# DISCRETE EXTENSION OPERATORS FOR MIXED FINITE ELEMENT SPACES ON LOCALLY REFINED MESHES

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ABSTRACT. The existence of uniformly bounded discrete extension operators is established for conforming Raviart-Thomas and Nédelec discretisations of $H(div)$ and $H(curl)$ on locally refined partitions of a polyhedral domain into tetrahedra.

## 1. INTRODUCTION

Many boundary value problems with non-homogeneous boundary data may be cast in the abstract variational form: find $u \in X$ such that $\gamma u = g \in M$ and

$$(1.1) \qquad B(u, v) = F(v) \quad \forall v \in X_0$$

where $X$ is a Hilbert space over a domain $\Omega$, $M$ is a Hilbert space over the boundary $\Gamma$ of $\Omega$, and $\gamma : X \to M$ is a trace operator with $X_0 = \{v \in X : \gamma v = 0\}$. We assume that the bilinear and linear forms $B : X \times X \to \mathbb{R}$ and $F : X \to \mathbb{R}$ satisfy suitable conditions for the problem to be well-posed; specific examples will be given later.

A Galerkin finite element approximation $u_h \approx u$ is obtained by selecting a finite dimensional subspace $X_h \subset X$, setting $M_h = \gamma X_h$, constructing a suitable approximation $g_h \approx g$ of the non-homogeneous boundary data, and seeking $u_h \in X_h$ such that $\gamma u_h = g_h \in M_h$ and

$$(1.2) \qquad B(u_h, v_h) = F(v_h) \quad \forall v_h \in X_{h0} = \{v_h \in X_h : \gamma v_h = 0\}.$$

The discrete problem (1.2) is well-posed provided that there exists a positive constant $\beta > 0$ such that

$$(1.3) \qquad \sup_{v_h \in X_h, \|v_h\|_X = 1} B(w_h, v_h) \geq \beta \|w_h\|_X \quad \text{for all } w_h \in X_h.$$

The issue of the accuracy of the resulting approximation is usually addressed by reference to the following classical Céa type estimate

$$(1.4) \qquad \|u - u_h\|_X \leq \left(1 + \frac{\kappa}{\beta}\right) \inf_{w_h \in X_h : \gamma w_h = g_h} \|u - w_h\|_X,$$

where $\kappa$ is the continuity constant of the bilinear form $B$. It is clear that the accuracy depends on both the choice of finite dimensional subspace $X_h \subset X$ and on the choice $g_h \approx g$ of the approximate Dirichlet boundary condition. Nevertheless, the bound (1.4) is somewhat unsatisfactory. In particular, whilst $X_h \subset X$ and $g_h \approx g$ can essentially be chosen independently of one another, the influence of each choice on the accuracy of the resulting finite element approximation is obscured through the

requirement that the choice of comparator $w_h \in X_h$ is constrained to satisfy the boundary condition $\gamma w_h = g_h$.

Under what conditions is it possible to obtain an error estimate of the form

$$(1.5) \qquad \|u - u_h\|_X \leq C\big(\inf_{v_h \in X_h} \|u - v_h\|_X + \|g - g_h\|_M\big),$$

in which the individual contributions to the error corresponding to the choice of $X_h \subset X$ and $g_h \approx g$ are isolated? Suppose that there exists a uniformly bounded *discrete extension* operator $L_h : M_h \to X_h$ such that

$$(1.6) \qquad (P1) \quad \gamma L_h \mu_h = \mu_h \quad \forall \mu_h \in M_h; \quad (P2) \quad \|L_h \mu_h\|_X \leq C_L \|\mu_h\|_M \quad \forall \mu_h \in M_h.$$

Let $v_h \in X_h$ be arbitrary, and set $w_h = v_h - L_h(\gamma v_h - g_h) \in X_h$, so that $\gamma w_h = g_h$ on $\Gamma$ and $u_h - w_h \in X_{0h}$. With this choice, estimate (1.4) then gives

$$(1.7) \qquad \|u - u_h\|_X \leq \left(1 + \frac{\kappa}{\beta}\right) \|u - v_h + L_h(\gamma v_h - g_h)\|_X.$$

With the aid of the triangle inequality and $(P2)$, the right hand side in the above estimate may be bounded by $\|u - v_h\|_X + C_L \|\gamma v_h - g_h\|_M$. The second term in this expression can be bounded by inserting $0 = \gamma u - g$, applying the triangle inequality and using the continuity of the trace operator $\gamma : X \to M$ to obtain $\|\gamma v_h - g_h\|_M \leq C\|v_h - u\|_X + \|g - g_h\|_M$. Combining the above estimates, we conclude that (1.5) holds whenever there exists a discrete extension operator satisfying $(P1)$-$(P2)$. Interestingly, the existence of an operator satisfying $(P1) - (P2)$ is also *necessary* for a bound of the form (1.5) to hold [9, 14].

The existence of uniformly bounded discrete extension operators satisfying $(P1)$-$(P2)$ is important in many areas of numerical analysis including the construction of domain decomposition preconditioners [17]. The main purpose of the current work is to establish the existence of discrete extension operators satisfying $(P1)$-$(P2)$ in the case where the discrete spaces are taken to be conforming discretisations of $H(\mathrm{div})$ and $H(\mathrm{curl})$, i.e. Raviart-Thomas and Nédelec spaces, on a possibly non-quasi-uniform partitioning of a polyhedral domain into tetrahedra.

Various results concerning stable extension operators are interspersed in the literature. There is a number of results available in the literature concerning discrete extensions from the boundary to the interior on a single isolated element [8], but the fact that the norms on the trace spaces are not additive means that one cannot prove the results for collections of elements by simply summing contributions from individual elements. The case of Raviart-Thomas elements on a two-dimensional domain appears in [3] applied to the analysis of weakly imposed essential boundary conditions for the mixed Laplacian for the case of quasi-uniform triangulations, and was subsequently extended [10] to cover meshes that are quasi-uniform in a neighborhood of the boundary. Subsequently, the case of general non-quasi-uniform meshes was covered in [11] although, unfortunately, the arguments used seem to be limited to the two-dimensional setting.

The approach employed in the present work for the treatment of Raviart-Thomas elements is similar to the idea used in [3, 10] without, however, requiring quasi-uniformity of the mesh. The key to relaxing the conditions on the mesh is to develop local regularity estimates along with discrete norm equivalences valid on locally refined meshes [1], and to show that the operator defined in [10] is, in fact, uniformly bounded on general shape regular meshes. The treatment of the Nédélec spaces is different again. The idea is to first split the discrete trace using a discrete Hodge decomposition and to then use our extension result for Raviart-Thomas elements to handle one component of the splitting, with the remaining component treated using an idea adapted from [11]. The resulting extension yields a divergence-free field which therefore belongs Brezzi-Douglas-Marini space. Consequently, our results for the Raviart-Thomas case (i.e. Nédélec spaces in the three dimensional case [12]) extend to the three dimensional counterpart of the Brezzi-Douglas-Marini finite element [5, 13].

The plan of the paper is as follows. The main results are stated in Section 2. Proofs are given in Sections 3 for the Raviart-Thomas elements, and in Section 4 for the Nédélec elements. The local regularity estimates needed in Section 3 are given in the Appendix. Basic results on the spaces $H^1(\Omega)$, $H(\mathrm{div}, \Omega)$, and $H(\mathrm{curl}, \Omega)$ will be assumed throughout. The symbol $\lesssim$ will be used as follows: for two quantities $a_h$ and $b_h$ depending on the triangulations (see below), we write $a_h \lesssim b_h$, whenever there exists $C > 0$ independent of $h$, such that $a_h \leq C b_h$. The quantity $C$ will be allowed to depend on: the polynomial degree, the shape-regularity of the triangulation, and the domain.

## 2. Main results

Let $\Omega \subset \mathbb{R}^3$ be a connected polyhedral Lipschitz domain. The unit outward pointing normal vector field on $\Gamma := \partial\Omega$ will be denoted by $\boldsymbol{n}$. Let now $\mathcal{T}_h$ be a shape regular simplicial triangulation of $\Omega$ which, however, need not be quasi-uniform. For each $K \in \mathcal{T}_h$ we let $h_K$ denote the diameter of $K$. We then consider the spaces of Raviart-Thomas and Nédélec finite elements:

$$\begin{aligned}
\boldsymbol{V}_h &:= \{\boldsymbol{q} \in H(\mathrm{div}, \Omega) : \boldsymbol{q}|_K \in [\mathcal{P}^k(K)]^3 + \mathcal{P}^k(K)\boldsymbol{x}, \text{ for all } K \in \mathcal{T}_h\}, \\
\boldsymbol{N}_h &:= \{\boldsymbol{q} \in H(\mathrm{curl}, \Omega) : \boldsymbol{q}|_K \in [\mathcal{P}^k(K)]^3 + [\mathcal{P}^k(K)]^3 \times \boldsymbol{x}, \text{ for all } K \in \mathcal{T}_h\},
\end{aligned}$$

where $\mathcal{P}_k(S)$ is the space of polynomials of degree $k$ or less defined on $S$. On the boundary $\Gamma$, we consider the induced triangulation,

$$\Gamma_h = \{\partial\Omega \cap \partial K : \text{ for all } K \in \mathcal{T}_h\}.$$

and two spaces

$$\begin{aligned}
M_h &:= \{\boldsymbol{q} \cdot \boldsymbol{n} : \boldsymbol{q} \in \boldsymbol{V}_h\} = \{v : v|_F \in \mathcal{P}^k(F) \text{ for all } F \in \Gamma_h\} \\
\boldsymbol{R}_h &:= \{\boldsymbol{q} \times \boldsymbol{n} : \boldsymbol{q} \in \boldsymbol{N}_h\} = \{\boldsymbol{r} \in H(\mathrm{div}_\Gamma, \Gamma) : \boldsymbol{r}|_F \in [\mathcal{P}^k(F)]^2 + \mathcal{P}^k(F)\boldsymbol{x}_t \text{ for all } F \in \Gamma_h\}.
\end{aligned}$$

In the last space $\boldsymbol{x}_t := \boldsymbol{x} - (\boldsymbol{x} \cdot \boldsymbol{n})\boldsymbol{n}$ is the tangential position vector, and $\mathrm{div}_\Gamma$ is the tangential divergence operator. We also consider $M_h^0$ to be the subset of $M_h$ consisting of elements whose average value vanishes.

Let $S$ be a $d$-dimensional domain, then the fractional Sobolev norms on $S$ are defined as follows: for non-negative integer $k$ and $0 < s < 1$, we define

$$\|v\|_{H^{k+s}(S)}^2 = \|v\|_{H^k(S)}^2 + |v|_{H^{k+s}(S)}^2$$

where

$$|v|_{H^{k+s}(S)}^2 = \sum_{|\alpha|=k} \int_S \int_S \frac{|\partial^\alpha v(x) - \partial^\alpha v(y)|^2}{|x-y|^{d+2s}} dx dy$$

is the Slobodetskij seminorm and $\|\cdot\|_{H^k(S)}$ is the usual Sobolev norm. For negative $s$, $H^{-s}(S)$ is the dual space of $H_0^s(S)$, the closure in $H^s(S)$ of the set of smooth compactly supported functions. In particular, in the case of the closed surface $\Gamma$, we can write for functions $v \in L^2(\Gamma)$:

$$\|v\|_{H^{-1/2}(\Gamma)} = \sup_{w \in C^\infty(\Gamma), \|w\|_{H^{1/2}(\Gamma)} = 1} \int_\Gamma vw$$

As noted before, the operator $\boldsymbol{V}_h \ni \boldsymbol{v} \mapsto \boldsymbol{v} \cdot \boldsymbol{n} \in M_h$ is surjective. The next result shows that there is a right-inverse of this operator that is bounded as an operator $H^{-1/2}(\Gamma) \to \boldsymbol{H}(\mathrm{div}; \Omega)$, uniformly in the mesh size. Using the result [11, Theorem 5.1] it is enough to establish the uniform extension for data in $M_h^0$.

**Theorem 2.1.** *There exists a constant $C$ depending only on the shape regularity of $\mathcal{T}_h$ and on $\Omega$ such that for any $g_h \in M_h^0$ there exists $\boldsymbol{\sigma}_h \in \boldsymbol{V}_h$ with the following properties:*

(a) $\boldsymbol{\sigma}_h \cdot \boldsymbol{n} = g_h$ *on* $\Gamma$,

(b) $\|\boldsymbol{\sigma}_h\|_{H(\mathrm{div};\Omega)} \lesssim \|g_h\|_{H^{-1/2}(\partial\Omega)}$,
(c) $\mathrm{div}\,\boldsymbol{\sigma}_h = 0$  *in*  $\Omega$.

The second result concerns the Nédélec space, asserting the existence of a uniformly bounded right-inverse of the operator $\boldsymbol{N}_h \ni \boldsymbol{w}_h \mapsto \boldsymbol{w}_h \times \boldsymbol{n} \in \boldsymbol{R}_h$.

**Theorem 2.2.** *There exists a constant $C$ depending only on the shape regularity of $\mathfrak{T}_h$ and on $\Omega$ such that for any $\boldsymbol{r}_h \in \boldsymbol{R}_h$ there exists $\boldsymbol{w}_h \in \boldsymbol{N}_h$ with the following properties:*

(a) $\boldsymbol{w}_h \times \boldsymbol{n} = \boldsymbol{r}_h$  *on*  $\Gamma$,
(b) $\|\boldsymbol{w}_h\|_{H(\mathrm{curl};\Omega)} \lesssim \|\boldsymbol{r}_h\|_{H^{-1/2}(\Gamma)} + \|\mathrm{div}_\Gamma \boldsymbol{r}_h\|_{H^{-1/2}(\Gamma)}$.

## 3. Discrete Extension Operators for Raviart-Thomas Finite Element Spaces

We first recall some properties of the Raviart-Thomas projection. Let $\boldsymbol{v} \in H^{1/2+s}(\Omega)$ for some $s > 0$. Then we define $\Pi\boldsymbol{v} \in \boldsymbol{V}_h$ satisfying

$$\int_F (\Pi\boldsymbol{v} \cdot \boldsymbol{n})w = \int_F (\boldsymbol{v} \cdot \boldsymbol{n})w \qquad \forall w \in \mathcal{P}^k(F) \qquad \forall F \in \mathcal{E}_h,$$

$$\int_K \Pi\boldsymbol{v} \cdot \boldsymbol{w} = \int_K \boldsymbol{v} \cdot \boldsymbol{w} \qquad \forall \boldsymbol{w} \in [\mathcal{P}^{k-1}(K)]^3 \quad \forall K \in \mathfrak{T}_h$$

(see [4, Example 2.5.3]). Here $\mathcal{E}_h$ is the set of all faces of the triangulation. The following classical result can be found in [4, Propositions 2.5.1, 2.5.2].

**Proposition 3.1.** *For every $\boldsymbol{v} \in (H^{1/2+s}(\Omega))^3$ one has*

(a) $\mathrm{div}\,\Pi\boldsymbol{v} = P\mathrm{div}\,\boldsymbol{v}$,
(b) $\|\Pi\boldsymbol{v} - \boldsymbol{v}\|_{L^2(K)} \lesssim h_K^{1/2+s}\|\boldsymbol{v}\|_{H^{1/2+s}(K)}$ *for all $K \in \mathfrak{T}_h$,*

*where $P$ is the $L^2(\Omega)$-orthogonal projection onto the space of piecewise $\mathcal{P}^k(K)$ functions.*

The following inverse inequality will play a key role in our analysis.

**Lemma 3.2.** *For any $g \in M_h$ we have*

$$\sum_{F \in \Gamma_h} h_F \|g_h\|_{L^2(F)}^2 \lesssim \|g_h\|_{H^{-1/2}(\Gamma)}^2$$

*Proof.* This result is a consequence of basic estimates given in [1]. Let $\{\varphi_{i,F} : i = 1, \ldots, \dim\mathcal{P}^k(F)\}$ be a basis for $\mathcal{P}^k(F)$ built by pushing forward the Lagrange basis on the reference element. Then we decompose $g_h = \sum_F \sum_i g_{i,F}\varphi_{i,F}$, and estimate

$$\begin{aligned}
\sum_{F \in \Gamma_h} h_F \|g_h\|_{L^2(F)}^2 &\lesssim \sum_F \sum_i h_F^2 |g_{i,F}|^2 \|\varphi_{i,F}\|_{L^2(F)}^2 \\
&\lesssim \sum_F \sum_i h_F^3 |g_{i,F}|^2 && \text{(simple computation)} \\
&\lesssim \sum_F \sum_i \|g_{i,F}\varphi_{i,F}\|_{H^{-1/2}(\Gamma)}^2 && \text{(by [1, Theorem 4.8])} \\
&\lesssim \|g_h\|_{H^{-1/2}(\Gamma)}^2, && \text{(by [1, Lemma 5.4])}
\end{aligned}$$

which finishes the proof. $\qquad\square$

We also need elliptic regularity results; see for example [7] for the case $g = 0$. Consider the Poisson problem with Neumann boundary conditions

(3.1a) $$-\triangle u = \quad f \qquad \text{on } \Omega$$

(3.1b) $$\nabla u \cdot \boldsymbol{n} = \quad g \qquad \text{on } \Gamma,$$

under the assumption that $\int_\Gamma g + \int_\Omega f = 0$ and $\int_\Omega u = 0$. Then, there exist $C > 0$ and $s \in (0, 1/2)$ such that

(3.2) $$\|u\|_{H^{3/2+s}(\Omega)} \leq C\big(\|f\|_{H^{-1/2+s}(\Omega)} + \|g\|_{H^s(\Gamma)}\big).$$

We can localize this regularity result to obtain:

**Theorem 3.3.** *Suppose that $f \equiv 0$ in (3.1) and let $\boldsymbol{\sigma} = \nabla u$. Then for each $K \in \mathcal{T}_h$ we have*

$$\|\boldsymbol{\sigma}\|_{H^{1/2+s}(K)} \leq C(h_K^{-1/2-s}\|\boldsymbol{\sigma}\|_{L^2(D_K)} + \|g\|_{H^s(\partial D_K \cap \Gamma)} + h_K^{-s}\|g\|_{L^2(\partial D_K \cap \Gamma)}),$$

*where*

$$D_K := \cup\{K' \in \mathcal{T}_h \ : \ \overline{K} \cap \overline{K'} \neq \emptyset\},$$

*is the collection of tetrahedra sharing one or more vertices with $K$.*

The proof of this result is contained in Appendix A.

*Proof of Theorem 2.1.* Let $u$ satisfy

$$-\triangle u = \quad 0 \qquad \text{on } \Omega,$$
$$\nabla u \cdot \boldsymbol{n} = \quad g_h \qquad \text{on } \Gamma,$$

and set $\boldsymbol{\sigma} = \nabla u \in (H^{1/2+s}(\Omega))^3$ (see (3.2)). Note that $\operatorname{div}\boldsymbol{\sigma} = 0$. We define $\boldsymbol{\sigma}_h = \Pi\boldsymbol{\sigma}$ and we note that Theorem 2.1 (a) holds, and that by Proposition 3.1(a), $\operatorname{div}\boldsymbol{\sigma}_h = 0$. Therefore, using elliptic regularity, Proposition 3.1(b) and Lemma 3.2,

$$
\begin{aligned}
\|\boldsymbol{\sigma}_h\|_{H(\operatorname{div};\Omega)} =& \|\boldsymbol{\sigma}_h\|_{L^2(\Omega)} \leq \|\boldsymbol{\sigma}\|_{L^2(\Omega)} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2(\Omega)} \\
\lesssim& \|g_h\|_{H^{-1/2}(\Gamma)} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2(\Omega)} \\
\lesssim& \|g_h\|_{H^{-1/2}(\Gamma)} + \big(\sum_{K \in \mathcal{T}_h} h_K^{2(1/2+s)}\|\boldsymbol{\sigma}\|_{H^{1/2+s}(K)}^2\big)^{1/2} \\
\lesssim& \|g_h\|_{H^{-1/2}(\Gamma)} \\
& + \bigg(\sum_{K \in \mathcal{T}_h} \|\boldsymbol{\sigma}\|_{L^2(D_K)}^2 + h_K^{2(1/2+s)}\|g_h\|_{H^s(\partial D_K \cap \Gamma)}^2 + h_K\|g_h\|_{L^2(\partial D_K \cap \Gamma)}^2\bigg)^{1/2}
\end{aligned}
$$

The shape regularity of the elements means that

$$\sum_{K \in \mathcal{T}_h} \|\boldsymbol{\sigma}\|_{L^2(D_K)}^2 \lesssim \|\boldsymbol{\sigma}\|_{L^2(\Omega)}^2.$$

Also, if we let $D_F$ to be the macro-element surrounding $F$ (triangles sharing a vertex with $F$), we have that

$$
\begin{aligned}
\sum_{K \in \mathcal{T}_h} h_K^{2(1/2+s)}\|g_h\|_{H^s(\partial D_K \cap \Gamma)}^2 &\lesssim \sum_{F \in \Gamma_h} h_F^{2(1/2+s)}\|g_h\|_{H^s(D_F)}^2 \\
&\lesssim \sum_{F \in \Gamma_h} h_F\|g_h\|_{L^2(D_F)}^2 \lesssim \sum_{F \in \Gamma_h} h_F\|g_h\|_{L^2(F)}^2,
\end{aligned}
$$

where we have used a standard local inverse estimate for piecewise polynomial functions. Applying Lemma 3.2 completes the proof.

$\square$

As an application we can get an error estimate for the Laplacian in mixed form with Neumann boundary conditions. In this case $X = H(\text{div}; \Omega) \times L^2(\Omega)$ and the trace space is $M = H^{-1/2}(\partial\Omega)$ with trace operator $\gamma(\boldsymbol{\sigma}, u) = \boldsymbol{\sigma} \cdot \boldsymbol{n}$. The bilinear form $B$ and the linear form $F$ are given by

$$B((\boldsymbol{\sigma}, u), (\boldsymbol{\eta}, w)) := \int_\Omega (\boldsymbol{\sigma} \cdot \eta - u \, \text{div} \, \boldsymbol{\eta} + w \, \text{div} \, \boldsymbol{\sigma})$$

$$F((\boldsymbol{\eta}, w)) := \int_\Omega fw$$

for a given $f \in L^2$.

Of course, the finite element space will be $X_h = \boldsymbol{V}_h \times U_h$ where

$$U_h = \{w : \Omega \to \mathbb{R} \, : \, w|_K \in \mathcal{P}^k(K) \quad \forall K \in \mathcal{T}_h\}.$$

In view of Theorem 2.1 and the introductory discussion, we have the following error estimates for Raviart-Thomas elements.

**Corollary 3.4.** *Let* $X = H(\text{div}; \Omega) \times L^2(\Omega)$ *,* $M = H^{-1/2}(\partial\Omega)$ *and* $X_h = \boldsymbol{V}_h \times U_h$*. Let* $(\sigma, u)$ *satisfy* (1.1) *and* $(\sigma_h, u_h)$ *satisfy* (1.2) *then the following holds*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\boldsymbol{H}(\text{div};\Omega)} + \|u - u_h\|_{L^2(\Omega)} \lesssim \inf_{\boldsymbol{v} \in \boldsymbol{V}_h} \|\boldsymbol{\sigma} - \boldsymbol{v}\|_{\boldsymbol{H}(\text{div};\Omega)} + \inf_{w \in U_h} \|u - w\|_{L^2(\Omega)} + \|g - g_h\|_{H^{-1/2}(\Gamma)}.$$

Finally, we note that other applications of the existence of an extension operator like $L_h$ are given in the analysis of a variety of discretization methods for the Stokes-Darcy problem [10, 11].

## 4. Discrete Extension Operators for Nédélec Finite Element Spaces

The proof of Theorem 2.2 relies on a Helmholtz-Hodge type decomposition of $\boldsymbol{R}_h$, two liftings (one for Lagrange finite elements and the one provided by Theorem 2.1), and local estimates in the space of Lagrange finite elements on the boundary

$$P_h := \{\phi_h \in \mathcal{C}(\Gamma) \, : \, \phi_h|_F \in \mathcal{P}_{k+1}(F) \text{ for all } F \in \Gamma_h\}.$$

We begin with some technical results:

**Lemma 4.1.** *If* $\boldsymbol{r}_h \in \boldsymbol{R}_h$ *satisfies* $\text{div}_\Gamma \boldsymbol{r}_h = 0$*, then* $\boldsymbol{r}_h = \text{curl}_\Gamma \phi_h$ *with* $\phi_h \in P_h$*.*

*Proof.* By definition of $\boldsymbol{R}_h$, there exists $\boldsymbol{w}_h \in \boldsymbol{N}_h$ such that $\boldsymbol{w}_h \times \boldsymbol{n} = \boldsymbol{r}_h$. Note now that

$$\boldsymbol{q}_h := \nabla \times \boldsymbol{w}_h \in \boldsymbol{V}_h, \qquad \boldsymbol{q}_h \cdot \boldsymbol{n} = \text{div}_\Gamma(\boldsymbol{w}_h \times \boldsymbol{n}) = 0.$$

Therefore, by the exactness of the discrete de-Rham complex with essential boundary conditions (see for example [2]), there exists $\boldsymbol{w}_h^0 \in \boldsymbol{N}_h$ such that $\nabla \times \boldsymbol{w}_h^0 = \boldsymbol{q}_h$ and $\boldsymbol{w}_h^0 \times \boldsymbol{n} = 0$. We then consider the difference $\boldsymbol{w}_h - \boldsymbol{w}_h^0$ and note, again, by the exactness of the discrete de-Rham complex, there exists $u_h$ in the finite element space

$$(4.1) \qquad W_h := \{u_h \in \mathcal{C}(\Omega) \, : \, u_h|_K \in \mathcal{P}_{k+1}(K) \text{ for all } K \in \mathcal{T}_h\},$$

satisfying $\nabla u_h = \boldsymbol{w}_h - \boldsymbol{w}_h^0$. Take $\phi_h = u_h|_\Gamma$. The result follows since $\boldsymbol{w}_h \times \boldsymbol{n} = \nabla u_h \times \boldsymbol{n} = \text{curl}_\Gamma \phi_h$. $\square$

**Lemma 4.2.** *It holds*

$$\|\phi_h\|_{H^{1/2}(\Gamma)} \lesssim \|\text{curl}_\Gamma \phi_h\|_{H^{-1/2}(\Gamma)} \qquad \text{for all } \phi_h \in P_h.$$

*Proof.* Let $\{\phi_j \,:\, j = 1, \ldots, \mathcal{N}\}$ be a basis of $P_h$, and $\boldsymbol{\theta}_j := h_j \mathrm{curl} \phi_j$, where $h_j$ is the diameter of any of the elements contained in the support of $\phi_j$. We decompose $\phi_h = \sum_{j \in \mathcal{N}} \alpha_j \phi_j$ and use [1, Formula (5.1)] and [1, Theorem 4.8 (1)] to bound

$$(4.2) \qquad \|\phi_h\|^2_{H^{1/2}(\Gamma)} \lesssim \sum_{j \in \mathcal{N}} \alpha_j^2 \|\phi_j\|^2_{H^{1/2}(\Gamma)} \approx \sum_{j \in \mathcal{N}} \alpha_j^2 h_j^2.$$

On the other hand, we can use a basis $\{\psi_\ell\}$ of $M_h$ and write

$$\boldsymbol{\theta}_j = \sum_{\ell \in S_j} \boldsymbol{c}_\ell \psi_\ell, \qquad |\boldsymbol{c}_\ell| \lesssim 1, \qquad \#S_j \approx 1,$$

and then use [1, Theorem 4.8 (1)] to show that

$$(4.3) \qquad h_j^2 \approx h_j^{-2} \sum_{\ell \in S_j} \|\psi_\ell\|^2_{H^{-1/2}(\Gamma)} \approx h_j^{-2} \|\boldsymbol{\theta}_j\|^2_{H^{-1/2}(\Gamma)} = \|\mathrm{curl}_\Gamma \phi_j\|^2_{H^{-1/2}(\Gamma)}.$$

By equations (4.2) and (4.3) it follows that

$$\|\phi_h\|^2_{H^{1/2}(\Gamma)} \lesssim \sum_{j \in \mathcal{N}} \alpha_j^2 \|\mathrm{curl}_\Gamma \phi_j\|^2_{H^{-1/2}(\Gamma)},$$

and the result is then a consequence of [1, Formula (5.13)]. $\qquad \square$

**Lemma 4.3.** *For all* $\boldsymbol{v}_h \in \boldsymbol{V}_h$ *with* $\mathrm{div}\, \boldsymbol{v}_h = 0$, *there exists* $\boldsymbol{w}_h \in \boldsymbol{N}_h$ *such that* $\nabla \times \boldsymbol{w}_h = \boldsymbol{v}_h$ *and* $\|\boldsymbol{w}_h\|_{L^2(\Omega)} \lesssim \|\boldsymbol{v}_h\|_{L^2(\Omega)}$.

*Proof.* By [6] uniformly bounded projections $\boldsymbol{\Pi}_h : H(\mathrm{curl}, \Omega) \to \boldsymbol{N}_h$ and $\boldsymbol{Q}_h : H(\mathrm{div}, \Omega) \to \boldsymbol{V}_h$ exist such that $\nabla \times \boldsymbol{\Pi}_h \boldsymbol{w} = \boldsymbol{Q}_h(\nabla \times \boldsymbol{w})$ for all $\boldsymbol{w} \in H(\mathrm{curl}, \Omega)$. On the other hand the curl operator is surjective from $H(\mathrm{curl}, \Omega)$ to $\{\boldsymbol{v} \in H(\mathrm{div}, \Omega) \,:\, \mathrm{div}\boldsymbol{v} = 0\}$ and has therefore a bounded right-inverse. We then apply this right-inverse to $\boldsymbol{v}_h$ to obtain $\boldsymbol{w} \in H(\mathrm{curl}, \Omega)$ and define $\boldsymbol{w}_h = \boldsymbol{\Pi}_h \boldsymbol{w}$. Then $\nabla \times \boldsymbol{w}_h = \boldsymbol{Q}_h(\nabla \times \boldsymbol{w}) = \boldsymbol{Q}_h \boldsymbol{v}_h = \boldsymbol{v}_h$ and the result is proved. $\qquad \square$

*Proof of Theorem 2.2.* Let $\boldsymbol{r}_h \in \boldsymbol{R}_h$. Then $\mathrm{div}_\Gamma \boldsymbol{r}_h \in M_h^0$ and by Theorem 2.1 we can find $\boldsymbol{v}_h \in \boldsymbol{V}_h$ such that

$$\mathrm{div}\, \boldsymbol{v}_h = 0, \qquad \boldsymbol{v}_h \cdot \boldsymbol{n} = \mathrm{div}_\Gamma \boldsymbol{r}_h,$$

and

$$\|\boldsymbol{v}_h\|_{H(\mathrm{div},\Omega)} \lesssim \|\mathrm{div}\, \boldsymbol{r}_h\|_{H^{-1/2}(\Gamma)}.$$

We then use Lemma 4.3 to obtain $\boldsymbol{w}_h \in \boldsymbol{N}_h$ such that $\nabla \times \boldsymbol{w}_h = \boldsymbol{v}_h$, and

$$(4.4) \qquad \|\boldsymbol{w}_h\|_{H(\mathrm{curl},\Omega)} \lesssim \|\boldsymbol{v}_h\|_{H(\mathrm{div},\Omega)} \lesssim \|\mathrm{div}\, \boldsymbol{r}_h\|_{H^{-1/2}(\Gamma)}.$$

Consider now the function $\boldsymbol{m}_h := \boldsymbol{r}_h - \boldsymbol{w}_h \times \boldsymbol{n} \in \boldsymbol{R}_h$ and note that

$$\|\boldsymbol{m}_h\|_{H^{-1/2}(\Gamma)} \lesssim \|\boldsymbol{r}_h\|_{H^{-1/2}(\Gamma)} + \|\mathrm{div}_\Gamma \boldsymbol{r}_h\|_{H^{-1/2}(\Gamma)}$$

by (4.4) and the continuity of the tangential trace operator from $H(\mathrm{curl}, \Omega)$ to $H^{-1/2}(\Gamma)^3$. Additionally

$$\mathrm{div}_\Gamma \boldsymbol{m}_h = \mathrm{div}_\Gamma \boldsymbol{r}_h - (\nabla \times \boldsymbol{w}_h) \cdot \boldsymbol{n} = 0,$$

by construction of $\boldsymbol{w}_h$. We then apply Lemma 4.1 to find $\phi_h \in P_h$ such that $\boldsymbol{r}_h - \boldsymbol{w}_h \times \boldsymbol{n} = \mathrm{curl}_\Gamma \phi_h$ and use Lemma 4.2 to bound

$$(4.5) \qquad \|\phi_h\|_{H^{1/2}(\Gamma)} \lesssim \|\boldsymbol{r}_h - \boldsymbol{w}_h \times \boldsymbol{n}\|_{H^{-1/2}(\Gamma)} \lesssim \|\boldsymbol{r}_h\|_{H^{-1/2}(\Gamma)} + \|\mathrm{div}_\Gamma \boldsymbol{r}_h\|_{H^{-1/2}(\Gamma)}.$$

We then take $u_h$ in the finite element space $W_h$ (see (4.1)) such that $u_h|_\Gamma = \phi_h$ and

$$(4.6) \qquad \|u_h\|_{H^1(\Omega)} \lesssim \|\phi_h\|_{H^{1/2}(\Gamma)}.$$

This can be accomplished by first taking $u \in H^1(\Omega)$ whose trace is $\phi_h$ and satisfying $\|u\|_{H^1(\Omega)} \lesssim \|\phi_h\|_{H^{1/2}(\Gamma)}$ and then applying the Scott-Zhang interpolation operator [16] to $u$. The desired lifting of $\boldsymbol{r}_h$ is the function $\boldsymbol{w}_h + \nabla u_h \in N_h$. The bound

$$\|\boldsymbol{w}_h + \nabla u_h\|_{H(\mathrm{curl},\Omega)} \leq \|\boldsymbol{w}_h\|_{H(\mathrm{curl},\Omega)} + \|\nabla u_h\|_{L^2(\Omega)} \lesssim \|\boldsymbol{r}_h\|_{H^{-1/2}(\Gamma)} + \|\mathrm{div}_\Gamma \boldsymbol{r}_h\|_{H^{-1/2}(\Gamma)}$$

is a direct consequence of (4.4), (4.5), and (4.6). The fact that it is a lifting follows from

$$(\boldsymbol{w}_h + \nabla u_h) \times \boldsymbol{n} = \boldsymbol{w}_h \times \boldsymbol{n} + \mathrm{curl}_\Gamma \phi_h = \boldsymbol{w}_h \times \boldsymbol{n} + \boldsymbol{m}_h = \boldsymbol{r}_h.$$

This finishes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

As an application to Theorem 2.2 we consider problem (1.1) with $X = H(\mathrm{curl}; \Omega)$, $M = H^{-1/2}(\mathrm{div}_\Gamma; \partial\Omega) := \{\mu \in H^{-1/2}(\partial\Omega) : \mathrm{div}_\Gamma \mu \in H^{-1/2}(\partial\Omega)\}$. The finite element space are the Nédélec elements $X_h = \boldsymbol{N}_h$. The bilinear form $B$ and linear form $F$ are as follows

$$B(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega (\nabla \times \boldsymbol{u}) \cdot (\nabla \times \boldsymbol{v}) + \boldsymbol{u} \cdot \boldsymbol{v},$$

$$F(\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}.$$

Theorem 2.2 and the Introductory discussion now gives the following error estimates for Nédélec elements.

**Corollary 4.4.** *Let $X = H(\mathrm{curl}; \Omega)$, $M = H^{-1/2}(\mathrm{div}_\Gamma; \partial\Omega)$ and $X_h = \boldsymbol{N}_h$. Let $\boldsymbol{u}$ satisfy (1.1) and $\boldsymbol{u}_h)$ satisfy (1.2) then the following holds*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{H(\mathrm{curl};\Omega)} \lesssim \inf_{\boldsymbol{v} \in \boldsymbol{N}_h} \|\boldsymbol{u} - \boldsymbol{v}\|_{H(\mathrm{curl};\Omega)} + \|\boldsymbol{g} - \boldsymbol{g}_h\|_{H^{-1/2}(\mathrm{div}_\Gamma;\Gamma)}.$$

## Appendix A. Proof of Theorem 3.3

For each $K \in \mathfrak{T}_h$ we can find a cut–off function $\omega = \omega_K \in C^\infty(D_K)$ with the following properties:

(A.1a) $\qquad\qquad\qquad\qquad\qquad \omega \equiv 1$ in $K$,

(A.1b) $\qquad\qquad\qquad\qquad\qquad \omega \equiv 0$ in $\Omega \setminus D_K$,

(A.1c) $\qquad\qquad\qquad\qquad\qquad \|D^s \omega\|_{L^\infty(D_K)} \lesssim h_K^{-s}$ for $s = 0, 1, 2$.

Note that

(A.2) $\qquad \|u\|_{H^{3/2+s}(K)} \leq \|\omega u\|_{H^{3/2+s}(\Omega)} \leq C(\| - \triangle(\omega u)\|_{H^{-1/2+s}(\Omega)} + \|\nabla(\omega u) \cdot \boldsymbol{n}\|_{H^s(\Gamma)}),$

by (3.2).

Let us first deal with elements $K$ such that $D_K$ does not contain a face in $\partial\Omega$. In this case $\nabla(\omega u) \cdot \boldsymbol{n} \equiv 0$ on $\partial\Omega$. To bound the first term we let $v \in H^{1/2-s}(\Omega)$, and define $m(v) = \frac{1}{|D_K|} \int_{D_K} v$. Then, we have

$$-\int_\Omega \triangle(\omega u)v = -\int_\Omega \triangle(\omega u)(v - m(v)) - \int_\Omega \triangle(\omega u)m(v) = -\int_\Omega (2\nabla\omega \cdot \nabla u + u\triangle\omega)(v - m(v))$$

where we used that $\triangle u = 0$ and that $\nabla(\omega u) \cdot \boldsymbol{n} \equiv 0$ on $\partial\Omega$. Therefore

$$-\int_\Omega \triangle(\omega u)v \leq (\|\nabla\omega \cdot \nabla u\|_{L^2(D_K)} + \|u\triangle\omega\|_{L^2(D_T)})\|v - m(v)\|_{L^2(D_K)}$$

$$\lesssim (h_T^{-1/2-s}\|\nabla u\|_{L^2(D_K)} + h_K^{-3/2-s}\|u\|_{L^2(D_K)})\|v\|_{H^{1/2-s}(D_K)},$$

where we have used (A.1c), the Poincaré inequality and an interpolation argument. Taking the supremum over $v$ we have

$$\| - \triangle(\omega u)\|_{H^{-1/2+s}(\Omega)} \leq (h_K^{-1/2-s}\|\nabla u\|_{L^2(D_K)} + h_K^{-3/2-s}\|u\|_{L^2(D_K)}).$$

Hence, in the case $D_K$ does not contain a face in $\partial\Omega$ we have

$$\|u\|_{H^{3/2+s}(K)} \lesssim (h_K^{-1/2-s}\|\nabla u\|_{L^2(D_K)} + h_K^{-3/2-s}\|u\|_{L^2(D_K)})$$

If we replace $u$ with $m(u)$, and note that

$$\|u - m(u)\|_{L^2(D_K)} \lesssim h_K\|\nabla u\|_{L^2(D_K)}$$

we get

$$\|\nabla u\|_{H^{1/2+s}(K)} \lesssim h_K^{-1/2-s}\|\nabla u\|_{L^2(D_K)}.$$

Next we consider the case when $\partial D_K$ contains one or more faces on $\Gamma$. To bound the first term in the right of (A.2) we get

$$
\begin{aligned}
\left|\int_\Omega \triangle(\omega u)v\right| &\leq (2\|\nabla\omega\cdot\nabla u\|_{L^2(D_K)} + \|u\triangle\omega\|_{L^2(D_T)})\|v\|_{L^2(D_K)} \\
&\lesssim (h_K^{-1/2-s}\|\nabla u\|_{L^2(D_K)} + h_K^{-3/2-s}\|u\|_{L^2(D_K)})h_K^{1/2-s}\|v\|_{H^{1/2-s}(D_K)},
\end{aligned}
$$

where we have used (A.1c) and the fact that $v$ vanishes on at least one face of $\partial D_K$, which allows us to use an inequality in the form

(A.3) 
$$\|v\|_{L^2(D_K)} \lesssim h_K^{1/2-s}\|v\|_{H^{1/2-s}(D_K)}.$$

Therefore, as above we have that

$$\| -\triangle(\omega u)\|_{H^{-1/2+s}(\Omega)} \lesssim h_K^{-1/2-s}\|\nabla u\|_{L^2(D_K)} + h_K^{-3/2-s}\|u\|_{L^2(D_K)}.$$

To bound the second term on the right of (A.2) we first use the product rule and get

$$
\begin{aligned}
\|\nabla(\omega u)\cdot\boldsymbol{n}\|_{H^s(\partial\Omega)} &\leq \|u\nabla\omega\cdot\boldsymbol{n}\|_{H^s(\partial\Omega)} + \|\omega g\|_{H^s(\Gamma)} \\
&\lesssim h_K^{-1/2-s}\|u\nabla\omega\|_{L^2(D_T)} + h_K^{1/2-s}\|\nabla(u\nabla\omega)\|_{L^2(D_T)} + \|\omega g\|_{H^s(\Gamma)} \\
&\lesssim h_K^{-3/2-s}\|u\|_{L^2(D_T)} + h_K^{-1/2-s}\|\nabla u\|_{L^2(D_T)} + \|\omega g\|_{H^s(\Gamma)},
\end{aligned}
$$

after using a localized version of the trace theorem and (A.1c). By a simple interpolation argument and (A.1c), we have

$$\|\omega g\|_{H^s(\Gamma)} \lesssim \|g\|_{H^s(\partial D_K\cap\Gamma)} + h_K^{-s}\|g\|_{L^2(\partial D_K\cap\Gamma)}.$$

Combining the above inequalities we have

$$
\begin{aligned}
\|u\|_{H^{3/2+s}(K)} \lesssim\ & (h_K^{-1/2-s}\|\nabla u\|_{L^2(D_K)} + h_K^{-3/2-s}\|u\|_{L^2(D_K)}) \\
& +(\|g\|_{H^s(D_K\cap\Gamma)} + h_K^{-s}\|g\|_{L^2(D_K\cap\Gamma)}).
\end{aligned}
$$

If we apply the above argument to $u - m(u)$, we obtain our result.

## References

[1] Mark Ainsworth, William McLean, and Thanh Tran. The conditioning of boundary element equations on locally refined meshes and preconditioning by diagonal scaling. *SIAM J. Numer. Anal.*, 36(6):1901–1932 (electronic), 1999.

[2] Douglas Arnold, Richard Falk and Raganar Winther. Multigrid in H(div) and H(curl). *Numer. Math.*, 85 (2), 197-217,2000.

[3] Ivo Babuška and Gabriel N. Gatica. On the mixed finite element method with Lagrange multipliers. *Numer. Methods Partial Differential Equations*, 19(2):192–210, 2003.

[4] Daniele Boffi, Franco Brezzi, and Michel Fortin. *Mixed finite element methods and applications*, volume 44 of *Springer Series in Computational Mathematics*. Springer, Heidelberg, 2013.

[5] Franco Brezzi, Jim Douglas, Jr., Ricardo Durán, and Michel Fortin. Mixed finite elements for second order elliptic problems in three variables. *Numer. Math.*, 51(2):237–250, 1987.

[6] Snorre H. Christiansen and Ragnar Winther. Smoothed projections in finite element exterior calculus. *Math. Comp.*, 77(262):813–829, 2008.

[7] Monique Dauge. *Neumann and mixed problems on curvilinear polyhedra*, Integral equations operator theory, 15(2), 227–261, 1992.

[8] Leszek Demkowicz, Jayadeep Gopalakrishnan, and Joachim Schöberl. Polynomial extension operators. I. *SIAM J. Numer. Anal.*, 46(6):3006–3031, 2008.

[9] Víctor Domínguez and Francisco-Javier Sayas. Stability of discrete liftings. *C. R. Math. Acad. Sci. Paris*, 337(12):805–808, 2003.

[10] Gabriel N. Gatica, Ricardo Oyarzúa, and Francisco-Javier Sayas. A twofold saddle point approach for the coupling of fluid flow with nonlinear porous media flow. *IMA J. Numer. Anal.*, 32(3):845–887, 2012.

[11] Antonio Márquez, Salim Meddahi, and Francisco-Javier Sayas. Strong coupling of finite element methods for the stokes-darcy problem. submitted (arXiv:1203:4717), 2012.

[12] J.-C. Nédélec. Mixed finite elements in $\mathbf{R}^3$. *Numer. Math.*, 35(3):315–341, 1980.

[13] J.-C. Nédélec. A new family of mixed finite elements in $\mathbf{R}^3$. *Numer. Math.*, 50(1):57–81, 1986.

[14] Francisco Javier Sayas. Infimum-supremum. *Bol. Soc. Esp. Mat. Apl. SeMA*, (41):19–40, 2007.

[15] Joachim Schöberl. A posteriori error estimates for Maxwell equations. *Math. Comp.*, 77(262):633–649, 2008.

[16] L. Ridgway Scott and Shangyou Zhang. Finite element interpolation of nonsmooth functions satisfying boundary conditions. *Math. Comp.*, 54(190):483–493, 1990.

[17] Olof B. Widlund. *An Extension Theorem for Finite Element Spaces with Three Applications*, Proceedings of a GAMM Seminar on Numerical Techniques in Continuum Mechanics, held in Kiel, Germany January 17-19, 1986 (Wolfgang Hackbusch and Kristian Witsch, eds), Friedr. Vieweg and Sohn, Braunschweig/Wiesbaden, 1987, 110–122.

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