Multi-scale discontinuous Galerkin method for solving elliptic problems with curvilinear unidirectional rough coefficients

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Abstract

In this paper, we propose a multi-scale discontinuous Galerkin (DG) method for second-order elliptic problems with curvilinear unidirectional rough coefficients by choosing a special non-polynomial approximation space. The key ingredient of the method lies in the incorporation of the local oscillatory features of the differential operators into the approximation space so as to capture the multi-scale solutions without having to resolve the finest scales. The unidirectional feature of the rough coefficients allows us to construct the basis functions of the DG non-polynomial approximation space explicitly, thereby greatly increasing the algorithm efficiency. Detailed error estimates for two-dimensional second-order DG methods are derived, and a general guidance on how to construct such non-polynomial basis is discussed. Numerical examples are also presented to validate and demonstrate the effectiveness of the algorithm.

Key words: multi-scale method, discontinuous Galerkin method, non-polynomial basis, rough coefficients, composite material.

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1 Introduction

In this paper, we propose a multi-scale DG method for a special class of two dimensional second order elliptic boundary value problems

$$-\nabla \cdot (A(\mathbf{x})\nabla u) = f(\mathbf{x}) \quad \text{in } \Omega \tag{1.1}$$

with Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded computational domain, f is a function in $L^2(\Omega)$ and $A(\mathbf{x})$ is a special diagonal coefficient matrix containing a small scale ϵ

$$A(\mathbf{x}) = \begin{pmatrix} a^\epsilon(x,y) & 0 \\ 0 & a^\epsilon(x,y) \end{pmatrix},$$

or equivalently,

$$-(a^\epsilon(x,y)u_x)_x - (a^\epsilon(x,y)u_y)_y = f(x,y). \tag{1.2}$$

As the typical situation in multi-scale modeling, we assume the elliptic coefficients $a^\epsilon(x,y)$ are highly oscillatory functions involving a small scale ϵ. Furthermore, $a^\epsilon(x,y)$ belongs to $L^\infty(\Omega)$ and satisfies

$$0 < \alpha \leq a^\epsilon(x,y) \leq \beta < \infty \tag{1.3}$$

for any $(x,y) \in \Omega$, where α and β are constants independent of ϵ. In particular, we are interested in a special subclass of rough coefficients, in which a^ϵ is curvilinear unidirectional rough, as defined originally by Babuška et al. in [6], and to be specified more precisely in Subsection 2.1.

Applications of this type of problems arise in modeling two phase flow in porous media (see [20, 17, 9]), where Eq. (1.1) is the pressure equation, with u and $A(\mathbf{x})$ representing the pressure and the relative permeability tensor, respectively. In particular, $A(\mathbf{x})$ would be a diagonal tensor if the stochastic permeabilities are upscaled.

Eq. (1.1) also captures the mechanics of steady state heat (electrical) conduction through a composite material (e.g. [17]), where $A(\mathbf{x})$ and u describe the thermal (electronic) conductivity and temperature (electronic potential).

It is a well known fact that if the coefficient $a^\epsilon(x, y)$ is rough, the solution to (1.2) will also be rough; to be more specific, we will in general have

$$||a^\epsilon||_{H^1(\Omega)} \to \infty, \qquad ||u||_{H^2(\Omega)} \to \infty, \qquad \text{as } \epsilon \to 0.$$

Thus, we will not have enough regularity in u with respect to ϵ, i.e., u is not bounded uniformly with respect to ϵ in $H^2(\Omega)$ or in $H^{1+\delta}(\Omega)$ for any $\delta > 0$. As a result, it requires a computational mesh size h at least comparable to ϵ to obtain reasonable resolutions and to observe any convergence rate. This intrinsic difficulty prohibits the application of traditional numerical methods like finite-element and finite-difference, due to the limitations of available computational resources, e.g. memory and CPU time.

There have been a lot of efforts to overcome or bypass this numerical difficulty. As early as in the 60s, Tikhonov and Samarskii [21] (see also [16]) designed a simple 3-point finite difference scheme which can solve the one-dimensional version of (1.2) exactly at the grid points by utilizing harmonic averages and the special solution structure. However, this method is difficult to extend to multi-dimensions. Later, various multi-scale finite element methods have been developed and extensively studied in the literature, including [7, 15, 6, 17, 18, 13, 14, 9, 24, 11, 4, 19, 1, 2, 3] and also the review book [12] etc. The universal idea therein is to have the local properties of the differential operators built in to the scheme and, as a result, allowing adequate resolutions on coarser meshes. In particular, the special finite element methods proposed by Babuška et al. in [6] provide an efficient and convenient framework of constructing local trial spaces for this specific subclass of two dimensional elliptic problem like Eq. (1.2) concerned in this paper. The key ingredients of special finite element methods in [7, 15, 6] can be summarized as:

- A regularity result for the rough solution in a transformed space with respect to the right hand side function, $f \in L^2(\Omega)$.

- Construction of trial spaces with good approximation properties, in particular, they utilized special shape functions for which the explicit constructions of local multi-scale approximation basis are available, thus making the whole algorithm more efficient and easy to implement.

- Construction of reasonable test spaces to ensure stability result.

By exploiting the special features of the oscillatory coefficients, the special finite element method is proven to yield second order convergence even if the coefficient a^ϵ is rough. Notice that for this particular subclass of two dimensional elliptic equations, unlike the method based on the homogenization theory, no assumptions as periodicity of a^ϵ or scale separation are imposed. The coefficients as well as the solution u can then have a continuum scale spectrum from $O(\epsilon)$ to $O(1)$. For more general cases of the coefficients, though the general idea of incorporating the oscillatory basis into the approximation spaces still applies, such regularity result which is the key step in the procedure above in general no longer holds. Local problems would have to be first solved on a finer grid in order to obtain an approximation for the oscillatory basis, thus significantly reducing the efficiency of the multi-scale numerical methods. More details regarding a general framework for error analysis and related applications could be found in a series of work [17, 18, 13, 9] by Hou et al.

One of the difficulties for designing higher order schemes for multi-dimensional continuous Galerkin method is to make the elements conforming. It would be appealing if one can extend the results in [7, 6] to discontinuous Galerkin (DG) methods which would not enforce continuity constraints across the element interfaces, and thus automatically overcome the difficulties of requiring the elements to be conforming for multi-dimensions of the continuous Galerkin method and the practical inconvenience

of using curvilinear elements. Motivated by [7, 15, 6], Yuan and Shu proposed a multi-scale Babuška-Zlámal DG method with a non-polynomial approximation DG space [24, 23] for the one dimensional version of Eq. (1.2), and they proved arbitrary high order error estimates. Later, in [22], Wang improved the one-dimensional proof of the multi-scale Babuška-Zlámal DG method by only assuming that the solution is uniformly bounded with respect to ϵ in $H^1(\Omega)$. More regularity was assumed in [24] while proving estimates for the multi-scale Babuška-Zlámal DG method. Followed by previous work, in [25], Wang et al. proposed a multi-scale symmetric interior penalty DG (IP-DG ([10, 26, 5])) method. This is proved to be uniformly high order accurate, with respect to the small scale ϵ in one space dimension, even in the case where there is no separation of scales (the problem could have a continuum of scales from $O(\epsilon)$ to $O(1)$) and the mesh size h is much larger than ϵ. Optimal order convergence for second order scheme in two space dimension given unidirectional oscillations was also proven.

As for the curvilinear unidirectional rough coefficients, in this paper, we extend the idea in [6] to multi-scale IP-DG [25] framework in two space dimension. In particular, we prove uniform optimal accuracy for a second order approximation space in the two-dimensional case, without assuming any properties such as periodicity or scale separation in the rough coefficients. Although the local mapping might put some constraints on the computational geometry, this turns out to be non-essential restrictions required only for error analysis. An easy implementation framework is also suggested for the construction of a local basis, assuming that the leading oscillatory direction does not change abruptly within each element. We remark that one of the difficulties in practice for multi-dimensional special finite element methods is that the methods are designed for curvilinear elements. It is ideal for the purpose of error estimates, but would cause a lot of inconvenience on the implementation side. On the other hand, for the multi-scale IP-DG methods, we could use regular triangulations

instead. Of course, there is a price to pay for this flexibility: we must carefully analyze the errors associated with these discontinuities across element interfaces, to obtain error estimates.

The paper is organized as follows: in Section 2, we first propose the IP-DG scheme for general nonlinear elliptic equations and discuss the construction of special approximation spaces to cope with curvilinear unidirectional rough coefficients. In Section 3, optimal error estimates are proven for a second order approximation space regardless of the fine scales. Numerical examples and a new way of implementation are presented in Section 4. Concluding remarks are given in Section 5.

2 Multi-scale DG method: the methodology

In this section, we focus on solving the elliptic multi-scale problem (1.2) on a bounded domain Ω, with additional features to be specified in Subsection 2.1. Let $\mathcal{T}_h$ be a collection of *quasi-uniform rectangular* partitions of Ω and $\mathcal{E}_h$ be the collection of edges of the $\mathcal{T}_h$. Define the following inner products

$$(v, w)_{\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} \int_K v(x, y)\, w(x, y) dx\, dy,$$
$$\langle v, w \rangle_{\mathcal{E}_h} = \sum_{e \in \mathcal{E}_h} \int_e v(s)\, w(s)\, ds$$

where K denotes a given element in the triangulation, and e represents the edges associated with it.

There are many different DG formulations for such elliptic equations. To fix ideas, in this paper we focus on the IP-DG formulation for (1.2): for any test function $v_h \in V_h$, the IP-DG scheme is to find $u_h \in V_h$, such that

$$B_h(u_h, v_h) = (f, v_h)_{\mathcal{T}_h} \quad \forall v \in V_h, \tag{2.1}$$

where the bilinear form $B_h(u_h, v_h)$ is defined as

$$B_h(u, v) := (A\nabla u, \nabla v)_{\mathcal{T}_h} - \langle \{\!\{A\nabla u\}\!\}, [\![v]\!] \rangle_{\mathcal{E}_h} - \langle \{\!\{A\nabla v\}\!\}, [\![u]\!] \rangle_{\mathcal{E}_h} + \frac{\eta}{h} \langle [\![u]\!], [\![v]\!] \rangle_{\mathcal{E}_h}$$

where η is a sufficiently large penalty coefficient to ensure stability. For the scalar valued function u, $\{\!\{u\}\!\}$, $[\![u]\!]$ represent the average and jump across the element interface, respectively. To be more precise, let e be an interior edge shared by elements K_1 and K_2, define the unit normal vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ on e pointing exterior to K_1 and K_2, respectively. We set

$$\{\!\{u\}\!\} = \frac{1}{2}(u_1 + u_2), \quad [\![u]\!] = u_1\boldsymbol{n}_1 + u_2\boldsymbol{n}_2 \quad \text{on } e \in \mathcal{E}_h^o, \tag{2.2}$$

with $u_i := u|_{\partial K_i}$, and $\mathcal{E}_h^o$ is the set of interior edges e. In a similar fashion, for a vector-valued function $\boldsymbol{q}$, we define $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ analogously and set

$$\{\!\{\boldsymbol{q}\}\!\} = \frac{1}{2}(\boldsymbol{q}_1 + \boldsymbol{q}_2), \quad [\![\boldsymbol{q}]\!] = \boldsymbol{q}_1 \cdot \boldsymbol{n}_1 + \boldsymbol{q}_2 \cdot \boldsymbol{n}_2 \quad \text{on } e \in \mathcal{E}_h^o. \tag{2.3}$$

For $e \in \mathcal{E}_h^\partial$, the set of boundary edges, we set

$$[\![u]\!] = u\boldsymbol{n}, \quad \{\!\{\boldsymbol{q}\}\!\} = \boldsymbol{q} \quad \text{on } e \in \mathcal{E}_h^\partial, \tag{2.4}$$

where $\boldsymbol{n}$ is the outward unit normal. We do not require either of the quantities $\{\!\{u\}\!\}$ or $[\![\boldsymbol{q}]\!]$ on boundary edges, and leave them undefined.

2.1 Curvilinear unidirectional rough coefficients

As Babuska et al. suggested in [6], the strict definition of "locally varying sharply in at most one direction" is outlined as follows. For the two dimensional elliptic equation (1.2) on the domain Ω, assume the existence of the following local change of variables, such that

- There exists an open cover $\{\Omega_i, i = 1, \ldots, n\}$ of Ω, with $\cup_{i=1}^n \Omega_i = \Omega$, and $\cup_{i=1}^n (\bar{\Omega}_i \cap \partial\Omega) = \partial\Omega$.
- For each Ω_i, there exists a local mapping: (ξ_i, η_i): $\Omega_i \to \tilde{\Omega}_i$ such that,
  - The mapping is one-to-one and onto, with $|\frac{\partial(\xi_i,\eta_i)}{\partial(x,y)}| \geq \gamma_i > 0$ on Ω_i,
  - $\nabla\xi_i \cdot \nabla\eta_i = 0$, and $\tilde{\Omega}_i$ is rectangular,

– The boundary of each open set Ω_i is constraint by the boundaries of the domain Ω:

$$\begin{aligned}\bar{\Omega}_i \cap \partial\Omega &= \emptyset, \quad \text{if all the edges of } \Omega_i \text{ are interior edges} \\ &= \text{boundary edges of } \Omega_i, \text{ and remaining are interior edges}\end{aligned}$$

- Under the mapping, $a^\epsilon(x,y) = \tilde{a}_i^\epsilon(\xi_i)$, $\forall (x,y) \in \Omega_i$.

where in general we define for any function w defined on Ω

$$\tilde{w}(\xi(x,y), \eta(x,y)) = w(x,y).$$

Under such conditions, the elliptic problem (1.2) is said to have curvilinear unidirectional coefficients. In other words, the local oscillation direction changes smoothly (no ϵ-scale in this change of oscillation direction) within the computational domain Ω. Furthermore, in [6], the authors proved a generalized version of the classical Bernstein theorem [8] for this local change of variables, which we quote here:

Theorem 2.1. ([6]) Let u be the solution in H_0^1 of (1.2), where we assume $f \in L^2(\Omega)$ and a^ϵ is locally varying sharply in at most one direction. Let $O_i \subset \Omega_i$ be open and satisfying $O_i \subset\subset \Omega_i$ if $\bar{O}_i \cap \partial\Omega = \emptyset$, and $(\partial O_i \cap \partial\Omega_i) \subset \partial\Omega$ if $\bar{O}_i \cap \partial\Omega \neq \emptyset$. Let $\tilde{O}_i$ be the image of O_i under the mapping (ξ_i, η_i). Then there is a constant C which does not depend on ϵ, such that

$$\int_{\tilde{O}_i} \left\{ \tilde{a}_i^\epsilon \left| \frac{\partial}{\partial \xi_i} (\tilde{a}_i^\epsilon \frac{\partial \tilde{u}}{\partial \xi_i}) \right|^2 + \tilde{a}_i^\epsilon \left| \frac{\partial^2 \tilde{u}}{\partial \xi_i \partial \eta_i} \right|^2 + \frac{1}{\tilde{a}_i^\epsilon} \left| \frac{\partial^2 \tilde{u}}{\partial \eta_i^2} \right|^2 \right\} d\xi_i d\eta_i \leq C ||f||_{L^2(\Omega)} \tag{2.5a}$$

The constant in the above result depends on the distance between O_i and $\partial\Omega_i \backslash \partial\Omega$. Hence, we assume that for each Ω_i in the above covering we can can pick an O_i with the following properties.

1) $O_i \subset \Omega_i$ for $i = 1, \ldots, n$.

2) $\text{dist}(O_i, \partial\Omega_i \backslash \partial\Omega) > 0$ for $i = 1, \ldots, n$.

3) $\Omega = \bigcup_{i=1}^{n} O_i$

4) Each $K \in \mathcal{T}_h$ is completely contained in at least one O_i.

Remark 2.2. As suggested in [6], the result can be generalized to coefficients which could locally be transformed into the form $\tilde{a}^\epsilon(\xi, \eta)$ where the dependency of the function $\tilde{a}^\epsilon$ on η is independent of the small scale ϵ. For example, $\tilde{a}^\epsilon(\xi, \eta) = \tilde{a}(\frac{\xi}{\epsilon}, \eta)$.

Remark 2.3. One may notice that due to the definition of curvilinear unidirectional coefficients and properties of the associated local change of coordinates specified above, some constraints may apply to the computational domain. For example, if we consider coefficients of the specific form $a^\epsilon(x, y) = \dfrac{1}{4 + \cos(\frac{x^2+y^2}{\epsilon})}$, the convenient coordinates mapping would be the polar one:

$$\xi = \sqrt{x^2 + y^2}, \quad \eta = \tan^{-1}(\frac{y}{x})$$

Thus a quarter torus domain such as

$$1 \leq x^2 + y^2 \leq 4, \quad 0 \leq \eta \leq \frac{\pi}{2}$$

would satisfy the requirement that the transformed domain $\tilde{\Omega}$ is rectangular in (ξ, η). However, the definition suggests that a square domain like $[-1, 1] \times [-1, 1]$ will be ruled out. We notice that this restriction is only required for the regularity result above, which simplifies the error analysis of the multi-scale DG scheme. In Section 4, extensive numerical results suggest that this requirement is not essential. We still observe an optimal convergence rate for the second order multi-scale DG scheme, even if the constraints associated with the local mapping are not strictly satisfied.

2.2 Approximation spaces

In this subsection, we first propose a way to construct the approximation space motivated by [6]. For each element $K_j \in \mathcal{T}_h$, the local oscillating approximation space is obtained as follows:

- Find $O_i \subset \Omega_i$ such that $K_j \subset O_i$ and the local mapping (ξ_i, η_i) associated with Ω_i.

- Define the second order local approximation space on K_j as:

$$V_j^1 = \left\{ v : v|_{K_j} \in \text{span} \left\{ 1, \int_{\xi_j^*}^{\xi_i} \frac{1}{\tilde{a}_i^\epsilon(s)} ds, \eta_i - \eta_j^* \right\} \right\}.$$

  Here $\xi_j^* = \xi_j(x_j^*, y_j^*), \eta_j^* = \eta_j(x_j^*, y_j^*)$ where (x_j^*, y_j^*) is the centroid of K_j.

- The multi-scale DG approximation space on Ω is $S_h^1 = \bigcup_{j=1}^N V_j^1$.

- Likewise, a higher order approximation space can be defined by including more basis function in, e.g. $S_h^2 = \bigcup_{j=1}^N V_j^2$, where

$$\begin{aligned} V_j^2 \qquad = \left\{ v : v|_{K_j} \in \text{span} \left\{ 1, \int_{\xi_j^*}^{\xi_i} \frac{1}{\tilde{a}_i^\epsilon(s)} ds, \eta_i - \eta_j^*, \int_{\xi_j^*}^{\xi_i} \frac{s}{\tilde{a}_i^\epsilon(s)} ds, \right.\right. \\ \left.\left. (\eta_i - \eta_j^*) \int_{\xi_j^*}^{\xi_i} \frac{1}{\tilde{a}_i^\epsilon(s)} ds, (\eta_i - \eta_j^*)^2 \right\} \right\}. \end{aligned}$$

This construction is vital to the success of the multi-scale DG method, which can be expected to yield the same convergence rate as the usual polynomial based DG methods when the coefficient a is smooth, while achieves greatly improved accuracy on a coarse mesh when a is rough. An easy implementation of the basis construction is proposed in Subsection 4.4, to address the cases where the local mapping is not obvious or directly available in an analytical form. Numerical experiments show that this practical implementation achieves optimal order of convergence with a second order approximation space.

3 Multi-scale DG method: the error estimates

Combining all the pieces together, we have completely defined the multi-scale DG method for solving two dimensional elliptic equations (1.2) with curvilinear unidirectional coefficients. In this section, we proceed to analyze the theoretical aspects

of the multi-scale DG methods. The first result to prove is the interpolation error associated with the special approximation space S_h^1 defined in the previous section.

3.1 Error estimates of the interpolation error

We first define the following straightforward energy norm

$$|||v|||^2 := (A\nabla v, \nabla v)_{\mathcal{T}_h} + \frac{1}{h}\langle [\![v]\!], [\![v]\!]\rangle_{\mathcal{E}_h} + h\langle \{\!\{A\nabla v\}\!\}, \{\!\{A\nabla v\}\!\}\rangle_{\mathcal{E}_h}.$$

Meanwhile, given $K_j \in \mathcal{T}_h$, find the containing partition Ω_i, such that $K_j \subset O_i \subset \Omega_i$. Suppose the corresponding local mapping (ξ_i, η_i) satisfies all the conditions specified in Subsection 2.1. Denote the images of K_j, Ω_i with such mapping as $\tilde{K}_j$, $\tilde{\Omega}_i$ respectively. Then define a further mapping as follows:

$$\begin{aligned}\hat{\xi}_i :=& \int^{\xi_i} \frac{1}{\tilde{a}_i^\epsilon(s)}\, ds\\ \hat{\eta}_i :=& \eta_i.\end{aligned}$$

which maps the element $\tilde{K}_j \in \tilde{\mathcal{T}}_h$ to $\hat{K}_j \in \hat{\mathcal{T}}_h$. Accordingly, for any function u defined on K_j, we could define functions $\tilde{u}$ on $\tilde{K}_j$ and $\hat{u}$ on $\hat{K}_j$ separately as

$$\tilde{u}(\xi_i, \eta_i) = \hat{u}(\hat{\xi}_i, \hat{\eta}_i) = u(x, y).$$

Before preceding to a proof of the interpolation error in the energy norm, we state and prove the following two lemmas, one certifying the equivalence of L^2-estimates after changing coordinates, the other assuring the trace inequality estimates, both regarding the transformed "hat" domain.

Lemma 1. Given K_j and the local mapping $(\hat{\xi}_i, \hat{\eta}_i)$ defined above, then for any function $w \in H^1(K_j)$, the following inequality holds

$$\|w\|^2_{L^2(K_j)} \le C'\|\hat{w}\|^2_{L^2(\hat{K}_j)} \tag{3.1}$$

$$\int_{K_j} A\nabla w \cdot \nabla w dx\, dy \le C\|\hat{\nabla}\hat{w}\|^2_{L^2(\hat{K}_j)} \tag{3.2}$$

where C' and C are constants depending only on $\alpha, \beta, \xi_i, \eta_i$.

Proof. (3.1) can be proved by straightforward calculus:

$$\|w\|^2_{L^2(K_j)} = \int_{\hat{K}_j} \hat{w}^2 \left| \frac{\partial(x,y)}{\partial(\hat{\xi}_i, \hat{\eta}_i)} \right| d\hat{\xi}_i \, d\hat{\eta}_i \leq C' \|\hat{w}\|^2_{L^2(\hat{K}_j)}$$

Similar for (3.2)

$$\begin{aligned}
\int_{K_j} A\nabla w \cdot \nabla w dx \, dy = & \int_{\tilde{K}_j} \left[\tilde{a}_i^\epsilon (\frac{\partial}{\partial \xi_i} \tilde{w})^2 |\nabla \xi_i|^2 + \tilde{a}_i^\epsilon (\frac{\partial}{\partial \eta_i} \tilde{w})^2 |\nabla \eta_i|^2 \right] |\frac{\partial(x,y)}{\partial(\xi_i, \eta_i)}| d\xi_i \, d\eta_i \\
\leq & C \int_{\tilde{K}_j} \left[\tilde{a}_i^\epsilon (\frac{\partial}{\partial \xi_i} \tilde{w})^2 + \tilde{a}_i^\epsilon (\frac{\partial}{\partial \eta_i} \tilde{w})^2 \right] d\xi_i \, d\eta_i \\
\leq & C \int_{\hat{K}_j} \left[(\frac{\partial}{\partial \hat{\xi}_i} \hat{w})^2 + (\tilde{a}_i^\epsilon \frac{\partial}{\partial \hat{\eta}_i} \hat{w})^2 \right] d\hat{\xi}_i \, d\hat{\eta}_i \\
\leq & C \|\hat{\nabla} \hat{w}\|^2_{L^2(\hat{K}_j)}
\end{aligned}$$

□

Lemma 2. For each element K_j and the associated mapping, given any function $w \in C^1(K_j)$, the following trace inequality holds:

$$\|\hat{w}\|_{L^2(\partial \hat{K}_j)} \leq C_1 h_{\hat{K}_j}^{-\frac{1}{2}} \|\hat{w}\|_{L^2(\hat{K}_j)} + C_2 h_{\hat{K}_j}^{\frac{1}{2}} \|\hat{\nabla} \hat{w}\|_{L^2(\hat{K}_j)}$$

where $h_{\hat{K}_j} = \text{diam}(\hat{K}_j)$. Note that both constants C_1 and C_2 do not depend on the small scale ϵ.

Proof. Denote M as the mapping defined on $\mathbf{x} \in \partial K_j$: $M(\mathbf{x}) = \hat{\mathbf{x}} \in \partial \hat{K}_j$. As a result,

$$\hat{w}(\hat{\mathbf{x}}) = (\hat{w} \circ M)(\mathbf{x}) = w(\mathbf{x})$$

Moreover, by variable substitution,

$$\int_{\partial \hat{K}_j} \hat{w}(\hat{\mathbf{x}}) d\hat{\mathbf{x}} = \int_{\partial K_j} (\hat{w} \circ M)(\mathbf{x}) |det(J_M)(\mathbf{x})| d\mathbf{x}$$

where J_M is the Jacobian of the mapping M. It is obvious that there exist constants independent of ϵ, $0 < m_1(\alpha, \beta, \xi_i, \eta_i) \leq m_2(\alpha, \beta, \xi_i, \eta_i)$, such that $m_1 \leq |det(J_M)| \leq m_2$. Therefore, by the trace inequality on K_j and Lemma 1,

$$\|\hat{w}\|_{L^2(\partial \hat{K}_j)} \leq C \|w\|_{L^2(\partial K_j)} \leq C_1 h^{-\frac{1}{2}} \|w\|_{L^2(K_j)} + C_2 h^{\frac{1}{2}} \|\nabla w\|_{L^2(K_j)}$$

$$\leq C_1 h^{-\frac{1}{2}} \|\hat{w}\|_{L^2(\hat{K}_j)} + C_2 h^{\frac{1}{2}} \|\hat{\nabla}\hat{w}\|_{L^2(\hat{K}_j)}$$
$$\leq C_1 h_{\hat{K}_j}^{-\frac{1}{2}} \|\hat{w}\|_{L^2(\hat{K}_j)} + C_2 h_{\hat{K}_j}^{\frac{1}{2}} \|\hat{\nabla}\hat{w}\|_{L^2(\hat{K}_j)}$$

We have used here that there exist constants M_1, M_2, independent of K_j, such that $M_1 h \leq h_{\hat{K}_j} \leq M_2 h$. □

We then proceed to prove the following approximation result.

Lemma 3. Let u be the exact solution of (1.2). Then, there exists $v \in S_h^1$ so that

$$|||u - v||| \leq Ch \|f\|_{L^2(\Omega)}.$$

Proof. Define $\hat{v}$ to be the piecewise linear function defined on $\hat{K}_j$ such that $\hat{v}|_{\hat{K}_j} \in P^1(\hat{K}_j)$ satisfies

$$h_{\hat{K}_j} \|\hat{\nabla}(\hat{u} - \hat{v})\|_{L^2(\hat{K}_j)} + \|\hat{u} - \hat{v}\|_{L^2(\hat{K}_j)} \leq C h_{\hat{K}_j}^2 |\hat{D}^2 \hat{u}|_{L^2(\hat{K}_j)}, \tag{3.3}$$

From the definition of $|||\cdot|||$ we have

$$\begin{aligned} |||u - v|||^2 =& (A\nabla(u - v), \nabla(u - v))_{\mathcal{T}_h} + \frac{1}{h} \langle [\![(u - v)]\!], [\![(u - v)]\!] \rangle_{\mathcal{E}_h} \\ &+ h \langle \{\!\{A\nabla(u - v)\}\!\}, \{\!\{A\nabla(u - v)\}\!\} \rangle_{\mathcal{E}_h} \end{aligned} \tag{3.4}$$

for which each term will be bounded individually.

As for the first term in the energy norm (3.4), by Lemma 1 and the linear approximation results (3.3)

$$\int_{K_j} A\nabla(u - v) \cdot \nabla(u - v) dx\, dy \leq C \|\hat{\nabla}(\hat{u} - \hat{v})\|_{L^2(\hat{K}_j)}^2 \leq C h_{\hat{K}_j}^2 |\hat{D}^2 \hat{u}|_{L^2(\hat{K}_j)}^2.$$

Hence, if we sum over $K_j \subset O_i$ we recover

$$\sum_{K_j \subset O_i} (A\nabla(u - v), \nabla(u - v))_{K_j} \leq (A\nabla(u - v), \nabla(u - v))_{O_i} \leq C h^2 |\hat{D}^2 \hat{u}|_{L^2(\hat{O}_i)}^2. \tag{3.5}$$

Of course, summing over $i = 1, \ldots, n$ we get

$$(A\nabla(u - v), \nabla(u - v))_{\mathcal{T}_h} \leq C h^2 \sum_{i=1}^{n} |\hat{D}^2 \hat{u}|_{L^2(\hat{O}_i)}^2. \tag{3.6}$$

For the second term in the energy norm (3.4), we use a trace inequality to get

$$\frac{1}{h}\langle[\![(u-v)]\!],[\![(u-v)]\!]\rangle_{\mathcal{E}_h} \leq \frac{C}{h^2}\|u-v\|^2_{L^2(\Omega)} + C\|\nabla(u-v)\|^2_{L^2(\Omega)} \tag{3.7}$$

By Lemma 1, the first term in (3.7) satisfies

$$\|u-v\|^2_{L^2(K_j)} \leq C\|\hat{u}-\hat{v}\|^2_{L^2(\hat{K}_j)} \leq Ch^4_{\hat{K}_j}|\hat{D}^2\hat{u}|^2_{L^2(\hat{K}_j)}.$$

Taking the sum over $K_j \subset O_i$ we get

$$\sum_{K_j\subset O_i}\|u-v\|^2_{L^2(K_j)} \leq Ch^4|\hat{D}^2\hat{u}|^2_{L^2(\hat{O}_i)}, \quad \frac{1}{h^2}\|u-v\|^2_{L^2(\Omega)} \leq \sum_{i=1}^{n} Ch^2|\hat{D}^2\hat{u}|^2_{L^2(\hat{O}_i)} \tag{3.8}$$

As for the second term in (3.7), it can be bounded by scaling argument,

$$\|\nabla(u-v)\|^2_{L^2(\Omega)} \leq C\frac{1}{\alpha}(A\nabla(u-v),\nabla(u-v))_{\mathcal{T}_h} \leq \sum_{i=1}^{n} Ch^2|\hat{D}^2\hat{u}|^2_{L^2(\hat{O}_i)}, \tag{3.9}$$

where in the last inequality we use (3.6). Therefore for the middle term in the energy norm (3.4),

$$\frac{1}{h}\langle[\![(u-v)]\!],[\![(u-v)]\!]\rangle_{\mathcal{E}_h} \leq \sum_{i=1}^{n} Ch^2|\hat{D}^2\hat{u}|^2_{L^2(\hat{O}_i)} \tag{3.10}$$

For the last term in the energy norm (3.4), following the mapping argument carried out in Lemma 1 and Lemma 2, we have

$$h\langle\{\!\{A\nabla(u-v)\}\!\},\{\!\{A\nabla(u-v)\}\!\}\rangle_{\mathcal{E}_h} \leq Ch\sum_{K_j}\|\hat{\nabla}(\hat{u}-\hat{v})\|^2_{L^2(\partial\hat{K}_j)} \tag{3.11}$$

By the trace inequality in Lemma 2, it is clear that

$$\begin{aligned} h\sum_{K_j}\|\hat{\nabla}(\hat{u}-\hat{v})\|^2_{L^2(\partial\hat{K}_j)} \leq & C\sum_{\hat{K}_j}\|\hat{\nabla}(\hat{u}-\hat{v})\|^2_{L^2(\hat{K}_j)} + C\,h^2\sum_{\hat{K}_j}\|\hat{D}^2\hat{u}\|^2_{L^2(\hat{K}_j)} \\ \leq & C\,h^2\sum_{\hat{K}_j}\|\hat{D}^2\hat{u}\|^2_{L^2(\hat{K}_j)}. \end{aligned}$$

Thus we bound the third term in the (3.4) as:

$$h\langle\{\!\{A\nabla(u-v)\}\!\},\{\!\{A\nabla(u-v)\}\!\}\rangle_{\mathcal{E}_h} \leq C\,h^2\sum_{i=1}^{n}\|\hat{D}^2\hat{u}\|^2_{L^2(\hat{O}_i)} \tag{3.12}$$

Combine (3.6), (3.10), (3.12) to obtain

$$|||u-v|||^2 \leq \sum_{i=1}^{n} Ch^2 |\hat{D}^2 \hat{u}|^2_{L^2(\hat{O}_i)}.$$

To complete the proof we argue that

$$\sum_{i=1}^{n} |\hat{D}^2 \hat{u}|_{L^2(\hat{O}_i)} \leq C\|f\|_{L^2(\Omega)}. \tag{3.13}$$

As indicated by Theorem 2.1, (3.13) is immediate and this completes the proof.

□

3.2 Coercivity result

Next, we are going to prove the coercivity of the bilinear form, again by referring to Lemma 2.

Lemma 4. The following coercivity result holds

$$|||v|||^2 \leq CB_h(v,v) \quad \forall v \in S_h^1. \tag{3.14}$$

Proof. Let $v \in S_h^1$. Then by the definition of $B_h(\cdot,\cdot)$ we get

$$B_h(v,v) = (A\nabla v, \nabla v)_{\mathcal{T}_h} - 2\langle \{\!\{A\nabla v\}\!\}, [\![v]\!]\rangle_{\mathcal{E}_h} + \frac{\eta}{h}\langle [\![v]\!], [\![v]\!]\rangle_{\mathcal{E}_h}. \tag{3.15}$$

By the arithmetic-geometric mean inequality we get that

$$2\langle \{\!\{A\nabla v\}\!\}, [\![v]\!]\rangle_{\mathcal{E}_h} \leq \frac{1}{\delta h}\langle [\![v]\!], [\![v]\!]\rangle_{\mathcal{E}_h} + h\delta\langle \{\!\{A\nabla v\}\!\}, \{\!\{A\nabla v\}\!\}\rangle_{\mathcal{E}_h}, \tag{3.16}$$

for any $\delta > 0$. Next, we bound $h\delta\langle \{\!\{A\nabla v\}\!\}, \{\!\{A\nabla v\}\!\}\rangle_{\mathcal{E}_h}$. By (3.11), it is immediate that

$$\langle \{\!\{A\nabla v\}\!\}, \{\!\{A\nabla v\}\!\}\rangle_{\mathcal{E}_h} \leq C \sum \|\hat{\nabla}\hat{v}\|^2_{L^2(\partial\hat{K}_j)} \tag{3.17}$$

Invoking Lemma 2, the inverse inequality for the transformed domain,

$$\langle \{\!\{A\nabla v\}\!\}, \{\!\{A\nabla v\}\!\}\rangle_{\mathcal{E}_h} \leq C \sum \|\hat{\nabla}\hat{v}\|^2_{L^2(\partial\hat{K}_j)} \leq Ch^{-1} \sum_{\hat{K}_j} \|\hat{\nabla}\hat{v}\|^2_{L^2(\hat{K}_j)} \tag{3.18}$$

In the last inequality, the higher order derivative term drops out since $\hat{v} \in P^1(\hat{K}_j)$.

Combining this with Lemma 1,

$$h\delta\langle\{\!\{A\nabla v\}\!\},\{\!\{A\nabla v\}\!\}\rangle_{\mathcal{E}_h} \leq C\delta \sum_{\hat{K}_j} \|\hat{\nabla}\hat{v}\|^2_{L^2(\hat{K}_j)} \leq C\delta(A\nabla v,\nabla v)_{\mathcal{T}_h}. \tag{3.19}$$

If we insert this result into (3.16) we get

$$2\langle\{\!\{A\nabla v\}\!\},[\![v]\!]\rangle_{\mathcal{E}_h} \leq \frac{1}{\delta h}\langle[\![v]\!],[\![v]\!]\rangle_{\mathcal{E}_h} + C\delta(A\nabla v,\nabla v)_{\mathcal{T}_h}. \tag{3.20}$$

Using (3.15) to follow that

$$\begin{aligned} B_h(v,v) \geq & (A\nabla v,\nabla v)_{\mathcal{T}_h} + \frac{\eta}{h}\langle[\![v]\!],[\![v]\!]\rangle_{\mathcal{E}_h} - \frac{1}{\delta h}\langle[\![v]\!],[\![v]\!]\rangle_{\mathcal{E}_h} \\ & - C\delta(A\nabla v,\nabla v)_{\mathcal{T}_h} \\ = & (1-C\delta)(A\nabla v,\nabla v)_{\mathcal{T}_h} + (\frac{\eta}{h} - \frac{1}{\delta h})\langle[\![v]\!],[\![v]\!]\rangle_{\mathcal{E}_h}. \end{aligned}$$

If we choose δ such that $(1-C\delta) \leq \frac{1}{2}$ and assume that η is sufficiently large so that $\eta \geq (\frac{1}{2} + \frac{1}{\delta})$, we recover

$$(A\nabla v,\nabla v)_{\mathcal{T}_h} + \frac{1}{h}\langle[\![v]\!],[\![v]\!]\rangle_{\mathcal{E}_h} \leq C\, B(v,v).$$

The proof is complete by using (3.19). □

3.3 Main result

Combining the interpolation error analysis and the coercivity result, we now state our main error estimate as follows:

Theorem 3.1. Let u be the solution of (1.2) and let $u_h \in V_h$ be the IP-DG approximation, then

$$|||u-u_h||| \leq C\, h\|f\|_{L^2(\Omega)}, \tag{3.21a}$$

$$\|u-u_h\|_{L^2(\Omega)} \leq C\, h^2\|f\|_{L^2(\Omega)}. \tag{3.21b}$$

Proof. We first prove (3.21a). To this end, by Lemma 4 we have, for any $v \in S_h^1$,

$$|||v - u_h|||^2 \leq C\, B_h(v - u_h, v - u_h),$$

By Galerkin orthogonality of the IP-DG method we have

$$|||v - u_h|||^2 \leq C\, B_h(v - u, v - u_h).$$

Clearly, $B_h(\cdot, \cdot)$ is a bounded bilinear form. That is,

$$B_h(v - u, v - u_h) \leq C\ |||\, v - u\, |||\ |||v - u_h\, |||\,.$$

Therefore,

$$|||v - u_h||| \leq C\ |||\, v - u\, |||\,.$$

The triangle inequality gives

$$|||u - u_h||| \leq C\ |||\, v - u\, |||\,.$$

Since this holds for any $v \in S_h^1$, Lemma 3 yields (3.21a).

To prove (3.21b), we will use a duality argument. Define the problem

$$-(a(x,y)\phi_x(x,y))_x - (a(x,y)\phi_y(x,y))_y = (u - u_h)(x,y), \quad (x,y) \in \Omega, \tag{3.22a}$$

$$\phi(x,y) = 0, \quad (x,y) \in \partial\Omega. \tag{3.22b}$$

By adjoint consistency of the IP-DG method we have

$$\|u - u_h\|^2_{L^2(\Omega)} = B_h(u - u_h, \phi) = B_h(u - u_h, \phi - v),$$

for any $v \in S_h^1$. Here we have used Galerkin orthogonality. Hence,

$$\|u - u_h\|^2_{L^2(\Omega)} \leq C\ |||\, u - u_h\, |||\ |||\phi - v||| \leq C\, h\ |||\, u - u_h\, |||\ \|u - u_h\|_{L^2(\Omega)},$$

where we have used Lemma 3. Inequality (3.21b) now follows from (3.21a). □

4 Multi-scale DG method: numerical results

In this section, various two-dimensional numerical examples are presented to demonstrate that the proposed multi-scale DG method is able to capture oscillatory solutions without having to completely resolve the finest scales therein. This is in contrast to the traditional DG method based on polynomial approximation spaces.

We consider two two-dimensional elliptic multi-scale examples on the domain $[-1,1]^2$. In the first example we do not enforce zero Dirichlet boundary condition, thus an exact solution can be obtained. The second example is a real two-dimensional Dirichlet problem, in which case explicit formulas for the exact solutions are not available. Thus we compute the reference solutions by a spectral Chebyshev collocation method with a 512×512 mesh to check the convergence rates of the DG methods.

We note that in the second example, due to the difficulty of computing a high resolution reference solution, we are unable to test for very small values of ϵ (we tested $\epsilon = 0.01$ and $\epsilon = 0.005$). Indeed, the smaller ϵ is, the more significant advantage of the multi-scale IP-DG method is expected to have. This is because the error of the multi-scale IP-DG method does not depend on ϵ, whereas the traditional IP-DG method only has convergence when the mesh size is comparable to ϵ.

Finally, an easy implementation is proposed. Though no theoretical proof for the error estimates has been obtained, we demonstrate the effectiveness of this new way of implementation through the same set of numerical tests.

4.1 2D Example I: Accuracy test

In the first example, we consider $a^\epsilon(x,y) = \dfrac{1}{4+\cos(\frac{x^2+y^2}{\epsilon})}$. By not imposing zero Dirichlet boundary condition, we can assume an exact solution and smooth right hand side function, i.e.,

$$u = (x^2+y^2)^2 + \frac{1}{2}\epsilon(x^2+y^2)\sin\left(\frac{x^2+y^2}{\epsilon}\right) + \frac{1}{2}\epsilon^2\cos\left(\frac{x^2+y^2}{\epsilon}\right), \quad f = -4(x^2+y^2). \tag{4.1}$$

The boundary condition comes from the exact solution which has oscillations.

Note that throughout the domain, the oscillations only appear in the radical direction. Thus we define the corresponding mapping $(\xi, \eta) : \xi(x, y) = \sqrt{x^2 + y^2}$, $\eta(x, y) = \tan^{-1}(\frac{y}{x})$. The second order local approximations space S_h^1 is just

$$\text{span}\left\{1, \int_{\xi_{K_j}^*}^{\xi} \frac{1}{\tilde{a}^\epsilon(s)} ds, \eta - \eta_{K_j}^*\right\}, \tag{4.2}$$

where $\tilde{a}(\xi) = \frac{1}{4+\cos(\frac{\xi^2}{\epsilon})}$. The third order local approximation space S_h^2 is

$$\text{span}\left\{1, \int_{\xi_j^*}^{\xi_i} \frac{1}{\tilde{a}^\epsilon(s)} ds, \eta_i - \eta_j^*, \int_{\xi_j^*}^{\xi_i} \frac{s}{\tilde{a}^\epsilon(s)} ds, (\eta_i - \eta_j^*) \int_{\xi_j^*}^{\xi_i} \frac{1}{\tilde{a}^\epsilon(s)} ds, (\eta_i - \eta_j^*)^2\right\}. \tag{4.3}$$

The integration in the construction of basis above is done numerically with a sufficient number of quadrature points to completely resolve $\epsilon = 0.01$. We first studied the L^2-projection error of the multi-scale bases S_h^1 and S_h^2 for a small $\epsilon = 0.01$. From Table 4.1, we can see a second and a third order for S_h^1 and S_h^2 independent of ϵ, respectively.

Table 4.1: L^2 projection error of the multi-scale bases S_h^1 and S_h^2 with $\epsilon = 0.01$.

	S_h^1		S_h^2	
N	error	order	error	order
10	2.06E-01	–	3.91E-03	–
20	4.28E-02	2.27	4.35E-04	3.17
40	9.70E-03	2.14	5.16E-05	3.09
80	2.31E-03	2.07	6.13E-06	3.06

Next, in Table 4.2, the L^2-errors and convergence rates of the multi-scale DG methods are listed, showing an almost second order convergence rate and a clear third order of convergence rate for the S_h^1 and S_h^2 spaces, respectively. The results for smaller ϵ, e.g., $\epsilon = 0.005$ are similar, thus we do not list the results here.

Table 4.2: L^2-errors and orders of accuracy by the multi-scale DG method: Accuracy test with $\epsilon = 0.01$.

	S_h^1		S_h^2	
N	error	order	error	order
10	5.59E-01	–	3.50E-03	–
20	1.77E-01	1.66	3.21E-04	3.45
40	4.84E-02	1.87	3.35E-05	3.26
80	1.22E-02	1.99	3.93E-06	3.09

4.2 2D Example II: Radical direction

In the second example, the multi-scale DG method is tested with the coefficients with no separation of scales:

$$a^\epsilon(x,y) = \frac{1}{4+\sin(\frac{\cos(x^2+y^2)}{\epsilon})}, \quad f = -2. \tag{4.4}$$

In this case, zero boundary condition is imposed, thus an explicit formula for the exact solution is no longer available. The numerical solutions are compared with reference solutions as mentioned above.

The same local mapping is used as in the first example, i.e. the polar one (ξ, η). Table 4.3 shows the L^2-errors and orders of convergence for the cases with $\epsilon = 0.01$ and $\epsilon = 0.005$. As expected, it shows a nearly second order of convergence rates for the S_h^1 space. Although a clear improvement in the errors could be observed, for the S_h^2 space, it is not clear what the convergence rate is, since the error decreases faster after only a few refinements, the reference solution is no longer reliable.

4.3 2D Example III: $x^6 + y^6$ direction

In the third example, we consider the problem with oscillating coefficient

$$a^\epsilon(x,y) = \frac{1}{4+\cos(\frac{x^6+y^6}{\epsilon})} \tag{4.5}$$

and smooth right hand side $f = -2$ with zero boundary condition.

Table 4.3: L^2-errors and orders of accuracy by the multi-scale DG method: Example II.

	$\varepsilon = 0.01$		$\varepsilon = 0.005$	
S_h^1				
N	error	order	error	order
10	3.53E-01	–	3.58E-01	–
20	1.25E-01	1.50	1.25E-01	1.52
40	3.72E-02	1.75	3.65E-02	1.78
80	1.03E-02	1.85	9.91E-03	1.88
160	2.68E-03	1.95	2.58E-03	1.94
S_h^2				
10	8.50E-03	–	7.06E-03	–
20	1.53E-03	2.48	6.94E-04	3.55
40	9.08E-04	0.75	1.98E-04	1.81
80	2.89E-04	1.65	1.88E-04	0.08

It is easy to see the oscillation is along $x^6 + y^6$ direction. So we can define the mapping $(\xi, \eta) : \xi(x, y) = \sqrt{x^6 + y^6}$, $\eta(x, y) = \tan^{-1}(\frac{y^3}{x^3})$. With the (ξ, η), the second and third order finite element space S_h^1 and S_h^2 can be defined as Eq. (4.2) and (4.3). Since there is no exact solution for this problem, the reference solution is computed same as before.

Table 4.4 shows the L^2-errors and orders of convergence by the multi-scale DG method with $\epsilon = 0.01$ and $\epsilon = 0.005$. For S_h^1 space, the order is approaching second order. For S_h^2 space, the order is a little more than second order. Theoretically we cannot prove the order of convergence for the S_h^2 space, but numerically we can see a good approximate by S_h^2 with a small error.

4.4 Easy implementations

In this subsection, an easy implementation is proposed for the multi-scale DG methods. From the assumption on a^ϵ, it is clear that within each local element, the oscillation direction of the coefficients changes smoothly. Intuitively the oscillation direction at each cell center should be able to capture and explain the majority part of

Table 4.4: L^2-errors and orders of accuracy by the multi-scale DG method: Example III.

	$\varepsilon = 0.01$		$\varepsilon = 0.005$	
S_h^1				
N	error	order	error	order
10	4.87E-01	–	4.90E-01	–
20	2.10E-01	1.22	1.25E-01	1.22
40	8.38E-02	1.32	8.27E-02	1.34
80	3.19E-02	1.39	3.12E-02	1.41
160	1.06E-02	1.59	1.03E-02	1.59
S_h^2				
10	2.44E-02	–	2.40E-02	–
20	5.78E-03	2.08	5.85E-03	2.04
40	1.20E-03	2.27	1.24E-03	2.24
80	2.45E-04	2.29	2.95E-04	2.07

the local oscillations. This approach would be especially helpful for practical purposes, when the local mapping (ξ, η), is not directly available, or can not be written out in an analytical form. A new approximation space could be defined as:

$$ES_h^1 = \bigcup_j V_j^h = \left\{ v : v|_{K_j} \in \text{span} \left\{ 1, \int_{\xi^*_{K_j}}^{\xi_j} \frac{1}{\tilde{a}_j^\epsilon(s, \eta^*_{K_j})} ds, \eta_j - \eta^*_{K_j} \right\} \right\},$$

$$ES_h^2 = \bigcup_j V_j^h = \left\{ v : v|_{K_j} \in \text{span} \left\{ 1, \int_{\xi^*_{K_j}}^{\xi_j} \frac{1}{\tilde{a}_j^\epsilon(s, \eta^*_{K_j})} ds, \eta_j - \eta^*_{K_j}, \right.\right.$$
$$\left.\left. (\eta_j - \eta^*_{K_j}) \int_{\xi^*_{K_j}}^{\xi_j} \frac{1}{\tilde{a}_j^\epsilon(s, \eta^*_{K_j})} ds, \int_{\xi^*_{K_j}}^{\xi_j} \frac{s}{\tilde{a}_j^\epsilon(s, \eta^*_{K_j})} ds, (\eta_j - \eta^*_{K_j})^2 \right\} \right\}$$

where ξ_j denotes the oscillation direction of $a^\epsilon(x, y)$ at cell center of K_j, and η_j is the direction perpendicular to ξ_j. So (ξ_j, η_j) is now just a rotation of the original coordinates (x, y). $(\xi^*_{K_j}, \eta^*_{K_j})$ denotes the element center of K_j under the new mapping (ξ_j, η_j).

Consider the 2D accuracy test case for example, for each cell K_j with cell center's coordinates denoted as $(x^*_{K_j}, y^*_{K_j})$. We know that the oscillation direction at the cell center is $\xi_j = \cos(\theta^*_{K_j})x + \sin(\theta^*_{K_j})y$, and in turn the orthogonal direction to

ξ_j is $\eta_j = -\sin(\theta^*_{K_j})x + \cos(\theta^*_{K_j})y$, where $\theta^*_{K_j} = \tan^{-1}(\frac{y^*_{K_j}}{x^*_{K_j}})$. Under the mapping $\xi_j(x,y), \eta_j(x,y)$ defined as above, the element center (x_j, y_j) can now be represented as $\xi^*_{K_j} = \cos(\theta^*_{K_j})x^*_{K_j} + \sin(\theta^*_{K_j})y^*_{K_j}$, $\eta^*_{K_j} = -\sin(\theta^*_{K_j})x^*_{K_j} + \cos(\theta^*_{K_j})y^*_{K_j}$. As a result, the new local oscillatory basis defined on K_j now is:

$$\int_{\xi^*_{K_j}}^{\xi_j} \frac{1}{\tilde{a}_j(s, \eta^*_{K_j})} ds = \int_{\xi^*_{K_j}}^{\xi_j} \left(4 + \cos(\frac{s^2 + (\eta^*_{K_j})^2}{\epsilon})\right) ds$$

which can be calculated through numerical integration as in the previous session.

We study the L^2-projection error of the oscillating solution (4.1) onto the new approximation space. As shown in Table 4.5, we observe second order of convergence for both the ES^1_h and ES^2_h spaces, indicating that for this easy implementation approach, enriching the basis does not help improve the convergence rates. This is understandable as we have only captured the major portion of the oscillations in the solution by freezing the oscillation direction to be that at the element center. The error committed by this approximation is expected to be second order. As a result, we just focus on ES^1_h and omit the results obtained from ES^2_h.

Table 4.5: L^2 projection error of the easy implementation basis with $\epsilon = 0.01$.

	ES^1_h		ES^2_h	
N	error	order	error	order
10	4.48E-01	–	4.59E-02	–
20	1.27E-01	1.82	1.06E-02	2.11
40	2.98E-02	2.09	2.50E-03	2.08
80	7.28E-03	2.03	5.98E-04	2.06

The tests on the same numerical examples I & II as in the previous subsections 4.1 and 4.2 are performed to validate and to show the effectiveness of this method based on an easy implementation. From Table 4.6, it is clear that the multi-scale DG with the approximation space ES^1_h also provide a second order of convergence rate.

Table 4.6: L^2 errors and orders of accuracy obtained by the easy implementation: Example I & II.

	Accuracy Test		Dirichlet B.C.			
ES_h^1	$\epsilon = 0.01$		$\epsilon = 0.01$		$\epsilon = 0.005$	
N	error	order	error	order	error	order
10	4.84E-01	–	4.71E-01	–	5.41E-01	–
20	1.50E-01	1.69	1.72E-01	1.45	1.83E-01	1.56
40	4.32E-02	1.80	5.12E-02	1.75	5.12E-02	1.84
80	1.13E-02	1.93	1.39E-02	1.88	1.36E-02	1.91
160	2.88E-03	1.97	3.60E-03	1.95	3.66E-03	1.93

5 Concluding remarks

In this paper, we propose a multi-scale DG method for solving a class of two dimensional second order elliptic equations with rough coefficients which locally vary sharply in at most one direction, or in other words, are curvilinear unidirectional. Solving this type of problems is computationally expensive for most traditional numerical methods, which motivates us to exploit the flexibility of DG methods in choosing a special local approximation basis that allows a much more efficient algorithm.

Motivated by previous work in [6, 25], we build our local approximation space by utilizing special oscillatory basis functions indicated by the differential operators. As a result, we prove that the scheme yields optimal accuracy for the second order approximation on coarse meshes. The convergence rate is uniform and independent of the small scale ϵ. Furthermore, the constraints on the computational geometry indicated in the error estimates appear to be artificial, in the sense of being assumed only for the convenience of analysis rather than enforcing restrictions on real application.

Various numerical tests have been performed to demonstrate the effectiveness of the proposed schemes, particularly on coarse meshes. In addition, an easy implementation is suggested and validated through numerical examples. The easy implementation technique is helpful especially when it is hard to find the explicit local mapping.

In future work, we plan to apply this methodology to real applications that arise in engineering problems.

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