al.* [15], Miller & Runggaldier [22], and Zhang [27]. In this setting, a *continuum* of measurements is available for estimation purposes.

Both the discrete-time and the continuous-time models are unsatisfactory whenever continuous-time information (estimates) about the system state needs to be drawn from measurements *sampled* at a rate comparable to that of the system dynamics. This is the case, for instance, of medical applications, where measurements (such as blood samples) can only be taken from patients at sparse time instants, though the evolution of a disease or the effects of a therapy need be monitored with as much time resolution as possible. In the present paper we introduce a model where the continuous state x evolves in continuous time according to a linear stochastic differential equation, whereas noisy measurements are acquired at given time instants. The parameters of both the state equation and the measurement equation depend on a discrete state q which evolves in time as a *continuous-time Markov chain*. We study the problem of estimating the pair (x, q), given the available measurements. Observe that the switching may occur (in principle, even more than once) *between* two consecutive measurements. This greatly complicates the estimation task. Indeed, the optimal (Bayesian) estimates of x will rely on a *continuous* mixture of Gaussian densities: in particular, their description inevitably becomes non-parametric. In this work we will formulate the problem mostly restricting our attention to a typical *fault detection* setting (see, e.g., [24]), where all but one discrete states are *absorbing*. In this context, as we shall see, the use of a mixed continuous/discrete-time model allows to make inference about the precise instant of a switch when just a few subsequent measurements are provided. Therefore, we believe that the use of such a scheme may significantly enrich the analysis and monitoring of processes subject to sudden changes, such as biochemical reactions, industrial plants (subject to ruptures), and other systems of interest.

The paper is organized as follows. In section 2 we introduce the general sampled-measurement stochastic hybrid model and formalize the state estimation problems of our concern. We then focus on the fault detection case and introduce the switching time $t^\star$ and the final discrete state $q^\star$ as a characterization of the trajectory of the discrete state. Section 3 investigates the interpretation of process $q(t)$ in terms of *subordinated Markov chains*. In section 4 we derive a statistically equivalent model based on formal system discretization along fixed trajectories of q, which we call *conditioned system*. Using this result, the interpretation of the estimation of x as averaging of *conditional Kalman filters* – i.e. ordinary Kalman filters conditioned on the switching time and the final discrete state – is discussed in section 5. The following section presents methods for the efficient computation of the *a posteriori* joint probability distribution of $t^\star$ and $q^\star$, and shows its intimate connection with the computation of the conditional Kalman estimates in the form of functions of $t^\star$ and $q^\star$. The latter problem is tackled in section 7, where closed-form parametric expressions of the estimates are given, along with finite complexity parameter update formulas. Section 8 briefly illustrates how to extend the previous techniques to state estimation *between* measures. Numerical results are illustrated in section 9, whereas final comments and perspectives of our work are reported in Section 10. Unless otherwise stated, proofs are reported in the Appendix.

2 Problem Formulation

Let $\mathcal{T} = \{t_k\}_{k\in\mathbb{N}_0}$ (where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$) be an arbitrary deterministic sequence, with $t_k < t_{k+1}$ and $t_k \to \infty$. Consider a finite *state space* $\mathcal{Q} = \{0, 1, 2, \ldots, N-1\}$ and let q denote its generic element. Assume that we are given matrix functions: $F : \mathcal{Q} \to \mathbb{R}^{n\times n}$, $G : \mathcal{Q} \to \mathbb{R}^{n\times m}$, $H : \mathcal{Q} \to \mathbb{R}^{p\times n}$, and $K : \mathcal{Q} \to \mathbb{R}^{p\times r}$, which assign to each value $q \in \mathcal{Q}$ a four-tuple of matrices (F_q, G_q, H_q, K_q).

Consider the following dynamical model:

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & F_{q(t)}x(t) + G_{q(t)}u(t) \\ y_k & = & H_{q(t_k)}x(t_k) + K_{q(t_k)}v_k \end{array}\right. , \quad t \in \mathbb{R},\ t_k \in \mathcal{T}, \tag{1}$$

where $x : \mathbb{R} \to \mathbb{R}^n$, $y : \mathbb{N}_0 \to \mathbb{R}^p$, are stochastic processes. We will refer to the first equation in (1) as *state equation*, and the second one as *measurement equation*. In the above model two different zero-mean, normalized, white Gaussian noise inputs appear: the continuous-time noise $u(t)$, $t \in \mathbb{R}$ and the discrete-time noise v_k indexed by $k \in \mathbb{N}_0$. We assume that $\{u(t)\}_{t\in\mathbb{R}}$, $\{v_k\}_{k\in\mathbb{N}_0}$ and the initial condition $x(t_0) \sim \mathcal{N}(\mu_0, \Sigma_0)$ are mutually independent; μ_0 and Σ_0 are given. Furthermore, we shall assume that $q(t)$, $t \in \mathbb{R}$, is a continuous-time, homogeneous *Markov chain* (independent of $\{u(t)\}$, $\{v_k\}$ and x_0) with assigned transition probabilities $T_{i,j}(\Delta) \triangleq \mathbb{P}[q(t + \Delta) = j \mid q(t) = i]$, $i, j \in \mathcal{Q}$, and initial probabilities $p_q \triangleq \mathbb{P}[q(t_0) = q]$, $q \in \mathcal{Q}$. As a consequence of our hypotheses, for $\Delta > 0$ it holds that $\mathbb{P}[q(t + \Delta) \mid q(t), x(t)] = \mathbb{P}[q(t + \Delta) \mid q(t)]$.

In simple terms, the system may be viewed as follows: the continuous-time Markov Chain q switches in time among a finite number of states (and the time interval between two subsequent jumps is a memoryless random variable), thus changing the parameters of both the state equation (which is a linear stochastic differential equation) and the (static) measurement equation. Noisy measurements of state x are taken discretely in time, although switches of q may occur among

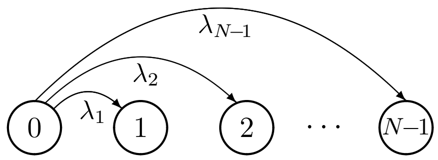

Figure 1: Graphical representation of Markov Process (4).

such measurements. Our problem is the following: given measurements up to time t_k, that is $y^k \triangleq \{y_0, \ldots, y_k\}$, we wish to compute the "best" estimate for the joint state (x, q). More precisely, for $\ell, k \in \mathbb{N}_0$ we wish to compute the least-squares estimate of the *continuous* state $x(t_\ell)$:

$$\hat{x}^a_{\ell|k} \triangleq \arg \min_{g(y^k): g \in \mathcal{M}} \mathbb{E}\left[||g(y^k) - x(t_\ell)||^2 \right] = \mathbb{E}\big[x(t_\ell) \,\big|\, y^k\big], \tag{2}$$

(where $\mathcal{M}$ is the set of measurable functions $g : \mathbb{R}^{(k+1)p} \to \mathbb{R}^n$) and at the same time the *a posteriori* probability distribution of the *discrete* state:

$$p_{\ell|k}(q) \triangleq \mathbb{P}\big[q(t_\ell) = q \,\big|\, y^k\big]\,. \tag{3}$$

We will mostly restrict our attention to the cases $\ell = k$ (filtering) and $\ell = k + 1$ (prediction). Estimate $\hat{x}^a$ is, as we shall see, a weighed *average* of a *continuum* of Bayesian estimates for different linear stochastic systems – hence the superscript "a". This is a consequence of the discrete parameters of the model changing in time in a random fashion, and will become clearer in the sections that follow.

According to our model the discrete state can switch among different values in $\mathcal{Q}$ *between* two successive measurements – in principle, even more than once: this makes the exact computation of the above estimates a formidable task. In the present paper, however, we shall limit ourselves to a *fault detection* setting. That is, we will assume that $\mathcal{Q} = \{0, 1, \ldots, N-1\}$ and that states $q \neq 0$ are *absorbing*: in other words, for a given set of positive parameters $\{\lambda_1, \ldots, \lambda_{N-1}\}$, the *generator matrix* $\mathbf{G} = \left[\frac{\mathrm{d}T}{\mathrm{d}\Delta}\right]_{\Delta=0} \in \mathbb{R}^{n \times n}$ of the Markov process (e.g., see [17]) is given by:

$$\mathbf{G} = \left[\begin{array}{c|ccccc} -(\lambda_1 + \ldots + \lambda_{N-1}) & \lambda_1 & \cdots & \lambda_j & \cdots & \lambda_{N-1} \\ \hline 0 & & & & & \\ \vdots & & & 0 & & \\ 0 & & & & & \end{array}\right] \tag{4}$$

that corresponds to a transition probability matrix $[T_{i,j}(\Delta)]$ of the form:

$$T(\Delta) = \left[\begin{array}{c|ccc} e^{-\Lambda\Delta} & \cdots & \frac{\lambda_j}{\Lambda}\left(1 - e^{-\Lambda\Delta}\right) & \cdots \\ \hline 0 & & & \\ \vdots & & I_{N-1} & \\ 0 & & & \end{array}\right] \tag{5}$$

where $\Lambda \triangleq \sum_{i=1}^{N-1} \lambda_i$. The graphical representation of such Markov process is shown in Figure 1.

The continuous-time Markov chain model of $q(t)$ enjoys an intriguing interpretation in terms of *subordinated Markov chains*, that is, discrete-time Markov chains whose switches are "enabled" by the arrivals of a Poisson process. The interested reader is referred to the discussion is Section 3.

As we said above, in the setting of fault detection there can be *at most* one switching of the discrete state from state 0 to any of the absorbing states $\{1,\ldots,N-1\}$. Hence, trajectory $q(t)$ is *characterized* by the *switching time* $t^\star$ (i.e. the time at which the event takes place) and the *new discrete state* $q^\star$ (i.e. the value of $q(t)$ for $t > t^\star$). The joint probability distribution of $t^\star$ and $q^\star$ is given by

$$F_{t^\star,q^\star}(t,q) \triangleq \mathbb{P}[t^\star \le t, q^\star = q] = \frac{\lambda_q}{\Lambda}(1-e^{-\Lambda t})p_0 + p_q \tag{6}$$

for $q = 1,\ldots,N-1$ and $t \ge 0$; it is zero for $q = 0$ and is undefined for $t < 0$. Therefore, the probability distribution of $t^\star$ is, for $t \ge 0$,

$$F_{t^\star}(t) = (1-e^{-\Lambda t})p_0 + (1-p_0), \tag{7}$$

and is undefined otherwise, while the probability distribution of $q^\star$ is, for $q = 1,\ldots,N-1$,

$$\mathbb{P}[q^\star = q] = \frac{\lambda_q}{\Lambda}p_0 + p_q,$$

and is zero for $q = 0$.

When $p_0 = 1$, in particular, $F_{t^\star,q^\star}(0,q) = 0$ for any $q \in \mathcal{Q}$. That is, the distribution of the joint variable $(t^\star, q^\star)$ is concentrated in $t^\star \ge 0$. Thus it makes sense to consider, for $t \ge 0$, the density function

$$f_{t^\star,q^\star}(t,q) \triangleq \frac{\mathrm{d}}{\mathrm{d}t}F_{t^\star,q^\star}(t,q) = \lambda_q e^{-\Lambda t} \tag{8}$$

(note that (8) is not properly a probability density function since one of the arguments is continuous while the other is discrete; we only differentiate with respect to the continuous argument). Similarly,

$$f_{t^\star}(t) \triangleq \frac{\mathrm{d}}{\mathrm{d}t}F_{t^\star}(t) = \Lambda e^{-\Lambda t}.$$

Therefore in this case ($p_0 = 1$) the switching time is exponentially distributed, $t^\star \sim \mathcal{E}(\Lambda)$. Moreover,

$$f_{t^\star,q^\star}(t,q) = \Lambda e^{-\Lambda t}\cdot\frac{\lambda_q}{\Lambda} = f_{t^\star}(t)\cdot\mathbb{P}[q^\star = q],$$

i.e. random variables $t^\star$ and $q^\star$ become *independent*. For the sake of clarity, the assumption $p_0 = 1$ will be maintained throughout the rest of the paper.

We shall compute the *a posteriori* probabilities

$$\mathbb{P}\big[\, t^\star \le t,\, q^\star = q \,\big|\, y^k \,\big] \tag{9}$$

from which probabilities (3) follow immediately:

$$p_{\ell|k}(q) = \begin{cases} \mathbb{P}[t^\star > t_\ell \,|\, y^k] & \text{for } q = 0 \\ \mathbb{P}[t^\star \le t_\ell, q^\star = q \,|\, y^k] & \text{for } q \neq 0 \end{cases}.$$

In practice, this will be done by computing and integrating the density

$$f_{t^\star,q^\star|y^k}(t,q) \triangleq \frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{P}\big[\, t^\star \le t,\, q^\star = q \,\big|\, y^k \,\big]. \tag{10}$$

The *a posteriori* switching time probability density may also be inferred from (10), since:

$$f_{t^\star|y^k}(t) \triangleq \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}[t^\star \leq t \,|\, y^k] = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{q=1}^{N-1}\mathbb{P}[t^\star \leq t, q^\star = q \,|\, y^k] = \sum_{q=1}^{N-1} f_{t^\star,q^\star|y^k}(t,q).$$

With an abuse of notation we shall indicate (10) by $f(t^\star, q^\star|y^k)$. A similar notation will be used for all density functions. Observe that, by Bayes' rule,

$$f(t^\star, q^\star|y^k) \propto f(y^k|t^\star, q^\star) \cdot f(t^\star, q^\star).$$

Part of the paper, as we shall see, will be devoted to the iterative computation of factor $f(y^k|t^\star, q^\star)$.

3 Subordinated Markov chains

An equivalent way of modelling the switching process $q(t)$ is as a *subordinated Markov chain*. Consider a Poisson process $\{\tau_k\}_{k\in\mathbb{N}_0}$, $\tau_k \geq 0$, of rate ν and let $c(t)$, $t \geq 0$, be the corresponding counting process:

$$c(t) \triangleq \sum_{k\in\mathbb{N}_0} \mathbb{1}_{[0,t]}(\tau_k),$$

where $\mathbb{1}_{\mathcal{B}}(\cdot)$ is the indicator function of set $\mathcal{B} \subseteq \mathbb{R}$. Next, consider a discrete-time homogeneous Markov chain $\{m_\ell\}_{\ell\in\mathbb{Z}}$, *independent of* τ_k, taking values in set $\mathcal{Q}$. Let P be the transition probability matrix of m_ℓ, where $P_{i,j} \triangleq \mathbb{P}[m_{\ell+1} = j|m_\ell = i]$. Now define

$$q^s(t) \triangleq m_{c(t)},$$

where superscript "s" stands for "subordinated". According to this definition, switches take place at the arrivals of a Poisson process and, when these occur, the switch itself follows the laws of a Markov chain. The following result is borrowed with minor modifications from [17].

Proposition 1 *Process $q^s(t)$ is a (continuous-time) homogeneous Markov chain with transition probabilities*

$$\mathbb{P}[q^s(t+\Delta) = j|q(t) = i] = e^{-\nu\Delta}\sum_{k=0}^{+\infty}\frac{(\nu\Delta)^k}{k!}\mathbb{P}[m_{\ell+k} = j|m_\ell = i]. \tag{11}$$

Define the transition probability matrix $T^s(\Delta)$, $\Delta > 0$, as $T^s_{i,j}(\Delta) \triangleq \mathbb{P}[q(t+\Delta) = j|q(t) = i]$, and recall that $\mathbb{P}[m_{\ell+k} = j|m_\ell = i] = (P^k)_{i,j}$. Then, equation (11) simply becomes

$$T^s(\Delta) = e^{-\nu\Delta}\sum_{k=0}^{+\infty}\frac{(\nu\Delta)^k}{k!}P^k = e^{\nu\Delta(P-I)}. \tag{12}$$

For certain choices of P, the rightmost matrix exponential may be computed explicitly. In the setting of *fault detection* (i.e., when all states $q \neq 0$ of the Markov chain m_ℓ are *absorbing*), in particular, matrix P takes the form

$$P = \left[\begin{array}{c|ccc} P_{0,0} & P_{0,1} & \cdots & P_{0,N-1} \\ \hline 0 & & & \\ \vdots & & I_{N-1} & \\ 0 & & & \end{array}\right]. \tag{13}$$

In the present framework, probability $P_{0,j}$, $j = 1, \ldots, N-1$ may be interpreted as that of switching from the initial state 0 to the absorbing (final) state j *whenever a switch is enabled by a Poisson arrival* τ_k, whereas $P_{0,0}$ is the probability of sticking to state 0.

Corollary 1 *Let P be defined as in (13). Then*

$$T^s(\Delta) = \left[\begin{array}{c|ccc} T^s_{0,0}(\Delta) & T^s_{0,1}(\Delta) & \cdots & T^s_{0,N-1}(\Delta) \\ \hline 0 & & & \\ \vdots & & I_{N-1} & \\ 0 & & & \end{array}\right], \tag{14}$$

where

$$T^s_{0,j}(\Delta) = \begin{cases} e^{-\nu\Delta(1-P_{0,0})}, & j = 0; \\ \frac{P_{0,j}}{1-P_{0,0}}(1 - e^{-\nu\Delta(1-P_{0,0})}), & j = 1, \ldots, N-1. \end{cases}$$

Thus, it is possible to establish a correspondence between processes $q(t)$ (as defined in the previous section) and $q^s(t)$. Indeed, one can write matrix (14) in the form of (5) by choosing $\lambda_j = \nu P_{0,j}$, $j = 1, \ldots, N$. On the other hand, in general, for a given process $q(t)$ there is not a *unique* subordinated Markov chain that corresponds to it; in fact, the latter is characterized by the $N-1$ parameters $\{\lambda_1, \ldots, \lambda_{N-1}\}$, whereas a subordinated Markov chain is defined by the rate ν *and* the $N-1$ probabilities $\{P_{0,1}, \ldots, P_{0,N-1}\}$, for a total of N independent parameters. However, if we fix $P_{0,0} = 0$ (that is, state $q^s(t)$ *has to change* its value at the first Poisson arrival) then the above probabilities become dependent (since they have to sum up to 1) and the total number of independent parameters for the subordinated Markov chain decreases to $N-1$. In this case matrix (4) uniquely defines the equivalent subordinated Markov chain as follows: $\nu = \lambda_1 + \ldots + \lambda_{N-1}$ and, for $1 \le j \le N-1$, $P_{0,j} = \lambda_j/(\lambda_1 + \ldots + \lambda_{N-1})$.

4 The Conditioned System

Fixing the values of $t^\star$ and $q^\star$, i.e. the whole trajectory $q(t)$, one obtains a standard linear time-varying Gaussian system with all the parameters determined by the sample trajectory of $q(t)$. Such a system can be discretized, i.e. one may define the *sampled state*

$$x_k \triangleq x(t_k)$$

and the *discrete-time*, time-varying linear Gaussian system

$$\begin{cases} x_{k+1} = A_k(t^\star, q^\star)\, x_k + u_k \\ y_k = C_k(t^\star, q^\star)\, x_k + D_k(t^\star, q^\star)\, v_k \end{cases}, \tag{15}$$

with $u_k \sim \mathcal{N}(0, Q_k(t^\star, q^\star))$ white and independent of $\{v_k\}$ and x_0, so that the joint statistical description of $\{x_k\}$ and $\{y_k\}$ is identical to that of the original system *given* $t^\star$ *and* $q^\star$. The equivalence is guaranteed by a suitable choice of the parameters $A_k(t^\star, q^\star)$, $Q_k(t^\star, q^\star)$, $C_k(t^\star, q^\star)$, $D_k(t^\star, q^\star)$, which will be discussed in the next section.

Clearly, (15) is a state-space representation of the random variables x_k and y_k *conditioned on* $t^\star$ and $q^\star$. Moreover, for changing values of $t^\star$ and $q^\star$, it describes a family of models corresponding to the different possible realizations of $q(t)$. Note that in the stochastic setting the above parameters are *random matrices.* In the sequel we will refer to (15) as the *conditioned system.*

4.1 Computation of the Conditioned System parameters

In this section we will assume that $q^\star$ is fixed and $t^\star$ takes a value in the interval (t_h, t_{h+1}), with $t_h, t_{h+1} \in \mathcal{T}$, for a certain index $h \in \mathbb{Z}_0$. The interval is assumed to be open without loss of generality (since $t^\star$ is a continuous random variable). Let us introduce a couple of preliminary results.

Lemma 1 *Let $q \in \mathcal{Q}$. If F_q and $-F_q$ have disjoint spectra, the* Lyapunov equation

$$F_q J_q + J_q F_q^T = -G_q G_q^T \tag{16}$$

admits a unique (symmetric) solution in $J_q \in \mathbb{R}^{n\times n}$.

A proof of this lemma may be found, for example, in [2].

Lemma 2 *Let $F \in \mathbb{R}^{n\times n}$, $G \in \mathbb{R}^{n\times m}$. For arbitrary values $a, b \in \mathbb{R}$, consider*

$$Q = \int_a^b e^{F(b-\sigma)} G G^T e^{F^T(b-\sigma)} \mathrm{d}\sigma. \tag{17}$$

If F and $-F$ have disjoint spectra, then

$$Q = J - e^{F(b-a)} J e^{F^T(b-a)}$$

where J is the unique (symmetric) solution to $FJ + JF^T = -GG^T$.

Integrals such as (17) arise in the discretization of stochastic systems. Therefore, with the aid of Lemmas 1 and 2, the parameters of the conditioned system (15) may be written in the following explicit form.

Proposition 2 *Assume that F_q and $-F_q$ have disjoint spectra, for all $q \in \mathcal{Q}$, and let J_q be the corresponding solution to Lyapunov equation (16). Assume also that $t^\star \in (t_h, t_{h+1})$. Let $\widehat{A}_{k,q} \triangleq e^{F_q(t_{k+1}-t_k)}$,*

$$\widetilde{A}_k(t^\star, q) \triangleq \begin{cases} e^{F_0(t^\star - t_k)} & \text{for } q = 0 \\ e^{F_q(t_{k+1}-t^\star)} & \text{for } q \neq 0 \end{cases}$$

and $\widetilde{Q}_k(t^\star, q) \triangleq J_q - \widetilde{A}_k(t^\star, q) J_q \widetilde{A}_k^T(t^\star, q)$ for all $k \in \mathbb{N}_0$ and $q \in \mathcal{Q}$. Then:

- *For $k < h$ (i.e. $t^\star > t_{k+1}$),*

$$\begin{aligned} A_k(t^\star, q^\star) &= \widehat{A}_{k,0} & Q_k(t^\star, q^\star) &= J_0 - \widehat{A}_{k,0}\, J_0\, \widehat{A}_{k,0}^T \\ C_k(t^\star, q^\star) &= H_0 & D_k(t^\star, q^\star) &= K_0\ ; \end{aligned}$$

- *For $k > h$ (i.e. $t^\star < t_k$),*

$$\begin{aligned} A_k(t^\star, q^\star) &= \widehat{A}_{k,q^\star} & Q_k(t^\star, q^\star) &= J_{q^\star} - \widehat{A}_{k,q^\star}\, J_{q^\star}\, \widehat{A}_{k,q^\star}^T \\ C_k(t^\star, q^\star) &= H_{q^\star} & D_k(t^\star, q^\star) &= K_{q^\star}\ ; \end{aligned}$$

- *For $k = h$ (i.e. $t^\star \in (t_k, t_{k+1})$),*

$$
\begin{aligned}
A_k(t^\star,q^\star) &= \widetilde{A}_k(t^\star,q^\star)\widetilde{A}_k(t^\star,0) & Q_k(t^\star,q^\star) &= \widetilde{A}_k(t^\star,q^\star)\widetilde{Q}_k(t^\star,0)\widetilde{A}_k(t^\star,q^\star) + \widetilde{Q}_k(t^\star,q^\star) \\
C_k(t^\star,q^\star) &= H_0 & D_k(t^\star,q^\star) &= K_0 \ .
\end{aligned}
$$

Remark 1. If the assumption on the spectrum of F_q fails to hold, computing the integral (17) may not reduce to solving equation (16). Yet the computation of parameters Q_k and $\widetilde{Q}_k$ (to which Lemma 17 applies) can be carried out, although not in explicit form.

Remark 2. For $k \neq h$, $A_k(t^\star,q^\star)$ and $Q_k(t^\star,q^\star)$ are *constant* w.r.t. $t^\star$. In fact, they only depend on which interval $t^\star$ belongs to, i.e. on index h. The same holds for $C_k(t^\star,q^\star)$, $D_k(t^\star,q^\star)$, for *any* k.

Remark 3. At this stage, all parameters of the conditioned system are expressed as explicit functions of $t^\star$, $q^\star$. Note that J_q, $q \in \mathcal{Q}$, may be computed *offline* using standard numerical techniques. Parameters A_k, Q_k, $k \neq h$ can be computed offline as well.

5 The filtering problem as averaging of Kalman filters

We shall now look at the estimation problems we stated in section 2. For any index ℓ, consider the computation of $\hat{x}^a_{\ell|k}$. Applying the Total Probability Law to the probability density of x_ℓ given y^k yields

$$
f(x_\ell|y^k) = \sum_{q^\star=1}^{N-1} \int_0^{+\infty} f(x_\ell|t^\star,q^\star,y^k) f(t^\star,q^\star|y^k)\,\mathrm{d}t^\star. \tag{18}
$$

We recognize $f(x_\ell|t^\star,q^\star,y^k)$ to be the *a posteriori* density of the state x_ℓ given y^k of the conditioned system (15).

In the light of the discussion of section 4, *for any fixed value of $t^\star$ and every $q^\star$* it must be the case that

$$
f(x_\ell|t^\star,q^\star,y^k) \sim \mathcal{N}\big(\hat{x}_{\ell|k}(t^\star,q^\star), P_{\ell|k}(t^\star,q^\star)\big), \tag{19}
$$

where mean and variance are the *linear* least square error estimate of x_ℓ given y^k and the associated error covariance matrix for the conditioned system (see, for instance, [16]). In particular,

$$
\hat{x}_{k|k}(t^\star,q^\star) \tag{20}
$$
$$
\hat{x}_{k+1|k}(t^\star,q^\star) \tag{21}
$$

are the *conditional Kalman filter* and the *conditional Kalman predictor* for the corresponding conditioned system, and

$$
P_{k|k}(t^\star,q^\star) \tag{22}
$$
$$
P_{k+1|k}(t^\star,q^\star) \tag{23}
$$

are the respective error covariance matrices. Of course, these may be computed by a *conditional Kalman recursion*:

Measurement update:

$$
\begin{aligned}
\hat{x}_{k|k} &= \hat{x}_{k|k-1} + L_k(t^\star,q^\star)[y_k - C_k(t^\star,q^\star)\hat{x}_{k|k-1}] \\
P_{k|k} &= P_{k|k-1} - L_k(t^\star,q^\star)C_k(t^\star,q^\star)P_{k|k-1}
\end{aligned} \tag{24}
$$

Time update:

$$\begin{aligned}\hat{x}_{k+1|k} &= A_k(t^\star, q^\star)\hat{x}_{k|k} \\ P_{k+1|k} &= A_k(t^\star, q^\star)P_{k|k}A_k^T(t^\star, q^\star) + Q_k(t^\star, q^\star)\end{aligned} \tag{25}$$

where the (conditional) Kalman gain is given by

$$L_k(t^\star, q^\star) \; = \; P_{k|k-1}C_k^T(t^\star, q^\star)\left[D_k(t^\star, q^\star)D_k^T(t^\star, q^\star) + C_k(t^\star, q^\star)P_{k|k-1}C_k^T(t^\star, q^\star)\right]^{-1}.$$

In general, as all the system parameters that appear in the recursion depend on $t^\star$ and $q^\star$, so do (20)÷(23). By equation (18), estimate (2) is therefore equal to the conditional average

$$\hat{x}^a_{\ell|k} = \sum_{q^\star=1}^{N-1}\int_0^{+\infty} \hat{x}_{\ell|k}(t^\star, q^\star)f(t^\star, q^\star|y^k)\mathrm{d}t^\star. \tag{26}$$

Hence, for $\ell = k$ (or $k+1$), we have a natural interpretation for $\hat{x}^a_{\ell|k}$ as *averaging of Kalman filters (or predictors)*. Note that (18) is an average of Gaussian densities (in x_ℓ, parameterized by $t^\star$ and $q^\star$) weighed by $f(t^\star, q^\star|y^k)$. Since x_ℓ is not Gaussian, there is no chance to compute $\hat{x}^a_{\ell|k}$ explicitly.

It is now evident that the *a posteriori* density $f(t^\star, q^\star|y^k)$ plays a fundamental role in the estimation (2). In fact, it is intimately related to the computation of (21), as the next section will clarify. For *filtering* ($\ell = k$) and *prediction* ($\ell = k+1$), we shall need *explicit* expressions of $f(t^\star, q^\star|y^k)$ (section 6) and of $f(x_\ell|t^\star, q^\star, y^k)$ (section 7).

6 Switching Time Estimation

In this section we shall present a technique for the computation of the *a posteriori* joint conditional probability density $f(t^\star, q^\star|y^k)$, introduced by equation (10). As we already discussed in section 2, one may compute the *a posteriori* density $f(t^\star|y^k)$ simply by:

$$f(t^\star|y^k) = \sum_{q^\star=1}^{N-1} f(t^\star, q^\star|y^k);$$

similarly, the *a posteriori* distribution (3) can be obtained by integration. As discussed in the previous section, density $f(t^\star, q^\star|y^k)$ is involved in the computation of the estimate of the continuous state $x(t_\ell)$ as well.

We will obtain $f(t^\star, q^\star|y^k)$ by first computing $f(y^k|t^\star, q^\star)$ and then applying Bayes'rule. Two different methods for the computation of $f(y^k|t^\star, q^\star)$ are presented, both making use of the results of section 4. We shall sometimes drop the dependency on $t^\star$ and $q^\star$ from our notation.

6.1 Direct computation of $f(y^k|t^\star, q^\star)$

Define vectors and matrices (remember that $x_0 \sim \mathcal{N}(\mu_0, \Sigma_0)$ by definition):

$$\widetilde{\mu}_k \triangleq \begin{bmatrix}0\\0\\\vdots\\0\\\mu_0\end{bmatrix} \in \mathbb{R}^{(k+1)n},\; \widetilde{\Sigma}_k(t^\star, q^\star) \triangleq \begin{bmatrix}Q_{k-1}(t^\star, q^\star) & 0 & \cdots & 0 & 0\\ 0 & Q_{k-2}(t^\star, q^\star) & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & Q_0(t^\star, q^\star) & 0\\ 0 & 0 & \cdots & 0 & \Sigma_0\end{bmatrix} \in \mathbb{R}^{(k+1)n\times(k+1)n}, \tag{27}$$

$$
\Theta_k(t^\star,q^\star) \triangleq \begin{bmatrix} I & A_{k-1} & A_{k-1}A_{k-2} & \cdots & A_{k-1}A_{k-2}\dots A_2A_1 & A_{k-1}A_{k-2}\dots A_1A_0 \\ 0 & I & A_{k-2} & \cdots & A_{k-2}A_{k-3}\dots A_2A_1 & A_{k-2}A_{k-3}\dots A_1A_0 \\ 0 & 0 & I & \cdots & A_{k-3}A_{k-4}\dots A_2A_1 & A_{k-3}A_{k-4}\dots A_1A_0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_1 & A_1A_0 \\ 0 & 0 & 0 & \cdots & I & A_0 \\ 0 & 0 & 0 & \cdots & 0 & I \end{bmatrix} \in \mathbb{R}^{(k+1)n\times(k+1)n} \; ; \tag{28}
$$

note that all the A_i's are functions of $t^\star$ and $q^\star$. Also define:

$$
\begin{aligned}
\Xi_k(t^\star,q^\star) &\triangleq \left[A_{k-1}\middle|A_{k-1}A_{k-2}\middle|\cdots\middle|A_{k-1}A_{k-2}\dots A_0\right] \in \mathbb{R}^{n\times kn} \; , \\
\Upsilon_k(t^\star,q^\star) &\triangleq \mathrm{diag}\{C_k(t^\star,q^\star),\dots,C_0(t^\star,q^\star)\} \in \mathbb{R}^{p\times(k+1)n} \; , \\
\Lambda_k(t^\star,q^\star) &\triangleq \mathrm{diag}\{D_k(t^\star,q^\star),\dots,D_0(t^\star,q^\star)\} \in \mathbb{R}^{p\times(k+1)n} \; .
\end{aligned}
$$

Proposition 3 *Let*

$$
y^k \triangleq [y_k^T, y_{k-1}^T, \dots, y_0^T]^T \in \mathbb{R}^{(k+1)p\times 1}
$$

be the vector *of all measurements up to time* t_k*. Then* y^k*, conditioned on* $t^\star$ *and* $q^\star$*, has the following multivariate Gaussian density:*

$$
f(y^k|\, t^\star,q^\star) \sim \mathcal{N}\left(\mu_{y^k}(t^\star,q^\star), \Sigma_{y^k}(t^\star,q^\star)\right) ,
$$

where

$$
\mu_{y^k}(t^\star,q^\star) = \Upsilon_k(t^\star,q^\star)\Theta_k(t^\star,q^\star)\,\widetilde{\mu}_k = \begin{bmatrix} C_kA_{k-1}\cdots A_0 \\ C_{k-1}A_{k-2}\cdots A_0 \\ \vdots \\ C_1A_0 \\ C_0 \end{bmatrix} \mu_0 \, , \tag{29}
$$

$$
\Sigma_{y^k}(t^\star,q^\star) \triangleq \Upsilon_k(t^\star,q^\star)\Theta_k(t^\star,q^\star)\,\widetilde{\Sigma}_k(t^\star,q^\star)\Theta_k^T(t^\star,q^\star)\Upsilon_k^T(t^\star,q^\star) + \Lambda_k(t^\star,q^\star)\Lambda_k^T(t^\star,q^\star)\, ; \tag{30}
$$

The above covariance matrix may be computed by the following iteration on k:

- $\Sigma_{y^k}(t^\star,q^\star)$ *is obtained by adding n rows and columns to* $\Sigma_{y^{k-1}}(t^\star,q^\star)$ *as follows:*

$$
\Sigma_{y^k} = \begin{bmatrix} \Phi_k & \Psi_k \\ \Psi_k^T & \Sigma_{y^{k-1}} \end{bmatrix}
$$

where matrices Φ_k *and* Ψ_k *are given by:*

$$
\Phi_k = C_k(\Xi_k\widetilde{\Sigma}_{k-1}\Xi_k^T + Q_{k-1})C_k^T + D_kD_k^T, \tag{31}
$$

$$
\Psi_k = C_k\Xi_k\widetilde{\Sigma}_{k-1}\Theta_{k-1}^T\Upsilon_{k-1}^T, \tag{32}
$$

with initialization: $\Sigma_{y^0} = C_0\Sigma_0C_0^T + D_0D_0^T$. □

6.2 Iterative formulation for the computation of $f(y^k|t^\star,q^\star)$

The above computation may look somewhat cumbersome: however it only requires the computation of the conditioned system's parameters, which was discussed in section 4.

Note that we may write density $f(y^k|t^\star, q^\star)$ simply as follows:

$$f(y^k|\, t^\star, q^\star) = f(y_k|\, t^\star, q^\star, y^{k-1})\, f(y^{k-1}|\, t^\star, q^\star); \tag{33}$$

this formula provides an iterative method for computing $f(y^k|t^\star, q^\star)$. Since y^k is a *given* vector of data, for fixed values of $t^\star$ and $q^\star$ the factor $f(y^{k-1}|t^\star, q^\star)$ is just a number that we carry on from the previous computation. Such number has to be multiplied by $f(y_k|\, t^\star, q^\star, y^{k-1})$, whose value is obtainable from $f(x_k|\, t^\star, q^\star, y^{k-1})$ in a straightforward manner. The latter quantity plays a fundamental role in the estimation of the continuous state x, as we saw in section 5; in section 7 we will show a technique for computing it. On the other hand, the direct method allows to evaluate $f(y^k|t^\star, q^\star)$ *without* referring to the underlying continuous state. Thus, if the estimation of x is not a concern, the use of Proposition 3 can be sufficient. Otherwise, once the sequence $f(x_j|\, t^\star, q^\star, y^{j-1})$, $j = 0, \ldots, k$, is known, the application of the iterative method is more appropriate. We will come back to this in section 7.1.

6.3 Application of Bayes' rule

The a posteriori density of $t^\star$ and $q^\star$ is given by:

$$f(t^\star, q^\star|y^k) = \frac{f(y^k|t^\star, q^\star) f(t^\star, q^\star)}{\sum_{q^\star=1}^{N-1} \int_0^\infty f(y^k|t^\star, q^\star) f(t^\star, q^\star)\, \mathrm{d}t^\star}, \tag{34}$$

with $f(t^\star, q^\star)$ as in equation (8). Since for $t^\star > t_k$ the density $f(y^k|t^\star, q^\star)$ is independent of the specific value assumed by $t^\star$ we have that the denominator of (34) is given by:

$$\sum_{q^\star=1}^{N-1} \left\{ \int_{t_0}^{t_k} f(y^k|t^\star, q^\star) f(t^\star, q^\star)\ \mathrm{d}t^\star + f(y^k|t^\star > t_k, q^\star)\, \mathbb{P}[t^\star > t_k]\, \mathbb{P}[q^\star] \right\}. \tag{35}$$

In principle, the integration above would require computing $f(y^k|t^\star, q^\star)$ for infinite values of $t^\star$. In practice, both the direct method and the iterative method allow straightforward evaluation of $f(y^k|t^\star, q^\star)$ at any $t^\star$ in the *finite interval* $(0, t_k)$. Thus, adaptive numerical integration methods – such as Simpson's adaptive quadrature, see [11] and references therein – may be applied, yielding approximations with arbitrarily small error at a reasonable computational cost.

7 Conditional Kalman Filtering

Following section 5, for any fixed values of $t^\star$ and $q^\star$ one may compute the a posteriori densities

$$f(x_k|t^\star, q^\star, y^k) \tag{36}$$

$$f(x_{k+1}|t^\star, q^\star, y^k) \tag{37}$$

by simply running the conditional Kalman recursion (24)÷(25) for the specific $t^\star$ and $q^\star$. However, such a procedure is not suited for the computation (e.g. by adaptive numerical methods) of integrals such as (18) and (35) (see equation (33) and subsequent comments). Indeed, for integration purposes, densities (36) and (37) need to be evaluated for a large set of values of $t^\star$, possibly changing with the index k. Therefore, an explicit expression of the above densities in terms of $t^\star$ (and $q^\star$) would be desirable.

Fix $q^\star$ and an index $h \in \mathbb{N}_0$, and let $t^\star$ take *any value* in the interval (t_h, t_{h+1}). Assume to iteratively update the conditional means (20), (21) and variance matrices (22),(23) according to steps (24) and (25). It may be observed that:

(i) (20), (22) and (21), (23) are *independent of* $t^\star$, $q^\star$ for $k \le h$ and $k < h$, respectively;

(ii) for $k \ge h+1$, (20)÷(23) depend on $t^\star$ *only through* $\hat{x}_{h+1|h}(t^\star, q^\star)$ and $P_{h+1|h}(t^\star, q^\star)$.

Indeed, by the results of section 4.1, the parameters $\{A_k(t^\star, q^\star), Q_k(t^\star, q^\star), C_k(t^\star, q^\star), D_k(t^\star, q^\star)\}$ of the conditioned system are constant w.r.t. $t^\star$ *and* $q^\star$ as long as $k < h$ (since $q(t) \equiv 0$ for all $t < t_h$). Similarly, they are constant w.r.t. $t^\star$ for all $k \ge h+1$ (since $q(t) \equiv q^\star$ for all $t > t_{h+1}$). Therefore, for *any* $t^\star \in (t_h, t_{h+1})$, the measurement update and time update equations (24) and (25) *evolve independently of the specific value* $t^\star$ for $k < h$ and $k \ge h+1$. The dependency on $t^\star$ is concentrated in the time update at step $k = h$, whereas $q^\star$ plays a role from the same time update step thereafter by fixing the new system parameters. Based on these remarks, in the rest of the section we will illustrate an iterative method for the computation of (20)÷(23) yielding explicit functions of $t^\star$, and its use for the construction of densities (36), (37). All results hold for any $q^\star \in \{1, \dots, N-1\}$.

Fix $h \in \mathbb{N}_0$ and assume $t^\star \in (t_h, t_{h+1})$. Let $\hat{x}^0_{\ell|k}$ and $P^0_{\ell|k}$, with $\ell = k, k+1$, denote the Kalman estimates associated to $q(t) \equiv 0$, i.e. *as if* there were *no switching*. The first result is just a formalization of remark (i) above.

Proposition 4 *For $k \le h$, it holds that*

$$\hat{x}_{k|k}(t^\star, q^\star) = \hat{x}^0_{k|k}, \quad P_{k|k}(t^\star, q^\star) = P^0_{k|k};$$

similarly, for $k < h$,

$$\hat{x}_{k+1|k}(t^\star, q^\star) = \hat{x}^0_{k+1|k}, \quad P_{k+1|k}(t^\star, q^\star) = P^0_{k+1|k}.$$

□

Next result states how $t^\star$ and $q^\star$ affect the time update (25) for $k = h$.

Proposition 5 *It holds that*

$$\begin{aligned}
\hat{x}_{h+1|h}(t^\star, q^\star) &= A_h(t^\star, q^\star)\hat{x}^0_{h|h}, \\
P_{h+1|h}(t^\star, q^\star) &= J_{q^\star} + \widetilde{A}_h(t^\star, q^\star)(J_0 - J_{q^\star})\widetilde{A}^T_h(t^\star, q^\star) + A_h(t^\star, q^\star)(P^0_{h|h} - J_0)A^T_h(t^\star, q^\star).
\end{aligned}$$

□

Recall that $A_h(t^\star, q^\star)$ and $\widetilde{A}_h(t^\star, q^\star)$ are known exponential matrices. Hence, new *initial conditions* for the recursion steps $k \ge h+1$ are given in terms of explicit functions of $t^\star$. Finally, let us show how (20)÷(23) depend on $t^\star$ for $k \ge h+1$.

Proposition 6 *Consider $k > h$. Denote with $\hat{x}^\dagger_{\ell|k}$ and $P^\dagger_{\ell|k}$, where $\ell = k, k+1$, the Kalman estimates associated to $\hat{x}_{h+1|h} = 0$, $P_{h+1|h} = 0$, and with $\Phi^\dagger_k$ the state transition matrix of the Kalman predictor $\hat{x}^\dagger_{k+1|k}$ (i.e. $\hat{x}^\dagger_{k+1|k} = \Phi^\dagger_k \hat{x}^\dagger_{k|k-1} +$ linear function of y_k). Also define $\Delta_k = C^T_k(D_k D^T_k)^{-1}D_k$. It holds that:*

$$\begin{aligned}
\hat{x}_{k|k}(t^\star, q^\star) &= \hat{x}^\dagger_{k|k} + A^{-1}_k U_k\{I + S_k P_{h+1|h}(t^\star, q^\star)\}^{-T}\{\hat{x}_{h+1|h}(t^\star, q^\star) + P_{h+1|h}(t^\star, q^\star)M_k\}, \\
P_{k|k}(t^\star, q^\star) &= P^\dagger_{k|k} + A^{-1}_k U_k\{I + S_k P_{h+1|h}(t^\star, q^\star)\}^{-T} P_{h+1|h}(t^\star, q^\star) U^T_k A^{-T}_k, \\
\hat{x}_{k+1|k}(t^\star, q^\star) &= \hat{x}^\dagger_{k+1|k} + U_k\{I + S_k P_{h+1|h}(t^\star, q^\star)\}^{-T}\{\hat{x}_{h+1|h}(t^\star, q^\star) + P_{h+1|h}(t^\star, q^\star)M_k\}, \\
P_{k+1|k}(t^\star, q^\star) &= P^\dagger_{k+1|k} + U_k\{I + S_k P_{h+1|h}(t^\star, q^\star)\}^{-T} P_{h+1|h}(t^\star, q^\star) U^T_k,
\end{aligned}$$

where the quantities U_k, M_k, S_k satisfy the recursions

$$\begin{aligned} U_k &= \Phi_k^\dagger U_{k-1}, \\ M_k &= U_{k-1}^T (C_k^T (D_k D_k^T)^{-1} y_k - (I + \Delta_k P_{k|k-1}^\dagger)^{-1} \Delta_k \widetilde{M}_k) + M_{k-1}, \\ S_k &= U_{k-1}^T (I + \Delta_k P_{k|k-1}^\dagger)^{-1} \Delta_k U_{k-1} + S_{k-1}, \end{aligned}$$

with initial conditions $U_h = I$, $M_h = 0$, $S_h = 0$. In turn, $\widetilde{M}_k$ satisfies

$$\widetilde{M}_k = \Phi_{k-1}^\dagger \widetilde{M}_{k-1} + P_{k|k-1}^\dagger C_k^T (D_k D_k^T)^{-1} y_k,$$

with initial condition $\widetilde{M}_h = 0$.

The proof of this result is not reported here. The interested reader is referred to [5].

Remark 1. Quantities U_k, S_k, $P^\dagger$, $\hat{x}^\dagger$ and M_k do not depend on the specific value of $t^\star$. However, they do depend on h (through their initialization) and on $q^\star$: so we may write $U_k(h, q^\star)$, $S_k(h, q^\star)$, $P^\dagger(h, q^\star)$, $\hat{x}^\dagger(h, q^\star)$ and $M_k(h, q^\star)$. The auxiliary matrix Δ_k depends on $q^\star$ only.

Remark 2. Following Remark 1, $\hat{x}_{\ell|k}$ and $P_{\ell|k}$, $\ell = k$ or $k+1$, depend on the specific value of $t^\star$ only through $\hat{x}_{h+1|h}(t^\star, q^\star)$ and $P_{h+1|h}(t^\star, q^\star)$.

Remark 3. In practice, $U_k(h, q^\star)$, $S_k(h, q^\star)$ and $P_{\ell|k}^\dagger(h, q^\star)$, with $\ell = k$ or $k+1$, can be computed *offline* for every possible $q^\star$ and every h of interest. In particular, if there exists $T > 0$ such that $(t_{k+1} - t_k) = T$ for all k (i.e. samples are equally spaced), then $U_k(h, q^\star) = U_{k-h}(0, q^\star)$, $S_k(h, q^\star) = S_{k-h}(0, q^\star)$, and so on, hence it suffices to carry out the computation for $h = 0$. On the contrary, $\hat{x}^\dagger(h, q^\star)$ and $M_k(h, q^\star)$ depend on the measurements, hence they need to be computed online for every $q^\star$ and h.

At this point, densities (36) and (37) may be computed for *arbitrary* values of $t^\star$, with no restriction to a specific interval (t_h, t_{h+1}). Indeed, for any index k and every $q^\star$, one may consider their restriction (w.r.t. $t^\star$) to each of the $k+2$ intervals

$$(t_0, t_1), \ldots, (t_h, t_{h+1}), \ldots, (t_k, t_{k+1}), (t_{k+1}, +\infty).$$

Then, the results of Propositions 4÷6 can be applied to evaluate the Gaussians (19), $\ell = k$, $k+1$, piecewise. The procedure only requires the offline computation of $U_k(h, q^\star)$, $S_k(h, q^\star)$, $P_{k|k}^\dagger(h, q^\star)$, $P_{k+1|k}^\dagger(h, q^\star)$ and the online computation of $\hat{x}_{k|k}^\dagger(h, q^\star)$, $\hat{x}_{k+1|k}^\dagger(h, q^\star)$, $M_k(h, q^\star)$ for each $h \leq k$ (restrictions to (t_h, t_{h+1}), Propositions 5 and 6), plus the iteration of a standard Kalman recursion up to step k (restriction to $(t_{k+1}, +\infty)$, Proposition 4).

Hence, we have the following recursive algorithm:

Offline: for $h, k \geq 0$, and $q^\star = 1, \ldots, N-1$, compute $U_k(h, q^\star)$, $S_k(h, q^\star)$, $P_{k+1|k}^\dagger(h, q^\star)$, $P_{k|k}^\dagger(h, q^\star)$; for $k \geq 0$, compute $P_{k|k}^0$, $P_{k|k-1}^0$;
Initialization: set $\hat{x}_{0|-1}^0 = \mu_0$;
Iteration ($k \geq 0$): as measurement y_k is collected,

1. for $h = 0, \ldots, k-1$ and $q^\star = 1, \ldots, N-1$, compute $\hat{x}_{k|k}^\dagger(h, q^\star)$ from $\hat{x}_{k|k-1}^\dagger(h, q^\star)$, $P_{k|k-1}^\dagger(h, q^\star)$ and $M_k(h, q^\star)$ from $M_{k-1}(h, q^\star)$; compute $\hat{x}_{k|k}^0$ from $\hat{x}_{k|k-1}^0$, $P_{k|k-1}^0$;

2. for $h = 0, \ldots, k-1$ and $q^\star = 1, \ldots, N-1$, compute $\hat{x}_{k+1|k}^\dagger(h, q^\star)$ from $\hat{x}_{k|k}^\dagger(h, q^\star)$; compute $\hat{x}_{k+1|k}^0$ from $\hat{x}_{k|k}^0$, $P_{k|k}^0$;

3. for $q^\star = 1, \ldots, N-1$, set $\hat{x}^\dagger_{k+1|k}(k, q^\star) = 0$ and $M_k(k, q^\star) = 0$;

Of course, the initialization step gives the parameters that are needed to represent $f(x_0|\cdot,\cdot,y^{-1})$, whereas points 1 and 2 of the iteration step yield the parameters to represent $f(x_k|\cdot,\cdot,y^k)$ and $f(x_{k+1}|\cdot,\cdot,y^k)$, respectively. Thus, *the scheme calculates densities* (36) *and* (37) *up to index k in the form of explicit functions of $t^\star$ by a computation of complexity $\mathcal{O}(k^2)$*. Note that no further effort is required for their evaluation in correspondence of *arbitrary* values of $t^\star$.

7.1 Application to switching time estimation

In addition to solving the continuous state estimation problem, the algorithm just presented for conditional Kalman filtering and prediction may also be applied to switching time and final state estimation. Consider the iterative method for the computation of the conditional density of y^k introduced in section 6, namely

$$\begin{aligned} f(y^k|t^\star,q^\star) &= f(y_k|t^\star,q^\star,y^{k-1})f(y^{k-1}|t^\star,q^\star) \\ &= f(y_k|t^\star,q^\star,y^{k-1})\cdot f(y_{k-1}|t^\star,q^\star,y^{k-2})\cdot\ldots\cdot f(y_0|t^\star,q^\star,y^{-1}). \end{aligned}$$

By exploiting the measurement equation of (15), it follows that, for a generic index ℓ,

$$f(y_\ell|t^\star,q^\star,y^{\ell-1}) = \mathcal{N}(C_\ell\hat{x}_{\ell|\ell-1}(t^\star,q^\star), C_\ell P_{\ell|\ell-1}(t^\star,q^\star)C_\ell^T + D_\ell D_\ell^T), \tag{38}$$

where the dependence of C_ℓ and D_ℓ on $t^\star$ and $q^\star$ has been dropped for simplicity. For given data y^k, the latter equation may be evaluated for $\ell = 0,\ldots,k$ at any value of $t^\star$ and $q^\star$ with no computational effort, provided the expressions of $\hat{x}_{\ell|\ell-1}(t^\star,q^\star)$ and $P_{\ell|\ell-1}(t^\star,q^\star)$ are known explicitly. Indeed, this is the case if the method described in the previous section is used. In particular, expression (38) is constant with respect to $t^\star$ for $t^\star > t_\ell$, and may be constructed piecewise over the intervals $(t_0,t_1),\ldots,(t_{\ell-1},t_\ell)$ for $t^\star < t_\ell$. Therefore, recalling equation (34), we can state the following result.

Proposition 7 *The* a posteriori *density $f(t^\star,q^\star|y^k)$ is given by:*

$$f(t^\star,q^\star|y^k) = \frac{\prod_{\ell=0}^k f(y_\ell|t^\star,q^\star,y^{\ell-1})f(t^\star,q^\star)}{f(y^k)},$$

where the normalization factor may be computed as

$$\begin{aligned} f(y^k) = \sum_{q^\star=1}^{N-1} \Bigg\{ & \prod_{\ell=0}^{k} f(y_\ell|t^\star > t_k, q^\star, y^{\ell-1})\,\mathbb{P}[t^\star > t_k]\,\mathbb{P}[q^\star] + \\ & + \sum_{h=0}^{k-1}\left[\prod_{\ell=0}^{h} f(y_\ell|t^\star > t_h, q^\star, y^{\ell-1})\cdot\int_{t_h}^{t_{h+1}}\prod_{\ell=h+1}^{k} f(y_\ell|t^\star,q^\star,y^{\ell-1})f(t^\star,q^\star)\,dt^\star\right]\Bigg\}; \end{aligned}$$

factors $f(y_\ell|t^\star,q^\star,y^{\ell-1})$ are given by equation (38).

Again, the solution requires to compute k integrals on compact sets, where the integrand functions may be efficiently evaluated at any value of $t^\star$ within the relevant domain. Hence, adaptive quadrature methods may be applied. By its nature, this formulation of $f(t^\star,q^\star|y^k)$ is also extremely well suited for a piecewise computation of integral (26) as well as of the probability function (3) and of the conditional expectation $\mathbb{E}[t^\star|y^k]$, the latter being of course the best estimate of $t^\star$ given y^k.

8 State estimation between measurements

The techniques that were illustrated in the previous paragraphs may be easily extended to the problem of state estimation *between* two successive measurement instants $t_k, t_{k+1} \in \mathcal{T}$.

First note that, analogously to (18),

$$f\big(x(t)|y^k\big) = \sum_{q^\star=1}^{N-1} \int_0^{+\infty} f\big(x(t)|t^\star, q^\star, y^k\big) f\big(t^\star, q^\star|y^k\big)\, \mathrm{d}t^\star,$$

for all $t \in (t_k, t_{k+1})$. We have that $f\big(x(t)|t^\star, q^\star, y^k\big)$ is a Gaussian density, whose mean and covariance matrix may be inferred from those of density $f(x_k|t^\star, q^\star, y^k)$, which was computed in the previous sections. We shall omit the proof of the proposition that follows. With an abuse of notation we will define $\hat{x}_{t|k}(t^\star, q^\star) \triangleq \mathbb{E}\big[x(t)\big|t^\star, q^\star, y^k\big]$ and $P_{t|k}(t^\star, q^\star) \triangleq \mathrm{Var}\big[x(t)\big|t^\star, q^\star, y^k\big]$, for $t_k < t < t_{k+1}$.

Proposition 8 *Assume that F_q and $-F_q$ have disjoint spectra for all $q \in \mathcal{Q}$, and let J_q be the corresponding solution to Lyapunov equation (16). Density $f\big(x(t)\big|t^\star, q^\star, y^k\big)$, $t_k < t < t_{k+1}$, is Gaussian, with mean and covariance matrix having the following expressions:*

- *for $t_k < t < t^\star$:*

$$\begin{aligned} \hat{x}_{t|k}(t^\star, q^\star) &= e^{F_0(t-t_k)} \hat{x}_{k|k}(t^\star, q^\star), \\ P_{t|k}(t^\star, q^\star) &= e^{F_0(t-t_k)} P_{k|k}(t^\star, q^\star) e^{F_0^T(t-t_k)} + J_0 - e^{F_0(t-t_k)} J_0 e^{F_0^T(t-t_k)}; \end{aligned}$$

- *for $t^\star < t_k < t$:*

$$\begin{aligned} \hat{x}_{t|k}(t^\star, q^\star) &= e^{F_{q^\star}(t-t_k)} \hat{x}_{k|k}(t^\star, q^\star), \\ P_{t|k}(t^\star, q^\star) &= e^{F_{q^\star}(t-t_k)} P_{k|k}(t^\star, q^\star) e^{F_{q^\star}^T(t-t_k)} + J_{q^\star} - e^{F_{q^\star}(t-t_k)} J_{q^\star} e^{F_{q^\star}^T(t-t_k)}; \end{aligned}$$

- *for $t_k < t^\star < t$:*

$$\begin{aligned} \hat{x}_{t|k}(t^\star, q^\star) &= e^{F_{q^\star}(t-t^\star)} e^{F_0(t^\star-t_k)} \hat{x}_{k|k}(t^\star, q^\star), \\ P_{t|k}(t^\star, q^\star) &= e^{F_{q^\star}(t-t^\star)} e^{F_0(t^\star-t_k)} (P_{k|k}(t^\star, q^\star) - J_0) e^{F_0^T(t^\star-t_k)} e^{F_{q^\star}^T(t-t^\star)} + \\ &\quad + e^{F_{q^\star}(t-t^\star)} (J_0 - J_{q^\star}) e^{F_{q^\star}^T(t-t^\star)} + J_{q^\star}. \end{aligned}$$

Note that functions $\hat{x}_{k|k}(t^\star, q^\star)$ and $P_{k|k}(t^\star, q^\star)$, defined in section 5, must be computed with the techniques that were exposed in the previous sections; for $t^\star > t_k$, in particular, they are simply given by $\hat{x}^0_{k|k}$ and $P^0_{k|k}$.

Incidentally, note that $\mathbb{P}\big[q(t) = j\big|y^k\big]$, $t_k < t < t_{k+1}$, is given by

$$\mathbb{P}\big[q(t) = j\big|y^k\big] = \begin{cases} e^{-\Lambda(t-t_k)} p_{k|k}(0) & \text{for } j = 0 \\ p_{k|k}(j) + \frac{\lambda_j}{\Lambda}\big(1 - e^{-\Lambda(t-t_k)}\big) p_{k|k}(0) & \text{for } j \neq 0 \end{cases} \tag{39}$$

where function $p_{k|k}(\cdot)$ was defined in section 2 and is computed using the methods that were described in the previous sections as well. This follows from the expression of the transition probability matrix (5), which yields

$$\mathbb{P}\big[q(t) = j\big|y^k\big] = \sum_{i=0}^{N-1} \mathbb{P}\big[q(t) = j\big|q(t_k) = i\big]\, p_{k|k}(i), \tag{40}$$

with

$$\mathbb{P}\big[q(t) = j\big|q(t_k) = i\big] = \begin{cases} \mathbb{1}_{\{j=i\}} & \text{for } i \neq 0 \\ \mathbb{1}_{\{j=0\}} e^{-\Lambda(t-t_k)} + \mathbb{1}_{\{j\neq 0\}} \frac{\lambda_j}{\Lambda}\big(1 - e^{-\Lambda(t-t_k)}\big) & \text{for } i = 0 \end{cases}$$

(where $\mathbb{1}$ is an indicator function). At this point (39) is proven simply by plugging the above expression into equation (40).

9 Numerical Example

In this section we will show numerical results concerning a specific example of stochastic hybrid system. Using the methods outlined in section 7, we will pursue a qualitative analysis of the estimation of both the switching time and the state of the system. In particular, we are interested in the probability of the current discrete state, i.e. $p_{k|k}(\cdot)$, and in the least-square estimates of $t^\star$, i.e. $\mathbb{E}[t^\star|y^k]$, and of the continuous state x_k, i.e. $\hat{x}^a_{k|k}$. In practice, this amounts to computing explicit expressions for $\hat{x}_{k|k}(t^\star, q^\star)$, $P_{k|k}(t^\star, q^\star)$, $\hat{x}_{k|k-1}(t^\star, q^\star)$ and $P_{k|k-1}(t^\star, q^\star)$, $k = 0, 1, 2, \ldots$, whence $f(y_k|t^\star, q^\star, y^{k-1})$, $f(y^k|t^\star, q^\star)$ and all the desired statistics by suitable integration.

Let $\mathcal{Q} = \{0, 1, 2\}$. Consider system (1) with $t_k \triangleq k \cdot T$, $T = 0.5$, and parameters $\mu_0 = 0$, $\Sigma_0 = 0.1 \cdot I_{2\times 2}$. We chose all 4-tuples (F_q, G_q, H_q, K_q) to be

$$\left(\begin{bmatrix} -0.4 & 0.6 \\ c_q & -0.5 \end{bmatrix}, \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \right)$$

where $c_0 = 0$, $c_1 = 1$, $c_2 = -2$. That is, only the state evolution matrix changes with q. This modifies the character of the continuous-time system from stable ($q = 0$, stable node) to unstable ($q = 1$, saddle) or oscillatory ($q = 2$, stable focus), according to the different spectra $\sigma(F_0) = \{-0.4, -0.5\}$, $\sigma(F_1) = \{-1.22, 0.32\}$, $\sigma(F_2) = \{-0.45 \pm i\,1.09\}$. In this setting y is simply a noisy version of the state x. The Markov chain underlying the evolution of $q(t)$ is set to start from $q(0) = 0$ with probability one, i.e. $p_0 = 1$; switching intensities are fixed to $\lambda_1 = 0.06$, $\lambda_2 = 0.08$. With this choice, $\mathbb{P}[q^\star = 1] \simeq 0.43$, $\mathbb{P}[q^\star = 2] \simeq 0.57$, i.e. jumps towards $q = 2$ are privileged. Moreover, the *a priori* expected switching time is $\mathbb{E}[t^\star] \simeq 7.14$.

In the simulations, we started off the system from $x(0) = 0$. We then randomly generated x and y up to time $k_{\max} \cdot T$, with $k_{\max} = 30$, for a jump of $q(t)$ occurring at time $\bar{t} = 5.25$, i.e. significantly before the expected time; we considered both $\bar{q} = 1$ and $\bar{q} = 2$ as final discrete states (in this context, we find it useful to explicitly distinguish the *sample values* of the switching time, $\bar{t}$, and final state, $\bar{q}$, from the corresponding *random variables* $t^\star$ and $q^\star$). Note that the values assumed by $t^\star$ (i.e. $\bar{t}$) and $q^\star$ (i.e. $\bar{q} = 1$ and $\bar{q} = 2$, in turn) have been chosen manually by the programmer, i.e. they have not been simulated as random variables. This simplifies the analysis and of course does not affect the validity of the proposed method. Also notice the exiguity of measurements, which is precisely the kind of situation the model is conceived to deal with. The algorithms of section 7 are then applied to the data y_k, $k = 0, \ldots, k_{\max}$. Numerical integrations are carried out by a standard iterative Simpson's adaptive quadrature algorithm [11]. Terminating conditions are chosen so to guarantee a relative error less than 10^{-6}.

Figure 2 shows, for different values of k, the *a posteriori* density of $t^\star$ given y^k computed by suitably applying Proposition 7, for the cases $\bar{q} = 1$ (left) and $\bar{q} = 2$ (right). The evolution from

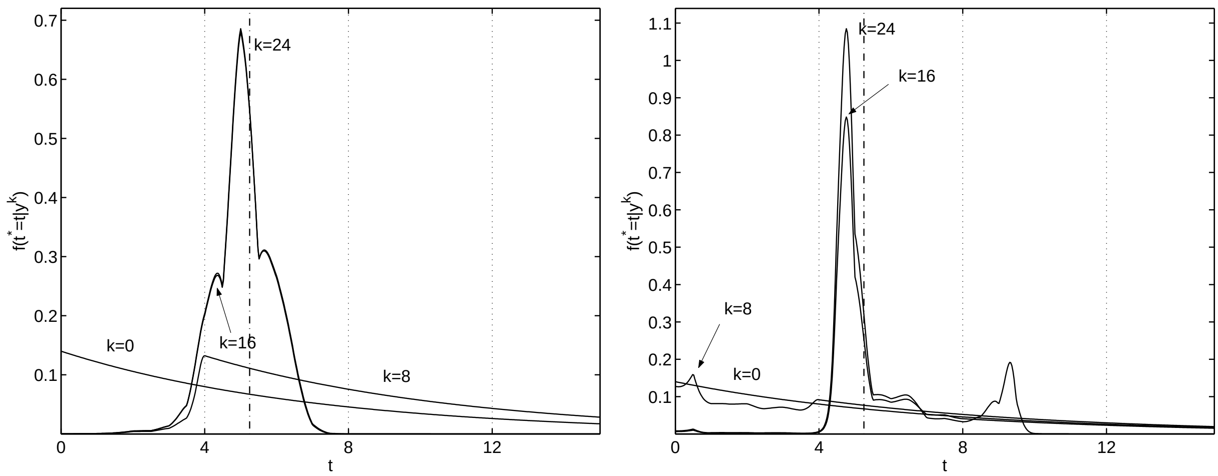

Figure 2: Density function $f(t^\star|y^k)$ plotted for $k = 0, 8, 16$ and 24. Left: $\bar{q} = 1$; Right: $\bar{q} = 2$. Dash-dotted lines mark the *actual* switching time $\bar{t} = 5.25$.

the exponential prior to a density roughly concentrated around the true switching instant may be observed. Notice the exponential tails of the curves for $t^\star > t_k$. For $\bar{q} = 1$, in particular, the almost indistinguishability of the curves associated to $k = 18$ and $k = 24$ reflects the fast convergence of $f(t^\star|y^k)$ to an almost invariant density.

The evolution of the conditional expectation of $t^\star$ and of the conditional probability distribution of $q(t_k)$ given y^k are reported in Figure 3 for $\bar{q} = 1$ and 2. One may note that, even before the switch happens, $p_{k|k}(0)$ adjusts to values that are significantly smaller than the prior probability $\mathbb{P}[q(0) = 0] = 1$. This fact reveals that certain fluctuations of the state due to the input noise u may also be explained in terms of a mode switch, and is accompanied by an increase in $p_{k|k}(1)$ and $p_{k|k}(2)$. Typically, the latter grows faster than the former, since $\lambda_2 > \lambda_1$. For $\bar{q} = 1$, where the initial fluctuations of $p_{k|k}(1)$ are rather limited, $\mathbb{E}[t^\star|y^k]$ grows in a quasi-linear fashion. This is primarily due to the memoryless nature of (unconditioned) random variable $t^\star$, and may be explained as follows. Under the condition that $t^\star > t_k$,

$$\mathbb{E}[t^\star|y^k, t^\star > t_k] = \int_0^{+\infty} t^\star f(t^\star|y^k, t^\star > t_k)\mathrm{d}t^\star = \int_{t_k}^{+\infty} t^\star \Lambda e^{-\Lambda(t^\star - t_k)}\mathrm{d}t^\star = \Lambda^{-1} + t_k,$$

where Λ^{-1} is the *a priori* expectation of $t^\star$. Thus, if condition $t^\star > t_k$ could be determined with certainty from data y^k, the estimate of $t^\star$ would just increase linearly in time (since $t_k = kT$). However, probability $\mathbb{P}[t^\star > t_k|y^k] = p_{k|k}(0)$ is generally different from one even before the actual switching time. In fact, the optimal estimate of $t^\star$ is easily found to be

$$\mathbb{E}[t^\star|y^k] = (\Lambda^{-1} + t_k)p_{k|k}(0) + \mathbb{E}[t^\star|y^k, t^\star \leq t_k](1 - p_{k|k}(0)).$$

Since $0 \leq \mathbb{E}[t^\star|y^k, t^\star \leq t_k] \leq t_k$ for all y^k, it also follows that

$$\Lambda^{-1}p_{k|k}(0) + t_k p_{k|k}(0) \leq \mathbb{E}[t^\star|y^k] \leq \Lambda^{-1}p_{k|k}(0) + t_k.$$

This provides bounds on the estimates of $t^\star$ in terms of the a posteriori probability of a switching event – the larger the value of $p_{k|k}(0)$, the tighter the bounds.

After the switch occurs, a small number of measurements suffice to detect the new discrete state

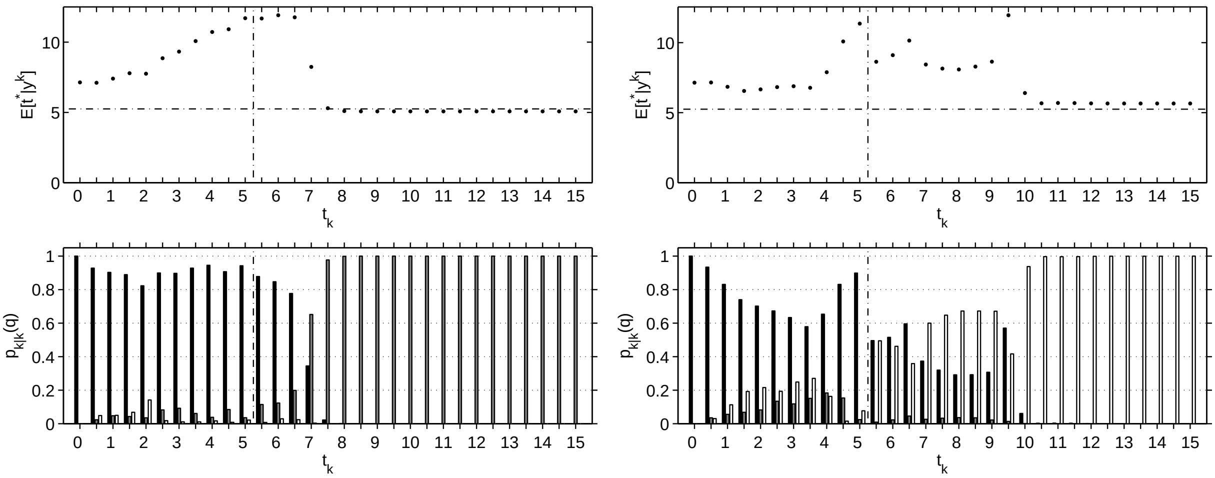

Figure 3: Evolution of the expectation $\mathbb{E}[t^\star|y^k]$ (above) and of the probability function $p_{k|k}(q)$ (below; left bar: $p_{k|k}(0)$, center bar: $p_{k|k}(1)$; right bar: $p_{k|k}(2)$). Left: $\bar{q} = 1$; Right: $\bar{q} = 2$. Dash-dotted lines mark the *actual* switching time $\bar{t} = 5.25$.

of the system. The estimates of the switch time $t^\star$ also converge to the true value $\bar{t}$ quite closely. However, comparison of the plots for $\bar{q} = 1$ and $\bar{q} = 2$ suggests that detecting a switch toward the unstable mode is "easier" than detecting a switch toward the damped oscillatory one. This is especially evident in the transient period following the switching event. In fact, contrary to the discrete state value $q = 1$, both $q = 0$ and $q = 2$ give rise to stable modes, which keep the state of the system close to zero. Thus, due to the stochastic nature of the system, the first few measurements taken after $\bar{t}$ are not sufficient to distinguish mode 0 from mode 2 when $\bar{q} = 2$. On the other hand, they are quite indicative of the new system's dynamics when $\bar{q} = 1$. In general, the more "different" the modes are, the quicker the algorithm is to detect the switch. Current research is aiming to make this statement mathematically more precise.

Plots of the estimates of the continuous state x are finally drawn in Figure 4. The optimal estimates $\hat{x}^a_{k|k}$ are compared with the true values x_k and with the best estimates $\hat{x}_{k|k}(\bar{t}, \bar{q})$ one could produce in case the switching event were known in advance. In both the cases $\bar{q} = 1$ and $\bar{q} = 2$, estimates $\hat{x}^a_{k|k}$ follow the benchmark $\hat{x}_{k|k}(\bar{t}, \bar{q})$ ones quite accurately, even in the "transient" between the actual switching instant and the time when $p_{k|k}(q)$ clearly singles out the final value of the discrete state. This is only partially surprising. Indeed, $\hat{x}^a_{k|k}$ is obtained by computing a weighted average of $\hat{x}_{k|k}(t^\star, q^\star)$. The weighting term, $f(t^\star, q^\star|y^k)$, is proportional to factors such as (38). Since in our case $y_k = x_k + Dv_k$ (with D fixed) independently of the discrete state, there is a direct connection between the quality of the state estimate $\hat{x}_{k|k}(t^\star, q^\star)$ and the weight associated to the specific values of $t^\star$ and $q^\star$. This is to say, the better a state estimate is, the more relevance is given to that estimate. Taking the analysis one step further, consider the update equation

$$\hat{x}_{k|k} = A_{k-1}\hat{x}_{k-1|k-1} + L_k[y_k - C_k A_{k-1}\hat{x}_{k-1|k-1}],$$

where the dependence on $(t^\star, q^\star)$ was omitted. In our case, the first term on the right hand side is close to zero, especially in the stable cases $q = 0$ and $q = 2$; the case $q = 1$ is made less relevant by the averaging for the reasons explained above. Therefore, most of the "mass" in the above formula is carried by the data-dependent term. Thus, in the transient period, estimate $\hat{x}_{k|k}(t^\star, q^\star)$, does not depend much on whether $\bar{q} = 1$ or $\bar{q} = 2$, and the averaged estimate $\hat{x}^a_{k|k}$ follows rather closely

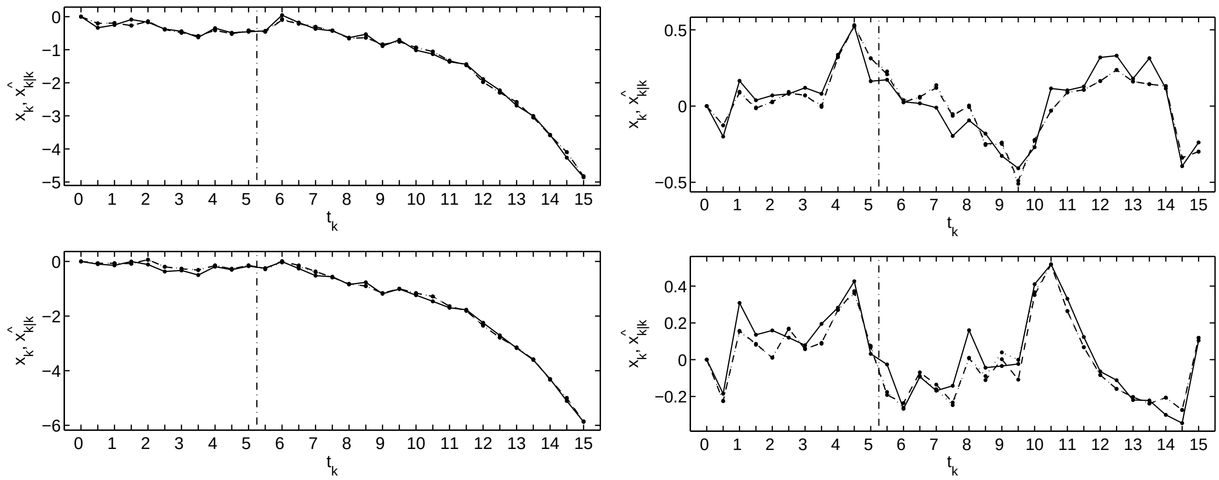

Figure 4: Evolution of the state x_k (solid line) and its estimates $\hat{x}_{k|k}(\bar{t})$ (dotted line) and $\hat{x}^a_{k|k}$; first and second component of state and relevant estimates are considered above and below, in the order. Left: $\bar{q} = 1$; Right: $\bar{q} = 2$. Dash-dotted lines mark the *actual* switching time $\bar{t} = 5.25$.

$\hat{x}_{k|k}(\bar{t}, \bar{q})$ (i.e. the Kalman Filter computed with respect to the actual trajectory of $q(t)$). In any case, this issue certainly calls for further investigation of the observability properties of stochastic hybrid systems.

10 Conclusions

In this paper we studied the problem of state estimation in a class of sampled-measurement stochastic hybrid systems. A switching state-space model was introduced in which the continuous state x satisfies a linear stochastic differential equation, noisy measurements are taken at known sample times and the parameters of the whole system change in time according to a continuous-time Markov chain q of known statistics. Contrary to the more common discrete-time jump Markov linear systems, such model accounts for switches *between* measurements, which makes it especially suited for hybrid systems whose discrete state switches at a rate comparable to that of measurements.

We focused on a fault detection setting where exactly one switch of the system from an initial mode to one out of several possible modes is allowed – i.e. the trajectory of q is characterized by a switching time $t^\star$ and a final state $q^\star$. We solved the problem of the recursive Bayesian estimation of the joint state (x, q) at sampling times t_k from the collection of measurements y^k by optimal averaging of the conditional state estimates associated to every possible switching event. In particular, for specific values of $t^\star$ and $q^\star$, the estimate of x is carried out by conditional Kalman filtering, whereas two different procedures were given for the computation of the weighing factor $f(t^\star, q^\star|y^k)$, whence of the *a posteriori* probability distribution of q. A direct extension of the method to the estimation (prediction) of the joint state *between* measurements was also proposed. Finally, numerical examples showing the effectiveness of our approach were presented.

Being the overall system non-Gaussian, the optimal estimates cannot be expressed in a simple parametric form. Moreover, due to the continuous-time nature of $t^\star$, numerical integrations over time intervals are unavoidable. Thus, one major challenge is to express the key quantities $f(t^\star, q^\star|y^k)$, $f(x_k|t^\star, q^\star, y^k)$ in a form that is convenient for both numerical quadrature and recursive update. We achieved this by a formal discretization of the system *conditioned* on the switching event. This allows to write the parameters of the conditioned *discrete-time* system as explicit functions of $t^\star$ and $q^\star$

with no loss of the information associated with the *continuous-time* variable $t^\star$. On top of this we developed tools for the recursive computation of the conditional *a priori* and *a posteriori* (Kalman filtering) system's statistics – in the form of parametric functions of $t^\star$ and $q^\star$ – allowing for efficient numerical evaluation of integrals and averages. The estimation algorithms we proposed update a finite number of parameters by way of exact matrix iterations, whereas integral approximations are isolated out of the recursion. This prevents accumulation of errors and yields an accurate computation of the estimates. Note, e.g., that any approximation (due to the numerical computation of integral (18)) that is introduced for the calculation of $f(x_k|y^k)$ does *not* influence the degree of approximation of $f(x_\ell|y^\ell)$ for $\ell > k$, since the latter density is not computed directly from the former.

The performance in the estimation of $t^\star$ and $q^\star$ strongly depends on the properties of the system modes. This is apparent in the numerical simulations reported in the previous section. However, the ways that the structural properties of the dynamical system affect the estimation performance still lack a rigorous understanding and will be investigated in the future. In this context, many interesting convergence issues arise and should be addressed. For example, it would be of utmost interest to assess the rate of convergence of conditional densities such as $f(t^\star|y^k)$ (or of the corresponding moments) as $k \to \infty$. Such rate would certainly depend on the *relative* dynamics (stability, modes of convergence or divergence, etc.) of the continuous-time systems that correspond to different values of the discrete state. Finally, it is our intention to extend the study of state estimation to more complex settings, such as systems that allow multiple Markovian jumps, e.g. starting from the case study of a discrete state q described by a two-state, continuous-time, irreducible Markov chain.

11 Appendix

11.1 Proof of Corollary 1

First of all, it is easily proven by induction that

$$P^k = \left[\begin{array}{c|ccc} P_{0,0}^k & c_k P_{0,1} & \cdots & c_k P_{0,N-1} \\ \hline 0 & & & \\ \vdots & & I_{N-1} & \\ 0 & & & \end{array}\right],$$

where

$$c_k \triangleq \sum_{\ell=0}^{k-1} P_{0,0}^\ell = \frac{P_{0,0}^k - 1}{P_{0,0} - 1}.$$

Therefore, by equation (12), matrix $T^s(\Delta)$ shall have the same structure as matrix P^k above (i.e. nonzero entries along the main diagonal and on the first row only). The upper left term is given by

$$T_{0,0}^s(\Delta) = e^{-\nu\Delta} \sum_{k=0}^{+\infty} \frac{(\nu\Delta P_{0,0})^k}{k!} = e^{-\nu\Delta} e^{\nu\Delta P_{0,0}} = e^{-\nu\Delta(1-P_{0,0})};$$

the remaining diagonal terms are given by

$$T_{i,i}^s(\Delta) = e^{-\nu\Delta} \sum_{k=0}^{+\infty} \frac{(\nu\Delta)^k}{k!} = e^{-\nu\Delta} e^{\nu\Delta} = 1;$$

finally, the remaining terms on the first row may be computed as

$$\begin{aligned} T^s_{0,j}(\Delta) &= e^{-\nu\Delta} \sum_{k=0}^{+\infty} \frac{(\nu\Delta)^k}{k!} P_{0,j} \frac{P^k_{0,0} - 1}{P_{0,0} - 1} \\ &= e^{-\nu\Delta} \frac{P_{0,j}}{P_{0,0} - 1} \Bigg(\sum_{k=0}^{+\infty} \frac{(\nu\Delta P_{0,0})^k}{k!} - \sum_{k=0}^{+\infty} \frac{(\nu\Delta)^k}{k!} \Bigg) \\ &= e^{-\nu\Delta} \frac{P_{0,j}}{P_{0,0} - 1} (e^{\nu\Delta P_{0,0}} - e^{-\nu\Delta}) \\ &= \frac{P_{0,j}}{P_{0,0} - 1} (e^{-\nu\Delta(1 - P_{0,0})} - 1). \end{aligned}$$

□

11.2 Proof of Lemma 2

First observe that the condition on the spectrum of F implies its invertibility. Also notice that F and e^{Ft} commute for any scalar t, and so do their transposed. Integration by parts of Q (reading $e^{F(b-\sigma)}G$ as derivative of $-F^{-1}e^{F(b-\sigma)}G$ and $G^T e^{F^T(b-\sigma)}$ as primitive of $-G^T e^{F^T(b-\sigma)} F^T$) followed by left multiplication by F yields

$$FQ = -(GG^T - e^{F(b-a)} GG^T e^{F^T(b-a)}) - QF^T.$$

By Lemma 1, such equation admits a unique solution (recall $\sigma(F) = \sigma(F^T)$) in Q. Precisely, it must be $Q = J + \bar{J}$, where J and $\bar{J}$ solve uniquely

$$\begin{aligned} FJ + JF^T &= -GG^T, \\ F\bar{J} + \bar{J}F^T &= e^{F(b-a)} GG^T e^{F^T(b-a)}. \end{aligned}$$

Last equation can be rewritten as

$$F(-e^{-F(b-a)} \bar{J} e^{-F^T(b-a)}) + (-e^{-F(b-a)} \bar{J} e^{-F^T(b-a)}) F^T = -GG^T,$$

hence it must be the case that

$$(-e^{-F(b-a)} \bar{J}(e^{-F^T(b-a)}) = J.$$

Eliciting $\bar{J}$ as a function of J yields the solution. □

11.3 Proof of Proposition 2

Discretization over the interval $T_k \triangleq (t_k, t_{k+1})$ amounts to solving for the dynamics from t_k to t_{k+1}. If $t^\star \notin T_k$ (*i.e.*, $k \neq h$), $F_q(t)$ and $G_q(t)$ are constant over T_k, hence

$$\begin{aligned} x(t_{k+1}) &= e^{F_q(t_{k+1} - t_k)} x(t_k) + u_k, \\ u_k &\triangleq \int_{t_k}^{t_{k+1}} e^{F_q(t_{k+1} - \sigma)} G_q u(\sigma) \mathrm{d}\sigma. \end{aligned}$$

As u_k is a linear function of $u(\sigma)$, it is zero-mean Gaussian with variance $Q_k(t^\star, q^\star)$ given by

$$\begin{aligned}
\mathbb{E}[u_k u_k^T] &= \iint_{T_k \times T_k} \mathbb{E}[u(\sigma)u(\sigma')] e^{F_q(t_{k+1}-\sigma)} G_q G_q^T e^{F_q^T(t_{k+1}-\sigma')} \mathrm{d}\sigma \mathrm{d}\sigma' \\
&= \iint_{T_k \times T_k} \delta(\sigma - \sigma') e^{F_q(t_{k+1}-\sigma)} G_q G_q^T e^{F_q^T(t_{k+1}-\sigma')} \mathrm{d}\sigma \mathrm{d}\sigma' \\
&= \int_{T_k} e^{F_q(t_{k+1}-\sigma)} G_q G_q^T e^{F_q^T(t_{k+1}-\sigma)} \mathrm{d}\sigma \\
&= J_q - e^{F_q(t_{k+1}-t_k)} J_q e^{F_q^T(t_{k+1}-t_k)},
\end{aligned}$$

where q is equal to 0 or $q^\star$ according to whether $t^\star > t_{k+1}$ or $t^\star < t_k$ (*i.e.*, $k < h$ or $k > h$), and Lemma 2 has been applied. If instead $t^\star \in T_k$ (*i.e.*, $k = h$), then $F_q(t)$ and $G_q(t)$ are piecewise constant:

$$(F_{q(t)}, G_{q(t)}) = \begin{cases} (F_0, G_0), & t \in (t_k, t^\star) \\ (F_{q^\star}, G_{q^\star}), & t \in (t^\star, t_{k+1}) \end{cases}.$$

Discretization over $(t_k, t^\star)$ yields

$$x(t^\star) = e^{F_0(t^\star - t_k)} x(t_k) + u_{k,0},$$

$$u_{k,0} \triangleq \int_{t_k}^{t^\star} e^{F_0(t^\star - \sigma)} G_0 u(\sigma) \mathrm{d}\sigma.$$

Discretization over $(t^\star, t_{k+1})$ yields

$$x(t_{k+1}) = e^{F_{q^\star}(t_{k+1}-t^\star)} x(t^\star) + u_{k,q^\star} = e^{F_{q^\star}(t_{k+1}-t^\star)} e^{F_0(t^\star - t_k)} x(t_k) + u_k$$

with $u_k \triangleq e^{F_{q^\star}(t_{k+1}-t^\star)} u_{k,0} + u_{k,q^\star}$ and

$$u_{k,q^\star} \triangleq \int_{t^\star}^{t_{k+1}} e^{F_{q^\star}(t_{k+1}-\sigma)} G_{q^\star} u(\sigma) \mathrm{d}\sigma.$$

Again, $u_{k,0}$ and $u_{k,q^\star}$ are both zero-mean Gaussian. Their variances are given by

$$\begin{aligned}
\widetilde{Q}_k(t^\star, 0) &= \int_{t_k}^{t^\star} e^{F_0(t^\star-\sigma)} G_0 G_0^T e^{F_0^T(t^\star-\sigma)} \mathrm{d}\sigma = J_0 - e^{F_0(t^\star-t_k)} J_0 e^{F_0^T(t^\star-t_k)}, \\
\widetilde{Q}_k(t^\star, q^\star) &= \int_{t^\star}^{t_{k+1}} e^{F_{q^\star}(t_{k+1}-\sigma)} G_{q^\star} G_{q^\star}^T e^{F_{q^\star}^T(t_{k+1}-\sigma)} \mathrm{d}\sigma = J_{q^\star} - e^{F_{q^\star}(t_{k+1}-t^\star)} J_{q^\star} e^{F_{q^\star}^T(t_{k+1}-t^\star)},
\end{aligned}$$

in the order, where again Lemma 2 has been applied. Moreover, as they depend on disjoint chunks of $u(t)$ and $u(t)$ is white, $u_{k,0} \perp u_{k,q^\star}$. Then u_k is zero mean Gaussian with variance

$$Q_k(t^\star, q^\star) = e^{F_{q^\star}(t_{k+1}-t^\star)} \widetilde{Q}_k(t^\star, 0) e^{F_{q^\star}^T(t_{k+1}-t^\star)} + \widetilde{Q}_k(t^\star, q^\star).$$

Finally, observe

$$(C_k(t^\star, q^\star), D_k(t^\star, q^\star)) = \begin{cases} (H_0, K_0), & t^\star > t_{k+1} \\ (H_{q^\star}, K_{q^\star}), & t^\star < t_k \end{cases}.$$

□

11.4 Proof of Proposition 3

Fix arbitrary values of $t^\star$ and $q^\star$. First of all note that

$$\begin{aligned}
x_1 &= A_0 x_0 + u_0 \\
x_2 &= A_1 x_1 + u_1 \;=\; u_1 + A_1 u_0 + A_1 A_0 x_0 \\
x_3 &= A_2 x_2 + u_2 \;=\; u_2 + A_2 u_1 + A_2 A_1 u_0 + A_2 A_1 A_0 x_0 \\
&\vdots \\
x_k &= u_{k-1} + A_{k-1} u_{k-2} + \cdots + A_{k-1} A_{k-2} \ldots A_2 A_1 u_0 + A_{k-1} A_{k-2} \ldots A_2 A_1 A_0 x_0 \,.
\end{aligned}$$

Define column vector $x^k \triangleq [x_k^T, x_{k-1}^T, \ldots, x_0^T] \in \mathbb{R}^{(k+1)n\times 1}$; define in an analogous way columns vectors $u^k \in \mathbb{R}^{(k+1)n\times 1}$ and $v^k \in \mathbb{R}^{(k+1)r\times 1}$, for all $k \in \mathbb{N}_0$. With this notation and the one introduced by equations (27) and (28) we have that:

$$x^k = \Theta_k(t^\star, q^\star) \begin{bmatrix} u^{k-1} \\ x_0 \end{bmatrix}, \quad \text{while} \quad y^k = \Upsilon_k(t^\star, q^\star) x^k + \Lambda_k(t^\star, q^\star) v^k,$$

whence

$$y^k = \Upsilon_k(t^\star, q^\star)\Theta_k(t^\star, q^\star) \begin{bmatrix} u^{k-1} \\ x_0 \end{bmatrix} + \Lambda_k(t^\star, q^\star) v^k.$$

Now, since $\begin{bmatrix} u^{k-1} \\ x_0 \end{bmatrix} \sim \mathcal{N}\big(\widetilde{\mu}_k, \widetilde{\Sigma}_k(t^\star, q^\star)\big)$, we have that y^k is multivariate Gaussian itself with mean μ_{y^k} and variance Σ_{y^k} given by (29) and (30), respectively. In particular, $\mu_{y^0} = C_0\mu_0$ and $\Sigma_{y^0} = C_0\Sigma_0 C_0^T + D_0 D_0^T$. Also note that:

$$\begin{aligned}
\Sigma_{y^k} &= \Upsilon_k \Theta_k \widetilde{\Sigma}_k \Theta_k^T \Upsilon_k^T + \Lambda_k \Lambda_k^T \\
&= \begin{bmatrix} C_k & 0 \\ 0 & \Upsilon_{k-1} \end{bmatrix} \begin{bmatrix} I & \Xi_k \\ 0 & \Theta_{k-1} \end{bmatrix} \begin{bmatrix} Q_{k-1} & 0 \\ 0 & \widetilde{\Sigma}_{k-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ \Xi_k^T & \Theta_{k-1}^T \end{bmatrix} \begin{bmatrix} C_k^T & 0 \\ 0 & \Upsilon_{k-1}^T \end{bmatrix} + \begin{bmatrix} D_k & 0 \\ 0 & \Lambda_{k-1} \end{bmatrix} \begin{bmatrix} D_k^T & 0 \\ 0 & \Lambda_{k-1}^T \end{bmatrix} \\
&= \begin{bmatrix} C_k\big(\Xi_k \widetilde{\Sigma}_{k-1} \Xi_k^T + Q_{k-1}\big) C_k^T + D_k D_k^T & C_k \Xi_k \widetilde{\Sigma}_{k-1} \Theta_{k-1}^T \Upsilon_{k-1}^T \\ \Upsilon_{k-1}\Theta_{k-1}\widetilde{\Sigma}_{k-1}\Xi_k^T C_k^T & \Upsilon_{k-1}\Theta_{k-1}\widetilde{\Sigma}_{k-1}\Theta_{k-1}^T \Upsilon_{k-1}^T + \Lambda_{k-1}\Lambda_{k-1}^T \end{bmatrix} \\
&= \begin{bmatrix} \Phi_k & \Psi_k \\ \Psi_k^T & \Sigma_{y^{k-1}} \end{bmatrix},
\end{aligned}$$

where Φ_k and Ψ_k were defined in equations (31) and (32). □

11.5 Proof of Proposition 5

In light of Proposition 4, the time update of the conditional Kalman filter is, for $k = h$,

$$\begin{aligned}
\hat{x}_{h+1|h}(t^\star, q^\star) &= A_h(t^\star, q^\star)\hat{x}^0_{h|h}, \\
P_{h+1|h}(t^\star, q^\star) &= A_h(t^\star, q^\star) P^0_{h|h} A_h^T(t^\star, q^\star) + Q_h(t^\star, q^\star),
\end{aligned}$$

where, by the definitions in Proposition 2,

$$
\begin{aligned}
Q_h(t^\star, q^\star) &= \widetilde{A}_h(t^\star, q^\star)(J_0 - \widetilde{A}_h(t^\star, 0)J_0\widetilde{A}_h^T(t^\star, 0))\widetilde{A}_h^T(t^\star, q^\star) + (J_{q^\star} - \widetilde{A}_h(t^\star, q^\star)J_{q^\star}\widetilde{A}_h^T(t^\star, q^\star)) \\
&= -A_h(t^\star, q^\star)J_0A_h^T(t^\star, q^\star) + \widetilde{A}_h(t^\star, q^\star)(J_0 - J_{q^\star})\widetilde{A}_h^T(t^\star, q^\star) + J_{q^\star},
\end{aligned}
$$

hence the assertion. □

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