Propagation Failure

Intro

Consider the system

$$ u_j = d(u_{j+1} + u_{j-1} - 2u_j) + f(u_j) \quad j \in \mathbb{Z} $$

where \( \cdot \) indicates differentiation wrt time \( t \), and \( f \) is a bistable nonlinearity, e.g.,

$$ f(u) = (1-u^2)(u-a), $$

$$ -1 < a < 1 $$

can either be considered as a coupling constant - for example, if derived from nerve signal propagation as Bjorn described previously - or if we write \( d = \frac{1}{h^2} \) as discretization of PDE

$$ u_t = u_{xx} + f(u) $$

Start by recalling how we analyzed the continuous case:

Considered potential

$$ F(u) := -\int_{-1}^{u} f(w) \, dw $$

Can write \( u_t = u_{xx} - F'(u) \)

If we first look for spatially homogeneous solutions

$$ u(x,t) = u(t) \forall x $$

$$ \Rightarrow u_t = -F'(u) \quad \Rightarrow \text{gradient system with energy} \ F(u) $$
Meanwhile, if look at $u_t = u_{xx}$, acts to smooth out gradients in IC

$u_{xx} < 0$

$u_{xx} > 0$

Now look for traveling front solutions

$\phi(\xi) = \phi(x-ct) = u(x,t)$

$\xi \to \xi - c t$

We want

$\phi(\xi) \to \begin{cases} 1 & \xi \to +\infty \\ -1 & \xi \to -\infty \end{cases}$

Substituting, we get

$-c\phi'(\xi) = \phi''(\xi) + f(\phi(\xi))$

which we can rewrite as a 1st order system

$\begin{pmatrix} u \\ v \end{pmatrix}_\xi = \begin{pmatrix} v \\ -cv - f(u) \end{pmatrix}$

Equilibria are $(-1,0), (1,0)$ [saddles] and $(0,0)$ [center]

Define energy

$H(u,v) = \frac{v^2}{2} + \int_1^u f(w)dw$

Level sets of $H$:

For $c < 0$, $H(u(t),v(t))$ will increase strictly along trajectories

So if we consider unit vectors tangent to unstable/stable manifolds at $(-1,0)$ and $(1,0)$ respectively:
Idea: fix b s.t. $a < b < 1$
show $h^u(c) := \text{first intersection of } w^u(-1,0) \mid v = b$
and $h^s(c) := \text{same } w^s(1,0)$
are well-defined, continuous in $c$, $h^u(0) < h^s(0)$,
and $h^u(c) > h^s(c)$ for $c \ll -1$.

$\Rightarrow \exists c_\ast \text{ s.t. } h^u(c_\ast) = h^s(c_\ast)$

Can also show this is unique, and $c_\ast$ varies continuously
as a function of $a$.

Note that since we have a continuous flow, if the
invariant manifolds $w^u(-1,0)$ \& $w^s(1,0)$ intersect, they
must be tangent where they intersect.

Now let’s return to our UDF

$$\dot{u}_j = d (u_{j+1} + u_{j-1} - 2u_j) + f(u_j) \quad j \in \mathbb{Z}$$

We again want to find traveling front solutions

$$u_j(t) = \varphi(j - ct)$$

where $\varphi$ is the wave profile, $c$ the wave speed, satisfying

$$\lim_{\xi \to \pm \infty} \varphi(\xi) = \pm 1$$

We find that $c = c(a)$ varies continuously and is nondecreasing.
wrt \( a \), but now we may find that \( c(a) = 0 \) on a nontrivial interval. Pictorially we have

\[
\begin{align*}
\text{continuous case} & \quad \text{LDE case} \\
\end{align*}
\]

Define \( [a_-, a_+] = \{ a \in (-1, 1) : c(a) = 0 \} \).

If \( a_- < a_+ \), we say propagation failure has occurred.

Some history:

Bell (1981)
Bell \& Cosner (1984)
[ Britton (1984) ]

Keener (1987) → where I'll start
\[
\frac{du_n}{dt} = d \left( u_{n+2} - 2u_n + u_{n+1} \right) + f(u_n) \quad n \in \mathbb{Z} \quad (1)
\]

where \( f(0) = f(a) = f(1) = 0 \quad 0 < a < 1 \quad f(u) < 0 \quad \text{for} \quad 0 < u < a \quad f(u) > 0 \quad \text{for} \quad a < u < 1 \)

**Thm 2.1** Suppose the function \( f(x) \) is continuously differentiable for \( x \in [0, 1] \) and that

(i) \( f(0) = f(a) = f(1) = 0 \quad 0 < a < 1 \); \( f(x) \neq 0 \quad x \neq 0, a, 1 \)

(ii) \( f(x) < 0 \quad 0 < x < a \)

\( 0 < a < 1 \) some \( x_0, x_1 \) with

(iii) \( f'(x_0) = f'(x_1) = 0 \) for \( 0 < x_0 < a < x_1 < 1 \)

\( f'(x) \neq 0 \) for \( x \neq x_0, x_1 \)

Then, for every \( d > 0 \) sufficiently small, to every doubly infinite sequence \( s_{n, 3} \) with \( s_n \in [0, 1] \), there corresponds at least one steady solution of \((2.1)\) with \( u_n \in [0, a) \) when \( s_n = 0 \), and \( u_n \in [a, 1] \) when \( s_n = 1 \).

**E.g.** \( 3, 1, 1, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, \ldots \)

<table>
<thead>
<tr>
<th>s_{-4}</th>
<th>s_{-3}</th>
<th>s_{-2}</th>
<th>s_0</th>
<th>s_1</th>
<th>s_2</th>
<th>s_3</th>
<th>s_4</th>
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\[
\begin{array}{c}
\vdots \\
D_{-1}, D_0, D_1, D_2, D_3, D_4, \ldots
\end{array}
\]
Will show all these solutions stable or block propagation.

But first part of 2.1

Rewrite as steady-state, 1st order system

\[
\begin{align*}
\dot{u}_{n+1} &= 2u_n - u_n - f(u_n)/c \\
\dot{v}_{n+1} &= v_n
\end{align*}
\]

Write \( \Phi : (u_n, v_n) \rightarrow (u_{n+1}, v_{n+1}) \)

Wts \( \Phi \) has as a subsystem the shift on the sequence of symbols \( 0, 1, \overline{1} \).

Def. \( Q := [0, 1] \times [0, 1] \)

A curve \( v = v(u) \) is a horizontal curve in \( Q \) if \( 0 \leq v(u) \leq 1 \) for \( 0 \leq u \leq 1 \). The set lying between two non-intersecting horizontal curves is called a horizontal strip.

Similarly, \( u = u(v) \) is called a vertical curve in \( Q \) if \( 0 \leq u(v) \leq 1 \) for \( 0 \leq v \leq 1 \), and the set between two non-intersecting vertical curves is a vertical strip.

Theorem (Moser) Suppose \( V_i, V_i, i = 0, 1 \), are disjoint horizontal and vertical strips (resp.) in \( Q \), and that \( \Phi(V_i) = U_i, i = 0, 1 \). Further suppose the vertical boundaries of \( V_i \) are mapped to vertical boundaries of \( U_i \), and the horizontal boundaries of \( V_i \) are mapped to horizontal boundaries of \( U_i \). Then \( \Phi \) possesses the shift on sequences \( 0, 1, \overline{1} \) as a subsystem.
We're using a somewhat weaker version of the theorem, not requiring contraction, since only require existence, not uniqueness.

For $\varphi$ as given by (2), note we have $\varphi^{-1}(u_n, v_n) \to (u_{n+1}, v_{n+1})$ given by

$$
\begin{align*}
    u_{n+1} &= v_n \\
    v_{n+1} &= 2v_n - u_n - \frac{f(v_n)}{d}
\end{align*}
$$

Recall we defined $x_0, x_1$ as the P3 s.t.

$$
f'(x_0) = f'(x_1) = 0, \quad 0 < x_0 < a < x_1 < 1
$$

So $v=0, 0 \leq u \leq x_0$ is a piece of a horizontal curve that is mapped by $\varphi$ to a monotone increasing horizontal curve $v = v_0(u)$ defined for $0 \leq u \leq F_0 := 2x_0 - \frac{f(x_0)}{d}$

with $v_0(0) = 0, v_0(F_0) = x_0$

Similarly, $v=1, 0 \leq u \leq x_0$ is mapped by $\varphi$ to $v = v_0(u+1)$ defined for $-1 \leq u \leq F_0 - 1$

$\Rightarrow$ If $F_0 > 2$, the curves $v = v_0(u)$ and $v = v_0(u+1), 0 \leq u \leq 1$ are the boundaries of a horizontal strip $U_0$.
Now let $x_0^*$ s.t. $2(x_0^*) - f(x_0^*)/d = 1 \quad \forall 0 \leq x_0^*, x_0^* < x_0$.

**Follow similar procedure for**

$v = 0, x_1 \leq u, u \leq 1$

$v = 1, u$

Define $F_1 := 2x_1 - \frac{f(x_1)}{d}$

Now require $F_1 < -1$

let $x_1^*$ s.t. $2(x_1^*) - \frac{f(x_1^*)}{d} = 1 \quad \forall x_1^* < \hat{x}_1, x_1^* < 1$

$\hat{x}_1$ s.t. $2(\hat{x}_1) - \frac{f(\hat{x}_1)}{d} = 0$
\[ y \text{ has the shift on 2 symbols as a subsystem.} \]

Also note in particular we have for any seq. \((s_i)_{i \in \mathbb{Z}}, s_i \in \{0, 1\}\)

\[ \exists (u_i)_{i \in \mathbb{Z}} \text{ which is a stationary solution if (1) with} \]

\[ u_i \in [0, \hat{x}_0] \text{ if } s_i = 0 \text{ and } u_i \in [\hat{x}_1, 1] \text{ if } s_i = 1. \]

\[ \square \]

As we saw above, needed \(F_0 > 2\) and \(F_1 < -1\), which can occur only for \(d\) suff. small; this leads us to following

Cor 2.2 Suppose \(f(x)\) is continuously differentiable for \(x \in [0, 1]\)

and suppose \(f(0) = f(1) = 0\). If \(\exists \hat{x}_0, \hat{x}_1\) s.t.

\[ (i) \ 2(\hat{x}_0 - 1) - \frac{f(\hat{x}_0)}{d} = 0 \quad \text{(clearly } \Rightarrow \ 2\hat{x}_0 - \frac{f(\hat{x}_0)}{d} = 2) \]

\[ (ii) \ 2\hat{x}_1 - \frac{f(\hat{x}_1)}{d} = 0 \]

\[ (iii) \ f'(x) < 2d \text{ on } 0 \leq x \leq \hat{x}_0 \text{ and } \hat{x}_1 \leq x \leq 1 \]

then the conclusions of Thm 2.1 hold.

[Note also weakens \(f\) slightly]

(i) \(\hat{x}_0\) and (ii) ensure existence of horizontal and vertical strips and (iii) ensures boundary curves are monotone.
To understand stability, we will use a comparison result:

**Thm 2.3** Suppose \( f(u(t)) \) and \( g(v(t)) \) satisfy the eqns

\[
\frac{du}{dt} \leq d(u_{n+1} - 2u_n + u_{n-1}) + f(u_n)
\]
\[
\frac{dv}{dt} \geq d(v_{n+1} - 2v_n - v_{n-1}) + f(v_n)
\]

If \( u_n(t_0) \leq v_n(t_0) \forall n \) then \( u_n(t) \leq v_n(t) \forall t \geq t_0 \).

[Skip proof.]

Now using hypotheses of Cor 2.2, establish global stability result.

**Thm 2.4** Suppose \( \exists \hat{x}, \bar{x}, \underline{x}, \bar{x} \) s.t.

(i) \( 2(\hat{x} - 1) - \frac{f(\hat{x})}{d} = 0 \), \( 2(\bar{x} - 1) - \frac{f(\bar{x})}{d} = 0 \)

(ii) \( 2\bar{x} - \frac{f(\bar{x})}{d} = 0 \), \( 2\underline{x} - \frac{f(\underline{x})}{d} = 0 \)

and \( f(x) - 2x > 0 \) for \( \underline{x} < x < \bar{x} < 1 \)

Suppose \( 0 \leq u_n(0) \leq 1 \forall n \). If \( u_k \in [0, \bar{x}] \) at \( t=0 \) then \( u_k(t) \in [0, \bar{x}] \forall t \geq 0 \), whereas if \( u_k \in [\underline{x}, 1] \) at \( t=0 \) then \( u_k(t) \in [\underline{x}, 1] \forall t \geq 0 \).

[Skip.]

Keener goes on to show how we can weaken the hypotheses somewhat by looking specifically at monotone solutions, and finds in particular that for \( f(u) = u(1-u)(u-a) \), if \( 0 < a < \frac{1}{2} \), the conditions of Cor 2.2 hold for \( d < \frac{a^2}{4} \) and weaker conditions for \( d < a^2/4 \).
So we've seen one very nice explanation of propagation failure, but from the point of view of dynamical systems, what's "really" going on?

- Compare with continuous case
- Connection to chaos/horseshoe map?

**Invariant manifolds**

Consider the equation

\[ c p(\xi) = dp(\xi+1) + p(\xi-1) - 2c p(\xi) + f(p(\xi); a) \]

Which we get if we substitute \( u(jt) = p(j-ct) = p(\xi) \) in our original ODE. If \( c = 0 \) this reduces to

\[ 0 = dp(j+1) + p(j-1) - 2p(j) + f(p(j); a) \quad j \in \mathbb{Z} \]

which is just a difference equation

We can rewrite it as

\[ p_{j+1} = r_j \]
\[ r_{j+1} = -p_j + 2r_j - \frac{f(r_j; a)}{d} \]

i.e., a discrete-time dynamical map. The front sol'ns were looking for are heteroclinic connections between \((-1, -1)\) and \((1, 1)\), and in general a heteroclinic connection may lie in the transverse intersection of \( W^u(-1, -1) \) and \( W^s(1, 1) \). If the intersection is transverse, stationary solutions will persist for \( a \) in some nbd of 0.
Of course, that’s not really the end of the story.

- Subtle dependence of behavior on form of f
- Interesting & more complicated phenomena when consider LDE posed on higher dimensional lattices.
- Predicting region \([a_+, a_-]\) on which failure (pinning) occurs?

Continuing on with papers / history

- Erneux & Nicolis (1993)

- Erneux & Laflante (1992)
  - Propagation failure in arrays of coupled bistable chemical reactors
  - Clever: hard to vary d if working with nerve cells or cardiac tissue
    - but with chemical reactors can vary exchange rate
  - Results show impact of noise?

- Anderson & Sleeman (1995)

Now sticking with lattice on \(Z\) / skipping ahead chronologically (come back to crystallographic pinning)
Start concretely, consider

\[ \dot{u}_j = \frac{1}{h^2} (u_{j+1} + u_{j-1} - 2u_j) + f(u, j, \alpha) \quad j \in \mathbb{Z} \quad (1) \]

with

\[ f(u, j, \alpha) = 2(1-u^2)(u-a) \quad -1 < a < 1 \quad (2) \]

Look for stationary solu's \( \nu \); rewrite as

\[ \begin{align*}
    P_{j+1} &= \nu_j \\
    \nu_{j+1} &= -P_j + 2\nu_j - h^2 f(\nu_j, \alpha)
\end{align*} \quad (3) \]

When \( a = 0 \), equilibria \((-1, -1)\) and \((+1, +1)\) are both saddles, and (3) admits solu's \( P^{(s)} \) and \( P^{(u)} \) satisfying

\[ \lim_{j \to \pm \infty} P_j^{(s)} = \lim_{j \to \pm \infty} P_j^{(u)} = \pm 1, \quad P_{j+1}^{(s)} = -P_j^{(s)}, \quad P_{j+1}^{(u)} = -P_j^{(u)} \quad (4) \]

These are referred to as \underline{site-centered} and \underline{bond-centered} solutions.

For each \( j \in \mathbb{Z} \), \((P_j^{(s)}, P_{j+1}^{(s)})\) and \((P_j^{(u)}, P_{j+1}^{(u)})\) lie in the intersection of the unstable manifold \( W^u(-1, -1) \) and the stable manifold \( W^s(1, 1) \).

There are now two possibilities: either the intersections of \( W^u(-1, -1) \) are transverse, in which case \( P^{(s)} \) and \( P^{(u)} \) persist for \( a \approx 0 \) (though only identify \( a \equiv (0) \) by continuation).
or else \( W^s(1,1) \) and \( W^u(-1,-1) \) coincide entirely at \( a=0 \), and \( p^{(u)} \) and \( p^{(w)} \) are part of a smooth family of stationary solutions.

**Case 1** → saddle-node bifurcation

(Stationary solutions persist for \( a \neq 0 \))

**Case 2** →

Do not expect the type of degeneracy in case 2 to exist for \( f \) as in (2), but give example where it does:

[think of as alternate discretization & \( u_t = u_{xx} + f(u,a) \)]

\[
  u_j = \frac{1}{h^2} (u_{j-1} + u_{j+1} - 2u_j) + (1 - u_j^4)(u_{j+1} + u_{j-1} - 2a), \quad j \in \mathbb{Z} \tag{5}
\]

Can directly verify that for any \( a, t \in \mathbb{R} \), (5) is satisfied by

\[
  u_j(t) = \tanh(\text{arc} \sinh(\frac{4}{3}(j - ct + l)), \quad c = \frac{2a}{\text{arc} \sinh(h)} \tag{6}
\]

so that we have a branch of stationary solutions at \( a=0 \) parameterized by \( t \in \mathbb{R} \), i.e., in case 2.

It follows from [M-P 1999, Thm 2.1] that \( a_- = a_+ = 0 \) holds for (5), so we do not see propagation failure.

Note this means the manifolds \( W^u(-1,-1) \) and \( W^s(1,1) \) separate completely for \( a \neq 0 \).

[Suggest using (5) as discretization for \( u_t = u_{xx} + f(u) \); Q: Daubechies]
Main Idea now is to show that if we are in case 2, for a broad class of nonlinearities \( f \), we'll have complete separation of the stable and unstable manifolds so that propagation failure does not occur.

In particular, consider
\[
\dot{u}_j = g(u_{j-1}, u_j, u_{j+1}) \quad j \in \mathbb{Z}, \quad u_j \in \mathbb{R}
\]

and define the conditions:

(Hg1) The nonlinearity \( g \) is \( C^3 \) smooth, with \( \partial u_1 g(u_1, u_2, u_3) \) and \( \partial u_3 g(u_1, u_2, u_3) > 0 \) for all \( (u_1, u_2, u_3) \in \mathbb{R}^3 \) and \( a \in (-1, 1) \).

In addition
\[
\partial u_2 g(u_1, u_2, u_3) < 0
\]
for all \( a \in (-1, 1) \) and \( (u_1, u_2, u_3) \in \mathbb{R}^3 \) with \(-1 < u_1 < u_2 < u_3 < 1\).

(Hg2) Setting \( \bar{g}(u; a) := g(u; 0, u; ja) \) we have
\[
\bar{g}(\pm 1; a) = 0, \quad \bar{g}(a; a) = 0
\]
\[
\bar{g}(u; ja) < 0 \text{ for } u \in (-1, a) \cup (1, a)
\]
\[
\bar{g}(u; ja) > 0 \text{ for } u \in (a, -1) \cup (a, 1)
\]
for every \( a \in (-1, 1) \). Also
\[
\partial u \bar{g}(\pm 1; a) < 0, \quad \partial u \bar{g}(a; a) > 0
\]
\[
\partial u a \bar{g}(-1; a) < 0, \quad \partial u a \bar{g}(1; a) > 0.
\]

[So \( \bar{g}(u; a) \) "looks like" original cubic.]

Also need to impose degeneracy condition on stationary solutions:
(Hp) There exists a $\bar{p} \in B_{C^3}(\Omega, \Omega)$ s.t. for any $\varepsilon \in \mathbb{R}$, the constant function $u(t) = p$ given by

$$p_j^{(\varepsilon)} = \bar{p}(j + \varepsilon)$$

satisfies (7) with $a = a_x$ for some $a_x \in \mathbb{R}$. In addition, $\bar{p}$ has $\bar{p}'(\varepsilon) > 0$ for all $\varepsilon \in \mathbb{R}$ and satisfies

$$\lim_{\varepsilon \to \pm\infty} \bar{p}(\varepsilon) = \pm 1$$

**Thm 1.1** Consider the system (7) and suppose (Hg1), (Hg2), (Hg3) hold. Then for every $a \in (-1, 1)$, (7) admits a solution of the form

$$u_j(t) = \overline{u}(j - ct)$$

for some $c \in \mathbb{R}$ and $\overline{u} \in C^1(\Omega, \Omega)$ with $\overline{u}'(\varepsilon) > 0$ for all $\varepsilon \in \mathbb{R}$ and

$$\lim_{\varepsilon \to \pm\infty} \overline{u}(\varepsilon) = \pm 1.$$ 

The wave speed $c = c(a)$ depends $C^1$-smoothly on $a$ with $c(a) = 0$ and $c(a) > 0$ for all $a \in (-1, 1)$.

[so we don't get prop. failure $u$ given conditions.]

**Cor 1.2** Consider (7) and suppose (Hg1), (Hg2), (Hg3) hold. There exists const. $\varepsilon_0 > 0$ s.t. the following holds: Suppose (7) admits a stationary soln

$$u_j(t) = u_j$$

for some $a \in (-1, 1)$. Suppose $u_j \leq u_j$ holds for $j_1 \leq j_2$ together with

$$\lim_{j \to \pm\infty} u_j = \pm 1,$$

or alternatively that $|a - a_x| < \varepsilon$ and $1 - p(\varepsilon) < \varepsilon$ for some $\varepsilon \in \mathbb{R}$. Then we must have $a = a_x$.

[So for $a \neq a_x$ can't have $j$-monotonic stationary solutions, also for $a$ close to $a_x$ if $u$ is stationary if close to $p(\varepsilon)$, must have $a = a_x$.]


Cor 1.3 (follows from M-P 1999) Consider (7) and suppose (Hq1), (Hq2) and (Hq3) are satisfied. Pick any $a \in (-1,1)$ and suppose (7) admits a soln of the form

$$u_j(t) = \bar{u}(j-ct)$$

for $c \neq 0$ and $\bar{u} \in C^1(\mathbb{R}, \mathbb{R})$ that satisfies

$$\lim_{\bar{u}(x) \to \pm 0} \bar{u}(x) = \pm 1$$

Then $u$ must be a temporal translate of the soln in Thm 1.1. [So waves in thm 1.1 unique among solns connecting $\pm 1$].

Idea of pf of Thm 1.1

Focus on dynamics of (7) for a near $a$

Define $M(a_x) = \{ p^{(e)} \}_{e \in \mathbb{R}}$, manifold of equilibria

Show persists as an invariant manifold $M(a)$ for a near $a$

To simplify, write $T$ for the right-shift operator with $(Tu)_j = u_{j+1}$ and note

$$p^{(e)} = Tp^{(e+1)}$$

$\forall e \in \mathbb{R}$. After factoring out the symmetry $T$, the manifold $M(a_x)$ is a circle of equilibria (really $M(a_x)/T$, see (i))

(i) every point represents an equivalence class $p^{(e)}$, $e \in \mathbb{R}$, all equilibria

(ii) one or more equilibria could survive for $a \neq a_x$

(iii) if (7) is a normal family equilibria don't survive
In theory some equilibria could survive for a finite time. But by computing flow on $M(a)$ to leading order, show situation in (iii) holds, so that the traveling waves described in 1.1 can be read off from the shift-periodic soln to (7) induced by the flow on $M(a)$.

In final section give 3 examples:
- normal family (no prop. fail)
- non-normal family w/o prop. fail
- non-normal family w/ prop. fail
Crystalllographic Pinning

Now consider higher dimensional lattices, for example:

\[ \dot{u}_{i,j} = (\Delta u)_{i,j} - F(u_{i,j}, a) \quad (i,j) \in \mathbb{Z}^2 \]

where \( \Delta \) is the discrete laplacian in \( \mathbb{Z}^2 \):

\[ (\Delta u)_{i,j} = u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} \]

Tuns out will again see propagation failure (pinning), but now direction matters.

(very) Loosely, M-P 1997 showed that for rational propagation directions \( \Theta \) (where rational means if \( x = \cos \Theta \), \( y = \sin \Theta \), then \( \frac{x}{y} \) is rational or infinite) we have a nontrivial pinning region, whereas for \( \Theta \) irrational the pinning region is trivial, as long as the nonlinearity \( F \) is sufficiently close to a sawtooth function. EMP proved directly in 1998 paper with Cahn and Van Vleck for sawtooth for itself.

In 2008, Hoffman and M-P showed that crystalllographic pinning occurs in the horizontal (or vertical) direction for almost all \( F \) with properties similar to our familiar bistable cubic nonlinearity (will of course make precise).
Focus on traveling wave solutions of LDEs posed on $\mathbb{Z}^2$.

Def. The nonlinearity $f$ is of bistable type if
\[ f(\pm 1) = 0 \quad f(a) = 0 \quad f'(\pm 1) > 0 \quad f'(a) < 0 \]
\[ f(u) > 0 \text{ for } u \in (-1, a) \cup (1, +\infty) \]
\[ f(u) < 0 \text{ for } u \in (-\infty, -1) \cup (a, 1) \]
for some $a \in (-1, 1)$.

Further consider a family of bistable functions $f = f(\cdot, a)$ parameterized by $a \in (-1, 1)$ satisfying the monotonicity condition
\[ \frac{df}{da}(u, a) > 0 \text{ for } u \in (-1, 1), \quad a \in (-1, 1) \]
(2)

Def. Let $N$ be the set of functions
\[ N = \{ f : [-1, 1] \times (-1, 1) \rightarrow \mathbb{R} : f(\cdot, \cdot) \text{ is } C^2 \text{ smooth with } f(\cdot, a) \text{ satisfying (1) for every } (u, a) \in [-1, 1] \times (-1, 1), \text{ and (2) holding}\} \]
$f$ is a normal family if $f \in N$.

Consider in particular the lattice $\mathbb{Z}^2$ and the system
\[ \dot{u}_{i,j} = (\Delta u)_{i,j} - f(u_{i,j}, a) \]
(3)

A traveling wave solution of (3) is of the form
\[ u_{i,j}(t) = \phi(iK + j\sigma - ct) \]
(4)
where \((k, \sigma) \in \mathbb{R}^2/\{0, 0\}\) is the direction vector, \(c\) is the wave speed, \(\phi\) the wave profile.

Substituting (4) in (3) we have

\[-c \phi'(\frac{\sigma}{c}) = \phi(\frac{\sigma}{c} + \frac{k}{c}) + \phi(\frac{\sigma}{c} - \frac{k}{c}) + \phi(\frac{\sigma}{c} + (\frac{k}{c}) - \frac{k}{c}) - 4 \phi(\frac{\sigma}{c}) - f(\phi(\frac{\sigma}{c}), a)\]

(5)

From M-P Global Struct. of Traveling Waves, we have that for each direction vector \((k, \sigma) \in \mathbb{R}^2/\{0, 0\}\) and each \(a \in (-1, 1)\) there exists a unique wave speed \(c = c(a, (k, \sigma))\) s.t. (5) admits a monotone soln satisfying the boundary conditions

\[\phi(-\infty) = -1, \quad \phi(+\infty) = 1\]

(6)

Moreover for \(c > 0\), this soln \(\phi = \phi(\frac{\sigma}{c}, a, (k, \sigma))\) is unique up to translation. The wavespeed \(c(a, (k, \sigma))\) depends continuously on \(a\) and \((k, \sigma)\). For each \((k, \sigma)\) it is nondecreasing in \(a\), smooth in \(a\), and satisfies \(\frac{dc(a, (k, \sigma))}{da} > 0\) when \(c(a, (k, \sigma)) \neq 0\). Also we have quantities \(a_\pm = a_\pm(k, \sigma)\) characterized by

\[\text{dom } a_\pm(k, \sigma) = \{a \in (-1, 1) : c(a, (k, \sigma)) > 0\}\]

Since we can check by rescaling \(\frac{\sigma}{c}\) by \(r\) that

\[a_\pm(r \cos \theta, r \sin \theta) = a_\pm(\cos \theta, \sin \theta)\]

\(r > 0\), we can write \(a_\pm(\theta)\) with slight abuse of notation. Now focus on \(a_+ (\theta)\) (though could do same for \(a_- (\theta)\)).
\( a_+(\theta) \) need not depend continuously on \( \theta \), but it is upper semi-continuous in \( \theta \), i.e.,
\[
\limsup_{\theta \to \theta_0} a_+(\theta) \leq a_+(\theta_0)
\]
(7)

For every \( \theta_0 \).

**Def.** Crystalllographic pinning occurs for \((3)\) in the direction of \( \theta_0 \) if the inequality in \((7)\) is strict or the analogous inequality for \( a_- (\theta) \) is strict.

The goal of paper is to give conditions on \( f \) s.t. crystalllographic pinning occurs for \( \theta_0 = 0 \) [\((k, \theta) = (1, 0)\)].

Use two conditions:

**HA** \( \exists p \in l^\infty(\mathbb{Z}) \) denoted \( p = \{ p_n \}_{n \in \mathbb{Z}} \) satisfying
\[
P_{n+1} + P_{n-1} - 2P_n = f(p_n, a_+(0)) \quad \forall n \in \mathbb{Z}
\]
and which also satisfies the boundary conditions:
\[
\lim_{n \to \pm \infty} P_n = \pm 1, \quad P_{n+1} = P_{n-1} \quad \forall n \in \mathbb{Z}
\]
Moreover, \( p \) is unique up to a shift in the index \( n \).

**HB** \( A \) holds. Further, if \( \nu \in l^\infty(\mathbb{Z}) \) satisfies
\[
\nu_{n+1} + \nu_{n-1} - 2\nu_n = f'(p_n, a_+(0))\nu_n \quad \forall n \in \mathbb{Z}
\]
where \( p \) is as in \( A \) and \( f'(0, a) = \frac{df(u, a)}{du} \), then
\[
B = \frac{1}{2} \sum_{n=-\infty}^{\infty} f''(p_n, a_+(0))\nu_n^3
\]
(11)
Satisfies \( B \neq 0 \).
Before stating main results, pause to see what these conditions mean / that they make sense.

First note (8) is the wave profile eqn with $c = 0$ and $(k, \sigma) = (1, 0):

$$-c \phi'(\xi) = \phi(\xi + \sigma) + \phi(\xi - \sigma) + \phi(\xi + k) + \phi(\xi - k) - 4 \phi(\xi) - f(\phi(\xi), a)$$

$$G = \phi(i) + \phi(i) + \phi(i + 1) + \phi(i - 1) - 4 \phi(i) - f(\phi(i), a)$$

$p_i = \phi(i)$

$$p_{i+1} + p_{i-1} - 2p_i = f(p_i, a)$$

Prop 1.8 Assume $c(a, (1, 0)) = 0$, so $a \in [a_-, a_+]$. Then there exists $p \in L^\infty$ satisfying

$$p_{n+1} + p_{n-1} - 2p_n = f(p_n, a) \forall n \in \mathbb{Z}$$

along with (9). Moreover, any such monotone $p$ is strictly monotone, i.e., $p_n < p_{n+1} \forall n \in \mathbb{Z}$.

(easy)

Pf Existence & monotonicity follow from previous comments.

To show strict monotonicity, suppose $p_{m+1} = p_m$ for some $m \in \mathbb{Z}$.

Then $f(p_m, a) = p_{m-1} - p_m \leq 0$ by monotonicity.

But also $f(p_m, a) = f(p_{m+1}, a) = p_{m+2} - p_{m+1} \geq 0 \Rightarrow f(p_m, a) = 0$.

Thus $p_{m-1} = p_{m} = p_{m+1} = p_{m+2}$, and continuing in this fashion we see $p_n$ is constant in $n$, which violates the fact that it is a heteroclinic connection between -1 and 1.

Now consider (4B)...
(10) can be expressed with the operator $L \in \mathcal{L}(l^p(\mathbb{Z}))$ given by

$$L = S + S^* - 2I - f'(p, a + (0))$$  \hspace{1cm} (12)

where $S \in \mathcal{L}(l^p(\mathbb{Z}))$ is the shift operator

$$(SX)_n = x_{n+1} \text{ for } X = \{x_n\}_{n \in \mathbb{Z}} \in l^p(\mathbb{Z})$$

and we denote by $f'(p, a + (0)) \in \mathcal{L}(l^p(\mathbb{Z}))$ the operator with entries $f'(p_n, a + (0))$ on the diagonal.

Prop 1.4: Assume (H4) with $p$ as stated there, and $L \in \mathcal{L}(l^p(\mathbb{Z}))$ as above. Then there exists $v \in l^p(\mathbb{Z}) \setminus \{0\}$ satisfying (10), i.e. $Lv = 0$. $v$ is unique up to a scalar multiple, and thus

$$\ker (L) = \{av : a \in \mathbb{R}\}$$

Further, $v$ can be chosen to satisfy

$$v_n > 0, \text{ } n \in \mathbb{Z}$$

and has the coordinatewise estimate

$$v_n \leq K|n|^{\mu - 1}, \text{ } n \in \mathbb{Z}$$  \hspace{1cm} (13)

for some $K > 0$, $0 < \mu < 1$. Thus $v \in L'(\mathbb{Z})$ and we may normalize $v$ to satisfy $<v, v> = 1$ where $<\cdot, \cdot>$ denotes the dot product (duality) between $L'(\mathbb{Z})$ and $l^p(\mathbb{Z})$. The operator $L$ is Fredholm with index zero, and range

$$\text{ran}(L) = \{w \in l^p(\mathbb{Z}) : <v, w> = 0\}$$

and its spectrum satisfies $\sigma(L) \cap (0, \infty) = \emptyset$.

[Proof later in paper.] Note estimate (13) implies sum in def. of $B$, as given in (11), is absolutely convergent.
Now state the main results:

**Thm 1.1** Assume (**HB**). Then the inequality (7) is strict at \( \Theta_0 = 0 \) and so crystallographic pinning occurs in the direction \( \Theta_0 = 0 \).

**Thm 1.2** (**HB**) is generic in the following sense: Fix any \( f_0 \in \mathcal{N} \) and define the set

\[
C^2_+ = \{ \gamma \in C^2[-1,1] : \gamma(u) > 0 \text{ for every } u \in [-1,1] \}
\]

noting that for every \( \gamma \in C^2_+ \), we have \( f \in \mathcal{N} \), where \( f(u,a) = \gamma(u) f_0(u,a) \). Let \( C^2_+ \) be endowed with the usual \( C^2 \) topology, so making it an open subset of the Banach space \( C^2[-1,1] \).

Then the set \( G(f_0) \subseteq C^2_+ \), defined as

\[
G(f_0) = \{ \gamma \in C^2_+ : \gamma f_0 \text{ satisfies (**HB**)} \}
\]

is a residual subset of \( C^2_+ \).

[residual in sense of Baire category theorem: complement of a meager set, i.e., countable union of nowhere dense sets]