1. **Phase portrait for the Pendulum.** We have seen in previous homework that the equation of the harmonic oscillator (that is $\ddot{x} = -\omega^2 x$) provides a good approximation only for small oscillations. A differential equation that better describes the pendulum’s oscillations (but still does not account for friction) is the following:

$$\ddot{x} = -\omega^2 \sin x,$$

where $\omega^2 = g/L$: for simplicity, assume $\omega^2 = 1$. The purpose of this problem is drawing, on the same graph, some orbits corresponding to solutions of system

$$\begin{cases}
\dot{x} = v \\
\dot{v} = -\sin x
\end{cases} \quad (1)$$

and the vector field associated to it (what are its equilibria?). The Matlab function you should use is `quiver` (type “help quiver” at the Matlab prompt to learn more about it). Create an m-file containing the following command lines:

```matlab
[X,V] = meshgrid(-6:.4:6,-4:.4:4);
dX=V;
dV=-sin(X);
quiver(X,V,dX,dV,'r');
```

you should be able to visualize the vector field corresponding to dynamical system (1). Note that the range of the grid we have chosen also contains equilibria $(-\pi,0)$ and $(\pi,0)$. You may choose to visualize a finer grid by decreasing grid step (fixed at 0.4 in the above program).

Now, plot on the same graph (using the “hold on” command) some trajectories corresponding to solutions of system (1), and verify that they are tangent to the vector field, as discussed in class. Remember that the numerical methods you should use for drawing these orbits were discussed in homework #3. Make sure to draw trajectories that are close to the origin (say, with initial values $x(1) = 0.2, v(1) = 0$, but also far from the origin $x(1) = -3, v(1) = 0$, and $x(1) = -\pi, v(1) = 0.2$); in fact, draw as many trajectories as you want. Discuss the stability of the equilibria (which ones are stable? Which ones are unstable?).

A differential equation that accounts for friction and describes the oscillations of a “real pendulum” even better is the following:

$$\ddot{x} = -k\dot{x} - \omega^2 \sin x,$$

i.e.:

$$\begin{cases}
\dot{x} = v \\
\dot{v} = -kv - \omega^2 \sin x
\end{cases} \quad \text{with } k > 0.$$

Fix $\omega^2 = 1, k = 0.2$ and repeat the exercise (if you want, you may vary the value of $k$ and see what happens). In particular, remember to modify the definition of $dV$ in the program above. Please hand in your programs as well as your graphs.

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1. Remember the notation we introduced in class: $\dot{x}(t)$ stands for $\frac{dx}{dt}(t)$, whereas $\ddot{x}(t)$ stands for $\frac{d^2x}{dt^2}(t)$. 

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2. **Predator-Prey model** (also known as **LOTKA-VOLTERRA** model). Consider the dynamical system we discussed in class:

\[
\begin{align*}
\dot{x} &= \alpha x - \beta xy \\
\dot{y} &= -\gamma y + \delta xy
\end{align*}
\]

with \( \alpha > 0, \ \beta > 0, \ \gamma > 0, \ \delta > 0, \) \hspace{1cm} (2)

where \( x(t) \) is the density of preys (rabbits) while \( y(t) \) is the density of predators (wolves). What are the equilibria of this system for \( x \geq 0 \) and \( y \geq 0 \)? Now, for simplicity, let \( \alpha = \beta = \gamma = \delta = 1 \), and repeat the previous problem: that is, draw the vector field in the \((x, y)\) plane and some orbits corresponding to solutions. Note that, in this case, the command lines you should use in order to draw the vector field should be something like the following:

\[
[X,Y] = \text{meshgrid}(0:.2:2,0:.2:2);
\]
\[
dX=X-X.*Y;
\]
\[
dY=-Y+X.*Y;
\]
\[
\text{quiver}(X,Y,dX,dY,’r’);
\]

(the dot in “.*” is necessary in order to perform element-by-element multiplication; also, note that we are only interested in positive values of \( x \) and \( y \)). Draw a sufficient number of orbits, including some with initial values close the equilibria (again, you should use numerical implementation methods similar to those described in homework #3). Comment on the stability of the equilibria. What do the orbits look like if we start on one of the axes (i.e. if our initial \((x, y)\)-values have a zero element)?

**Extra credit.** Repeat the same problem for this slightly modified Lotka-Volterra model:

\[
\begin{align*}
\dot{x} &= \alpha x - \beta xy - \varepsilon x^2 \\
\dot{y} &= -\gamma y + \delta xy
\end{align*}
\]

we have included the effects of competition of preys among themselves, due to the limited amount of resources (in the previous model, preys could die only if caught by predators; in this one they can starve too). Assume \( \alpha = \beta = \gamma = \delta = 1 \) and \( \varepsilon = 0.2 \). Comment on the qualitative behavior of orbits and on the stability of equilibria for model (3), as opposed to model (2). Please hand in both your programs and your graphs.

\[\text{Alfred LOTKA (1880–1949), chemist, demographer, ecologist and mathematician, was born in Lviv (Lemberg), at that time situated in Austria, now in Ukraine. He came to the United States in 1902 and wrote a number of theoretical articles on chemical oscillations during the early decades of the twentieth century, and wrote the first book on mathematical biology (1925). He is best known for the predator-prey model he proposed, at the same time but independently from Volterra, still the basis of many models used in the analysis of population dynamics. Vito VOLterra (1860–1940) was born in Ancona, Papal States (now Italy). Volterra’s mathematical skills were the connecting thread in a diverse range of interests as he undertook close mathematical study of phenomena in fields as diverse as optics, celestial mechanics, and biology. In 1926 Volterra proposed his model to explain the oscillatory levels of certain fish catches in the Adriatic sea. Always active as a scientist, after entering the Italian Senate in 1905, Volterra also became increasingly dedicated to the cause of democracy. In 1925, as president of the Accademia dei Lincei, a post once held by Galileo, Volterra was one of the principal signatories of the Intellectuals’ Declaration against Fascism. In the end, his bold and unsparing opposition to Fascism cost Volterra his professorship, his membership in learned societies, and, ultimately, his seat in the Senate. Always, however, he remained among the most respected men in Italy and Europe.}\]