MAX-NORM STABILITY OF LOW ORDER TAYLOR-HOOD ELEMENTS IN THREE DIMENSIONS

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ABSTRACT. We prove stability in $W^{1,\infty}(\Omega)$ and $L^\infty(\Omega)$ for the velocity and pressure approximations, respectively, using the lowest-order Taylor-Hood finite element spaces to solve the three dimensional Stokes problem. The domain Ω is assumed to be a convex polyhedra.

Keywords: maximum norm, finite element, optimal error estimates, Stokes.

1. Introduction

Consider the Stokes problem on a convex polyhedral domain $\Omega \subset \mathbb{R}^3$

$$-\Delta\vec{u} + \nabla p = \vec{f} \qquad \text{in } \Omega \tag{1.1a}$$

$$\nabla \cdot \vec{u} = 0 \qquad \text{in } \Omega \tag{1.1b}$$

$$\vec{u} = \vec{0} \qquad \text{on } \partial\Omega. \tag{1.1c}$$

Here $\vec{u}$ is the velocity and p is the pressure. The aim of this paper is to prove $W^{1,\infty}$ stability of the lowest order Taylor-Hood (see for example [1]) approximation in three dimensions. More specifically, we prove the bound

$$\|\nabla\vec{u}_h\|_{L^\infty(\Omega)} + \|p_h\|_{L^\infty(\Omega)} \;\leq\; C(\|\nabla\vec{u}\|_{L^\infty(\Omega)} + \|p\|_{L^\infty(\Omega)}).$$

where $\vec{u}_h \in \vec{V}_h$, $p_h \in M_h$ are the Taylor-Hood approximations.

In previous papers, $W^{1,\infty}$ [18, 5] stability was proven for many inf-sup stable pair of spaces, but one major exception was the lowest order Taylor-Hood pair in three dimensions. The reason for this is that in both papers it was assumed that there exists a Fortin projection Π_h (i.e. it commutes with the divergence operator) to the finite element velocity space that is quasi-local, i.e. $\Pi_h \in \mathcal{L}(H_0^1(\Omega)^3, \vec{V}_h)$ satisfies the following properties

$$(q_h, \nabla\cdot(\Pi_h(\vec{w}) - \vec{w}))_\Omega \;=\; 0, \qquad \forall \vec{w} \in H_0^1(\Omega)^3, \quad \forall q_h \in M_h.$$

$$|\Pi_h(\vec{v}) - \vec{v}|_{W^{m,q}(T)} \;\leq\; Ch_T^{s-m+3(\frac{1}{q}-\frac{1}{p})}|v|_{W^{s,p}(\Delta T)}, \qquad \forall T \in \mathcal{T}_h, \quad \forall \vec{v} \in W^{s,p}(\Omega)^3$$

for all real numbers $1 \leq sk+1, 1 \leq p, q \leq \infty$, and integer $m = 0$ or 1 such that $W^{s,p}(\Omega) \subset W^{m,q}(\Omega)$. The constant C is independent of h and T, and ΔT is a suitable macro-element containing T. Although such a Fortin projection exists for many inf-sup pair of spaces [16], existence of a quasi-local Fortin projection for the lowest-order Taylor-Hood element in three

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dimensions is still open. In this paper, we instead use a quasi-local inf-sup condition which holds for the Taylor-Hood element and avoid the use of a Fortin projection.

The local inf-sup condition has been used before by Arnold and Liu [20] to prove local energy estimates for Stokes problem. The local energy results in Arnold and Liu were proven only for interior domains. Chen [19] assuming local energy results (both interior domains and also subdomains touching the boundary $\partial\Omega$) proved $W^{1,\infty}$ stability for finite element approximations to the Stokes problem for domains Ω that have a smooth boundary.

The techniques used by Chen [19] cannot easily be extended to our setting where we assume that Ω is a convex polyhedral domain. First, higher elliptic regularity results were used by Chen, which do not hold in our setting. Second, we cannot use directly the local energy estimates that Chen assumed because this will require us to estimate the pressure error in a negative order norm which we do not know how to estimate with the given regularity of the problem. Instead we prove a local energy estimate that does not contain the error of the pressure which is very similar to the estimates obtained in [5] (see also [6]). Of course, the estimates derived in [5] assumed the existence of a quasi-local Fortin projection.

There will be many similarities between the proofs in this paper and the proofs in article [5]. In order to make our paper self contained we provide many details. However, we will compare the individual results below to corresponding results in [5]. We prove max-norm estimates for Stokes elements which satisfy assumptions A1-A6 below. As a corollary we show that the lowest-order Taylor-Hood element in three dimensions satisfies these assumptions. For simplicity we only consider Stokes elements that use continuous pressures.

2. $W^{1,\infty}$ stability result

In this section we state our main result in Theorem 1. The finite element approximation problems, and the assumptions of our result are presented bellow.

2.1. **Preliminaries and Assumptions.** For the finite element approximation of the problem, let $\mathcal{T}_h$, $0 < h < 1$, be a sequence of partitions of Ω, $\overline{\Omega} = \cup_{T\in\mathcal{T}_h}\overline{T}$, with the elements T mutually disjoint. Let h_T denote the diameter of the element T and $h := \max_T h_T$. The partitions are face-to-face so that simplices meet only in full lower-dimensional faces or not at all. The family of triangulation are shape regular and quasi-uniform. The finite element velocity space is denoted by $\vec{V}_h \subset [H_0^1(\Omega)]^3$ and the pressure space is denoted by $M_h \subset L^2(\Omega)$. We assume that $\vec{V}_h$ contains the space of piecewise polynomials of degree k ($k \geq 2$) and is contained is the space of piecewise polynomials of degree l. We assume that M_h contains the space of *continuous* piecewise polynomial of degree $k-1$.

The finite element approximation $(\vec{u}_h, p_h) \in \vec{V}_h \times M_h$ solves

$$(\nabla\vec{u}_h, \nabla\vec{v}) - (p_h, \nabla\cdot\vec{v}) = (\vec{f}, \vec{v}) \qquad \forall \vec{v} \in \vec{V}_h \tag{2.1a}$$

$$(q, \nabla\cdot\vec{u}_h) = 0 \qquad \forall q \in M_h \tag{2.1b}$$

where $(\cdot,\cdot)$ denotes the usual $L^2(\Omega)$ inner product. The approximation to the pressure p_h is unique up to a constant. We can for example require $p, p_h \in L_0^2(\Omega)$, i.e., $\int_\Omega p(x)dx = \int_\Omega p_h(x)dx = 0$. Instead, we will require

$$\int_\Omega p(x)\phi(x)dx = \int_\Omega p_h(x)\phi(x)dx = 0, \tag{2.2}$$

where $\phi(x)$ is an infinitely differentiable function on Ω that vanishes in a neighborhood of the edges and satisfies

$$\int_\Omega \phi(x)dx \;=\; 1. \tag{2.3}$$

Without loss of generality, we fix ϕ as above and assume p, p_h satisfy (2.2). In other words, we let p and p_h belong to the space L^2_ϕ.

We assume the existence of two projection operators $\mathbf{P} : [H^1_0(\Omega)]^3 \to \vec{V}_h$ and $\mathbf{R} : L^2(\Omega) \to M_h$ with following properties

A1 (Stability). There exists constants C_1, C_2 independent of h such that

$$\|\mathbf{P}\vec{v}\|_{H^1(\Omega)} \;\le\; C_1\|\vec{v}\|_{H^1(\Omega)}, \qquad \forall \vec{v} \in [H^1_0(\Omega)]^3. \tag{2.4a}$$

$$\|\mathbf{R}q\|_{L^2(\Omega)} \;\le\; C_2\|q\|_{L^2(\Omega)}, \qquad \forall q \in L^2(\Omega). \tag{2.4b}$$

A2 (Local Approximation) Let $Q \subset Q_d \subset \Omega$ with $d \ge \kappa h$, for some fixed κ sufficiently large and $Q_d = \{x \in \Omega : dist(x, \Omega) \le d\}$. For any $\vec{v} \in [H^l(Q_d)]^3$ there exists C independent of h and $\vec{v}$ such that

$$\|\vec{v} - \mathbf{P}\vec{v}\|_{L^2(Q)} + h\|\vec{v} - \mathbf{P}\vec{v}\|_{H^1(Q)} \;\le\; Ch^l\|\vec{v}\|_{H^l(Q_d)} \quad \text{for } l = 1, 2. \tag{2.5a}$$

For any $\vec{v} \in [C^{1+\sigma}(Q_d)]^3$ there exists a constant C independent of h such that

$$\|\vec{v} - \mathbf{P}\vec{v}\|_{W^t_\infty(Q)} \;\le\; Ch^{1+\sigma-t}\|\vec{v}\|_{C^{1+\sigma}(Q_d)} \quad \text{for } t = 0, 1, \tag{2.5b}$$

where

$$\|\vec{v}\|_{C^{1+\sigma}(Q)} = \|\vec{v}\|_{C^1(Q)} + \sup_{\substack{x,y\in Q\\ i\in\{1,2,3\}}} \frac{|\vec{e}_i \cdot (\nabla\vec{v}(x) - \nabla\vec{v}(y))|}{|x-y|^\sigma}$$

For any $q \in H^1(Q_d)$ there exists a constant C independent of h and Q such that

$$\|q - \mathbf{R}q\|_{L^2(Q)} \;\le\; Ch\|q\|_{H^1(Q_d)}. \tag{2.5c}$$

For any $q \in C^\sigma(Q_d)$ there exists a constant C independent of h such that

$$\|q - \mathbf{R}q\|_{L^\infty(Q)} \;\le\; Ch^\sigma\|q\|_{C^\sigma(Q_d)}. \tag{2.5d}$$

A3 (Superapproximation). Let $\omega \in C^\infty_0(Q_d)$ be a smooth cut-off function such that $\omega \equiv 1$ on Q and

$$|D^s\omega| \;\le\; Cd^{-s}, \quad s = 0, 1. \tag{2.6a}$$

We assume that

$$\|\omega^2\vec{v} - \mathbf{P}(\omega^2\vec{v})\|_{L^2(Q)} \;\le\; Chd^{-1}\|\vec{v}\|_{L^2(Q_d)}, \quad \forall \vec{v} \in \vec{V}_h \tag{2.6b}$$

$$\|\nabla(\omega^2\vec{v} - \mathbf{P}(\omega^2\vec{v}))\|_{L^2(Q)} \;\le\; Cd^{-1}\|\vec{v}\|_{L^2(Q_d)}, \quad \forall \vec{v} \in \vec{V}_h \tag{2.6c}$$

and

$$\|\nabla(\omega^2 q - \mathbf{R}(\omega^2 q))\|_{L^2(Q)} \leq Chd^{-1}\|q\|_{L^2(Q_d)}, \quad \forall q \in M_h. \tag{2.6d}$$

A4 (Inverse inequality). There exists a constant C independent of h such that

$$\|\vec{v}\|_{H^1(Q)} \leq Ch^{-1}\|\vec{v}\|_{L^2(Q_d)} \tag{2.7a}$$

A5 (Local inf-sup condition). There exists $\beta > 0$ and $\ell \geq 1$ such that for every set $B \subset \Omega$ there exist $B_h \supseteq B$, with $dist(B, \partial B_h \backslash \partial\Omega) \leq \ell h$, and $\beta > 0$ such that

$$\sup_{\substack{\vec{v} \in \vec{V}_h \backslash \{\vec{0}\} \\ \text{supp}(\vec{v}) \subset B_h}} \frac{(q, \nabla \cdot \vec{v})}{\|\vec{v}\|_{H^1(B_h)}} \geq \beta h \|\nabla q\|_{L^2(B)}, \qquad \forall q \in M_h. \tag{2.8}$$

A6 (L^1 inf-sup condition). There exists a constant $\gamma > 0$ independent of h such that

$$\sup_{\vec{v} \in \vec{V}_h \backslash \{\vec{0}\}} \frac{(q, \nabla \cdot \vec{v})}{\|\vec{v}\|_{W^1_\infty(\Omega)}} \geq \gamma h \|\nabla q\|_{L^1(\Omega)}, \qquad \forall q \in M_h. \tag{2.9}$$

When $B = \Omega$ property **A5** is the standard inf-sup condition for Stokes finite element spaces. We now state the main result of the paper.

Theorem 1. *Let $(\vec{u}, p)$ and $(\vec{u}_h, p_h)$ satisfy (1.1) and (2.1), respectively. Under the Assumptions 1-6, there exists a constant C independent of h such that*

$$\|\nabla \vec{u}_h\|_{L^\infty(\Omega)} + \|p_h\|_{L^\infty(\Omega)} \leq C(\|\nabla \vec{u}\|_{L^\infty(\Omega)} + \|p\|_{L^\infty(\Omega)}).$$

Of course, as a corollary we have

$$\|\nabla(u - \vec{u}_h)\|_{L^\infty(\Omega)} + \|p - p_h\|_{L^\infty(\Omega)} \leq C(\sup_{\vec{v} \in \vec{V}_h} \|\nabla(\vec{u} - \vec{v})\|_{L^\infty(\Omega)} + \sup_{q \in Q_h} \|p - q\|_{L^\infty(\Omega)}).$$

The proof of Theorem 1 is presented in section 4. In section 4.1 we state some Green's function estimates, established in [9, 7, 8, 11] which are used in section 4.2 to prove a key estimate for the gradient of the finite element approximation of the Green's function in the L^1 norm. Finally in section 4.3 we prove the stability in L^∞ norm of the velocity and the pressure.

3. Local energy estimate

An essential ingredient of our proof is the local energy estimate that we derive in this section. Consider $(\vec{v}, q) \in [H^1_0(\Omega)]^3 \times L^2(\Omega)$ and $(\vec{v}_h, q_h) \in \vec{V} \times M_h$ satisfying the following orthogonality relation:

$$(\nabla(\vec{v} - \vec{v}_h), \nabla\vec{\chi}) - (q - q_h, \nabla \cdot \vec{\chi}) = 0 \qquad \forall \vec{\chi} \in \vec{V}_h \tag{3.1a}$$

$$(w, \nabla \cdot (\vec{v} - \vec{v}_h)) = 0 \qquad \forall w \in M_h \tag{3.1b}$$

Theorem 2. *Suppose $(\vec{v}, q) \in [H^1_0(\Omega)]^3 \times L^2(\Omega)$ and $(\vec{v}_h, q_h) \in \vec{V} \times M_h$ satisfy (3.1). Then, there exists a constant $C > 0$ such that for every pair of sets $A_1 \subset A_2 \subset \Omega$ such that $dist(\overline{A_1}, \partial A_2 \backslash \partial\Omega) \geq d \geq \kappa h$ (for some fixed large enough constant κ) and for any $\varepsilon \in (0, 1)$, the following bound holds:*

$$\begin{aligned}\|\nabla(\vec{v}-\vec{v}_h)\|_{L^2(A_1)} &\leq C\left(\varepsilon^{-1}\|\nabla(\vec{v}-\mathbf{P}\vec{v})\|_{L^2(A_2)}+(\varepsilon d)^{-1}\|(\vec{v}-\mathbf{P}\vec{v})\|_{L^2(A_2)}+\|q-\mathbf{R}q\|_{L^2(A_2)}\right)\\ &\quad+\varepsilon\|\nabla(\vec{v}-\vec{v}_h)\|_{L^2(A_2)}+\frac{C}{\varepsilon d}\|(\vec{v}-\vec{v}_h)\|_{L^2(A_2)}\end{aligned}$$

The above result is similar to Theorem 2 in [5]. The main difference is that the term $\varepsilon^{-1}\|\nabla(\vec{v}-\mathbf{P}\vec{v})\|_{L^2(A_2)}$ appears in our result.

Proof. We first prove the statement with the following assumption for the sets A_1 and A_2.

A7 Redefine the sets as $A_s = B_{sd/2}\cap\Omega$ for $s = 1, 2$, where $B_{sd/2}$ is a ball of radius $sd/2$ centered at $x_0 \in \bar{\Omega}$ and assume that there exists a ball $B \subset A_1$, such that $diam(A_1) \leq d < \rho\, diam(B)$, where ρ is a fixed constant that only depends on Ω.

We will compete the proof for general sets by a covering argument.

Consider $\omega \in C_0^\infty(A_{3/2})$ the cut-off function defined in assumption **A3**, for $Q = A_1$ and $Q_d = A_2$. Define $\vec{e} = \vec{v}-\vec{v}_h$, $\vec{\eta} = \vec{v}-\mathbf{P}\vec{v}$, $\vec{\xi} = \mathbf{P}\vec{v}-\vec{v}_h$, $e_q = q-q_h$, $\eta_q = q-\mathbf{R}q$ and $\xi_q = \mathbf{R}q-q_h$ then

$$\|\nabla\vec{e}\|_{L^2(A_1)} \leq \|\omega\nabla\vec{e}\|_{L^2(\Omega)} = (\nabla\vec{e},\nabla(\omega^2\vec{e}))-(\nabla\vec{e},\nabla(\omega^2)\otimes\vec{e}) \tag{3.2}$$

Throughout this proof we will estimate the middle term of (3.2). We first obtain an estimate for the second term on the right hand side of (3.2), by Cauchy-Schwartz (C-S.) inequality and the property of ω (2.6a) we obtain

$$-(\nabla\vec{e},\nabla(\omega^2)\otimes\vec{e}) \leq \frac{C}{d}\|\omega\nabla\vec{e}\|_{L^2(\Omega)}\|\vec{e}\|_{L^2(A_{3/2})}.$$

Applying the arithmetic-geometric mean (a-g.m.) inequality and (3.2), we get

$$\frac{1}{2}\|\omega\nabla\vec{e}\|_{L^2(\Omega)} \leq (\nabla\vec{e},\nabla(\omega^2\vec{e}))+\frac{C}{d^2}\|\vec{e}\|^2_{L^2(A_{3/2})}. \tag{3.3}$$

Now for the first term on the right hand side of (3.3), we use $\vec{e} = \vec{\eta}+\vec{\xi}$, obtaining

$$\begin{aligned}(\nabla\vec{e},\nabla(\omega^2\vec{e})) &= (\nabla\vec{e},\nabla(\omega^2\vec{\xi}))+(\nabla\vec{e},\nabla(\omega^2\vec{\eta}))\\ &\leq (\nabla\vec{e},\nabla(\omega^2\vec{\xi}))+C\|\omega\nabla\vec{e}\|_{L^2(\Omega)}(\|\nabla\vec{\eta}\|_{L^2(A_{3/2})}+\frac{1}{d}\|\vec{\eta}\|_{L^2(A_{3/2})}),\end{aligned} \tag{3.4}$$

in the last line we have estimated the second term using (2.6a). The term $(\nabla\vec{e},\nabla(\omega^2\vec{\xi}))$ is more involved, we decompose it as follows

$$(\nabla\vec{e},\nabla(\omega^2\vec{\xi})) = (\nabla\vec{e},\nabla\mathbf{P}(\omega^2\vec{\xi}))+(\nabla\vec{e},\nabla(\omega^2\vec{\xi})-\mathbf{P}(\omega^2\vec{\xi})) =: I_1+I_2. \tag{3.5}$$

Summarizing, by (3.4), the a-g.m. inequality , the definition of I_1 and I_2 and (3.3) we have

$$\frac{1}{4}\|\omega\nabla\vec{e}\|_{L^2(\Omega)} \leq I_1+I_2+C\|\nabla\vec{\eta}\|^2_{L^2(A_{3/2})}+\frac{C}{d^2}\|\vec{\eta}\|^2_{L^2(A_{3/2})}+\frac{C}{d^2}\|\vec{e}\|^2_{L^2(A_{3/2})}. \tag{3.6}$$

We estimate I_2 applying C-S. inequality, the superapproximation assumption **A3** (2.6b) and the a-g.m. inequality for $0<\varepsilon<1$, obtaining

$$
\begin{aligned}
I_2 &\leq \|\nabla \vec{e}\|_{L^2(A_{3/2})} \|\nabla(\omega^2\vec{\xi} - \mathbf{P}(\omega^2\vec{\xi}))\|_{L^2(A_{3/2})} \leq \|\vec{e}\|_{L^2(A_{3/2})} \frac{C}{d} \|\vec{\xi}\|_{L^2(A_2)} \\
&= \varepsilon \|\nabla \vec{e}\|^2_{L^2(A_{3/2})} + \frac{C}{\varepsilon d^2}(\|\vec{\eta}\|^2_{L^2(A_2)} + \|\vec{e}\|^2_{L^2(A_2)}),
\end{aligned}
$$

To estimate I_1 we use (3.1a), then adding and subtracting $\mathbf{R}q$ we break I_1 into three parts

$$
\begin{aligned}
I_1 &= -(e_q, \nabla \cdot \mathbf{P}(\omega^2\vec{\xi})) \\
&= -(e_q, \nabla \cdot (\omega^2\vec{\xi})) - (\eta_q, \nabla \cdot (\mathbf{P}(\omega^2\vec{\xi}) - \omega^2\vec{\xi})) - (\xi_q, \nabla \cdot (\mathbf{P}(\omega^2\vec{\xi}) - \omega^2\vec{\xi})) = I_3 + I_4 + I_5
\end{aligned}
$$

Similar to the estimate for I_2, we estimate I_4

$$
\begin{aligned}
I_4 &\leq \|\eta_q\|_{L^2(A_{3/2})} \|\nabla \cdot (\mathbf{P}(\omega^2\vec{\xi}) - \omega^2\vec{\xi})\|_{L^2(A_{3/2})} \leq \|\eta_q\|_{L^2(A_{3/2})} \frac{C}{d} \|\vec{\xi}\|_{L^2(A_2)} \\
&= \|\eta_q\|^2_{L^2(A_{3/2})} + \frac{C}{d^2}(\|\vec{\eta}\|^2_{L^2(A_2)} + \|\vec{e}\|^2_{L^2(A_2)}),
\end{aligned}
$$

Next we estimate I_5. We use integration by parts (taking into account that M_h is continuous), C-S. inequality, superapproximation assumption **A3**

$$
\begin{aligned}
I_5 &= (\nabla \xi_q, \mathbf{P}(\omega^2\vec{\xi}) - \omega^2\vec{\xi}) \leq \|\nabla \xi_q\|_{L^2(A_{3/2})} \|\mathbf{P}(\omega^{\mathbf{2}}\vec{\xi}) - \omega^2\vec{\xi}\|_{L^2(A_{3/2})} \\
&\leq \|\nabla \xi_q\|_{L^2(A_{3/2})} \frac{Ch}{d} \|\vec{\xi}\|_{L^2(A_2)}
\end{aligned}
$$

Using the local inf-sup condition assumption **A5** we know there exists $A_{3/2} \subset B_h$ with $\operatorname{dist}(A_{3/2}, \partial B_h \backslash \partial\Omega) \leq \ell h$ such that

$$
\beta \|\nabla \xi_q\|_{L^2(A_{3/2})} \leq \sup_{\substack{\vec{z} \in \vec{V}_h \\ \operatorname{supp} \vec{z} \subset B_h}} \frac{(\xi_q, \nabla \cdot \vec{z})}{\|\vec{z}\|_{H^1(B_h)}}.
$$

Since $d \geq \kappa h$ and we can choose $\kappa > 2\ell$ then we have that $B_h \subset A_2$, and so

$$
\beta \|\nabla \xi_q\|_{L^2(A_{3/2})} \leq \sup_{\substack{\vec{z} \in \vec{V}_h \\ \operatorname{supp} \vec{z} \subset A_2}} \frac{(\xi_q, \nabla \cdot \vec{z})}{\|\vec{z}\|_{H^1(A_2)}}
$$

Now using equation (3.1a) and a-g.m. inequality to obtain

$$
\begin{aligned}
I_5 &\le \frac{C}{d} \sup_{\substack{\vec{z}\in\vec{V}_h\\ \operatorname{supp}\vec{z}\subset A_2}} \frac{(\xi_q, \nabla\cdot\vec{z})}{\|\vec{z}\|_{H^1(A_2)}} \|\vec{\xi}\|_{L^2(A_2)} \\
&\le \frac{C}{d}\left(\|\eta_q\|_{L^2(A_2)} + \sup_{\substack{\vec{z}\in\vec{V}_h\\ \operatorname{supp}\vec{z}\subset A_2}} \frac{(e_q, \nabla\cdot\vec{z})}{\|\vec{z}\|_{H^1(A_2)}} \right) \|\vec{\xi}\|_{L^2(A_2)} \\
&\le \frac{C}{d}\left(\|\eta_q\|_{L^2(A_2)} + \sup_{\substack{\vec{z}\in\vec{V}_h\\ \operatorname{supp}\vec{z}\subset A_2}} \frac{(\nabla e_q, \vec{z})}{\|\vec{z}\|_{H^1(A_2)}} \right) \|\vec{\xi}\|_{L^2(A_2)} \\
&\le \frac{C}{d}\left(\|\eta_q\|_{L^2(A_2)} + \sup_{\substack{\vec{z}\in\vec{V}_h\\ \operatorname{supp}\vec{z}\subset A_2}} \frac{(\nabla\vec{e}, \nabla\vec{z})}{\|\vec{z}\|_{H^1(A_2)}} \right) \|\vec{\xi}\|_{L^2(A_2)} \\
&\le \frac{C}{d}\left(\|\eta_q\|_{L^2(A_2)} + \|\nabla\vec{e}\|_{L^2(A_2)} \right) \|\vec{\xi}\|_{L^2(A_2)} \\
&\le \|\eta_q\|^2_{L^2(A_2)} + \varepsilon\|\nabla\vec{e}\|^2_{L^2(A_2)} + \frac{C}{d^2}(1+\varepsilon^{-1})(\|\vec{e}\|^2_{L^2(A_2)} + \|\vec{\eta}\|^2_{L^2(A_2)})
\end{aligned}
$$

Until now, combining the estimates for I_2, I_4 and I_5 in (3.6) we have

$$
\frac{1}{4}\|\omega\nabla\vec{e}\|_{L^2(\Omega)} \le I_3 + C\|\nabla\vec{\eta}\|^2_{L^2(A_2)} + \|\eta_q\|^2_{L^2(A_{3/2})} + \frac{C}{\varepsilon d^2}(\|\vec{\eta}\|^2_{L^2(A_2)} + \|\vec{e}\|^2_{L^2(A_2)}) + \varepsilon\|\nabla\vec{e}\|^2_{L^2(A_2)}
$$

It remains to estimate I_3. Again we use that $e_q = \eta_q + \xi_q$ decomposing I_3 into two terms

$$
I_3 = -(e_q, \nabla\cdot(\omega^2\vec{\xi})) = -(\eta_q, \nabla\cdot(\omega^2\vec{\xi})) - (\xi_q, \nabla\cdot(\omega^2\vec{\xi})) =: I_6 + I_7
$$

The estimate for I_6 is obtained applying C-S. inequality, property (2.6a) for $s = 0$ and $s = 1$, and the a-g.m. inequality, resulting

$$
I_6 \le C\|\eta_q\|^2_{L^2(A_{3/2})} + \frac{1}{8}\|\omega\nabla\vec{e}\|^2_{L^2(A_{3/2})} + C\|\nabla\vec{\eta}\|^2_{L^2(A_{3/2})} + \frac{C}{d^2}\|\vec{\eta}\|^2_{L^2(A_{3/2})} + \frac{C}{d^2}\|\vec{e}\|^2_{L^2(A_{3/2})}.
$$

In order to estimate I_7 we note that, by definition of ω

$$
(c, \nabla\cdot(\omega^2\vec{\xi})) = 0
$$

for c constant. Set $\hat{\xi}_q = \xi_q - c$ and choose c such that $\hat{\xi}_q$ has zero mean on $A_{3/2}$. Then by product rule and adding and subtracting $\mathbf{R}(\omega^2(\hat{\xi}_q))$ we have

$$
\begin{aligned}
I_7 &= -(\hat{\xi}_q, \nabla(\omega^2)\cdot\vec{\xi}) - (\hat{\xi}_q, \omega^2\nabla\cdot\vec{\xi}) \\
&= -(\hat{\xi}_q, \nabla(\omega^2)\cdot\vec{\xi}) - (\omega^2\hat{\xi}_q - \mathbf{R}(\omega^2\hat{\xi}_q), \nabla\cdot\vec{\xi}) - (\mathbf{R}(\omega^2\hat{\xi}_q), \nabla\cdot\vec{\xi}) =: I_8 + I_9 + I_{10}
\end{aligned}
$$

We estimate I_8 using C-S. inequality and property (2.6a)

$$
I_8 \le \frac{C}{d}\|\hat{\xi}_q\|_{L^2(A_{3/2})}\|\vec{\xi}\|_{L^2(A_{3/2})}.
$$

Using the superapproximation property (2.6c) and the inverse estimate assumption **A4** we estimate I_9 as follows

$$I_9 \;\leq\; \frac{Ch}{d}\|\hat{\xi}_q\|_{L^2(A_{3/2})}\|\nabla\vec{\xi}\|_{L^2(A_{3/2})} \;\leq\; \frac{C}{d}\|\hat{\xi}_q\|_{L^2(A_{3/2})}\|\vec{\xi}\|_{L^2(A_2)}.$$

To estimate I_{10} we apply the equation (3.1b), the property of $\mathbf{R}$ (2.4b), (2.6a), the local inf-sup condition **A5** and C-S. inequality obtaining

$$I_{10} \;=\; (\mathbf{R}(\omega^2\hat{\xi}_q), \nabla\cdot\vec{\eta}) \;\leq\; C\|\hat{\xi}_q\|_{L^2(A_{3/2})}\|\nabla\vec{\eta}\|_{L^2(A_{3/2})}.$$

We claim that $\|\hat{\xi}_q\|_{L^2(A_{3/2})} \leq C(\|\nabla\vec{e}\|_{L^2(A_2)} + \|\eta_q\|_{L^2(A_2)})$. We prove this claim in Lemma 3.1. Therefore, we have

$$\begin{aligned} I_7 \;&\leq\; C(\|\eta_q\|_{L^2(A_2)} + \|\nabla\vec{e}\|_{L^2(A_2)})(\|\nabla\vec{\eta}\|_{L^2(A_{3/2})} + \frac{1}{d}\|\vec{\xi}\|_{L^2(A_2)}) \\ &\leq\; \varepsilon(\|\eta_q\|^2_{L^2(A_2)} + \|\nabla\vec{e}\|^2_{L^2(A_2)}) + \frac{C}{\varepsilon d^2}(\|\vec{\eta}\|^2_{L^2(A_2)} + \|\vec{e}\|^2_{L^2(A_2)}) + \frac{C}{\varepsilon}\|\nabla\vec{\eta}\|^2_{L^2(A_2)}. \end{aligned}$$

The estimates for I_6 and I_7 yield

$$\frac{1}{8}\|\omega\nabla\vec{e}\|_{L^2(\Omega)} \;\leq\; C(\frac{1}{\varepsilon}\|\nabla\vec{\eta}\|^2_{L^2(A_2)} + \|\eta_q\|^2_{L^2(A_2)} + \frac{1}{\varepsilon d^2}\|\vec{\eta}\|^2_{L^2(A_2)}) + \frac{C}{\varepsilon d^2}\|\vec{e}\|^2_{L^2(A_2)} + \varepsilon\|\nabla\vec{e}\|^2_{L^2(A_2)}.$$

The exact statement of Theorem 2 is attached using ε^2. This completes the proof under Assumption **A7**.

Now we extend the result for general sets $A_1 \subset A_2 \subset \Omega$ with $dist(\overline{A_1}, \partial A_2\backslash\partial\Omega) \geq d \geq \kappa h$. It is not difficult to construct a covering $\{G_i\}_{i=1}^M$ of A_1, where $G_i = B_{d/2}(x_i)\cap\Omega$ with the following properties:

(1) $A_1 \subset \bigcup_{i=1}^M G_i$.
(2) $x_i \in A_1$ for each $1 \leq i \leq M$.
(3) Let $H_i = B_d(x_i)\cap\Omega$. There exists a fixed number L such that each point $x \in \bigcup_{i=1}^M H_i$ is contained in at most L sets from $\{H_j\}_{j=1}^M$.
(4) There exists a $\rho > 0$ such that for each $1 \leq i \leq M$ there exists a ball $B \subset G_i$ such that $diam(G_i) \leq \rho diam(B)$.

Since $dist(\overline{A_1}, \partial A_2\backslash\partial\Omega) \geq d$, using property 2 we have that $\bigcup_{i=1}^M H_i \subset A_2$.
Applying the result proved above and using properties 1 and 4 we have

$$\begin{aligned} \|\nabla(\vec{v}-\vec{v}_h)\|^2_{L^2(A_1)} \;&\leq\; \sum_{i=1}^M \|\nabla(\vec{v}-\vec{v}_h)\|^2_{L^2(G_i)} \;\leq\; \sum_{i=1}^M C\Big(\|\nabla(\vec{v}-\mathbf{P}\vec{v})\|^2_{L^2(H_i)} + \|q-\mathbf{R}q\|^2_{L^2(H_i)} \\ &+ (\frac{1}{\varepsilon d})^2\|\vec{v}-\mathbf{P}\vec{v}\|^2_{L^2(H_i)}\Big) + \varepsilon^2\|\nabla(\vec{v}-\vec{v}_h)\|^2_{L^2(H_i)} + (\frac{C}{\varepsilon d})^2\|\vec{v}-\vec{v}_h\|^2_{L^2(H_i)}. \end{aligned}$$

Using property 3 we have

$$
\begin{aligned}
\|\nabla(\vec{v}-\vec{v}_h)\|^2_{L^2(A_1)} \;\le\; & CL\Big(\|\nabla(\vec{v}-\mathbf{P}\vec{v})\|^2_{L^2(A_2)}+\|q-\mathbf{R}q\|^2_{L^2(a_2)} \\
& +(\frac{L}{\varepsilon d})^2\|\vec{v}-\mathbf{P}\vec{v}\|^2_{L^2(A_2)}\Big)+L\varepsilon^2\|\nabla(\vec{v}-\vec{v}_h)\|^2_{L^2(A_2)}+(\frac{CL}{\varepsilon d})^2\|\vec{v}-\vec{v}_h\|^2_{L^2(A_2)}.
\end{aligned}
$$

The exact statement of Theorem 2 is attached using ε^2.

□

The next result is exactly the same as Lemma 3.2 in [5]. However, the proof in [5] used the existence of a quasi-local Fortin projection.

Lemma 3.1. *Under the assumption* ***A7****, there exists a constat C independent of $A_{3/2}$ and $\hat{\xi}_q$, but depends on ρ sucht that*

$$\|\hat{\xi}_q\|_{L^2(A_{3/2})} \;\le\; C(\|\nabla\vec{e}\|_{L^2(A_2)}+\|\eta_q\|_{L^2(A_2)}).$$

Proof. Define $\vec{w}\in H^1_0(A_{3/2})$ as the solution of the problem

$$
\begin{aligned}
\nabla\cdot\vec{w} &= \hat{\xi}_q \quad \text{in } A_{3/2}\\
\vec{w} &= \vec{0} \quad \text{on } \partial A_{3/2}
\end{aligned}
$$

We can choose $\vec{w}$ so that $\|\vec{w}\|_{H^1(A_{3/2})}\le C\|\hat{\xi}_q\|_{L^2(A_{3/2})}$.

By Lemma 3.1 in Chapter III.3 in [11], the constant C is independent of $\hat{\xi}_q$ and depends only on the ratio of the diameter of $A_{3/2}$ and the radius of the largest ball that can be inscribed into $A_{3/2}$ and hence by our hypothesis only depends on ρ. Let us extend $\vec{w}$ on all of Ω by zero outside of $A_{3/2}$. We note that this implies that $\mathbf{P}\vec{w}$ vanishes outside of A_2 by **A3**. Then,

$$
\begin{aligned}
\|\hat{\xi}_q\|^2_{L^2(A_{3/2})} &= (\hat{\xi}_q,\hat{\xi}_q)_{A_{3/2}}=(\hat{\xi}_q,\nabla\cdot\vec{w})=(\xi_q,\nabla\cdot\vec{w})\\
&= (e_q,\nabla\cdot\vec{w})-(\eta_q,\nabla\cdot\vec{w}).
\end{aligned}
$$

Using (3.1a),

$$
\begin{aligned}
(e_q,\nabla\cdot\vec{w}) &= (e_q,\nabla\cdot\mathbf{P}\vec{w})+(e_q,\nabla\cdot(\vec{w}-\mathbf{P}\vec{w}))\\
&= (\nabla\vec{e},\nabla\mathbf{P}\vec{w})+(\eta_q,\nabla\cdot(\vec{w}-\mathbf{P}\vec{w}))+(\xi_q,\nabla\cdot(\vec{w}-\mathbf{P}\vec{w}))\\
&= (\nabla\vec{e},\nabla\mathbf{P}\vec{w})+(\eta_q,\nabla\cdot(\vec{w}-\mathbf{P}\vec{w}))-(\nabla\xi_q,\vec{w}-\mathbf{P}\vec{w})\\
&\le \|\nabla\vec{e}\|_{L^2(A_2)}\|\nabla\mathbf{P}\vec{w}\|_{L^2(A_2)}+\|\eta_q\|_{L^2(A_2)}\|\nabla(\vec{w}-\mathbf{P}\vec{w})\|_{L^2(A_2)}\\
&\qquad \|\nabla\xi_q\|_{L^2(A_{3/2})}\|\vec{w}-\mathbf{P}\vec{w}\|_{L^2(A_{3/2})}\\
&\le C(\|\nabla\vec{e}\|_{L^2(A_2)}+\|\eta_q\|_{L^2(A_2)}+h\|\nabla\xi_q\|_{L^2(A_{3/2})})\|\vec{w}\|_{H^1(A_{3/2})}
\end{aligned}
$$

Using the local inf-sup condition **A5** we have

$$h\|\nabla\xi_q\|_{L^2(A_{3/2})} \;\le\; C(\|\eta_q\|_{L^2(A_2)}+\|\nabla\vec{e}\|_{L^2(A_2)})$$

Therefore

$$
\begin{aligned}
\|\hat{\xi}_q\|^2_{L^2(A_{3/2})} &\le C(\|\eta_q\|_{L^2(A_2)}+\|\nabla\vec{e}\|_{L^2(A_2)})\|\vec{w}\|_{H^1(A_{3/2})}\\
&\le C(\|\eta_q\|_{L^2(A_2)}+\|\nabla\vec{e}\|_{L^2(A_2)})\|\hat{\xi}_q\|_{H^1(A_{3/2})}
\end{aligned}
$$

which implies the result. □

4. Proof of Theorem 1

4.1. **Green's function estimates.** In this section we recall pointwise estimates for the Green's matrix. Let $\phi(z)$ be an infinitely differentiable function in Ω which vanishes in a neighborhood of the edges of Ω such that

$$\int_{\Omega} \phi(x)dx = 1. \tag{4.1}$$

Consider the Stokes problem with non-zero divergence. Let $(\vec{u}, p) \in [H_0^1(\Omega)]^3 \times L_\phi^2(\Omega)$ solve

$$-\Delta\vec{u} + \nabla p = \vec{f} \qquad \text{in } \Omega \tag{4.2a}$$

$$\nabla \cdot \vec{u} = q \qquad \text{in } \Omega \tag{4.2b}$$

$$\vec{u} = \vec{0} \qquad \text{on } \partial\Omega. \tag{4.2c}$$

for arbitrary $\vec{f} \in [H^{-1}(\Omega)]^3$ and $q \in L_0^2(\Omega)$ with q vanishing on the singular points of Ω (see [2]). If $q \in H^1(\Omega) \cap L_0^2(\Omega)$ with q vanishing on the edges of Ω and $\vec{f} \in [L^2(\Omega)]^3$ we have the following elliptic regularity result (see [2])

$$\|\vec{u}\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)} \ \leq \ C(\|\vec{f}\|_{L^2(\Omega)} + \|q\|_{H^1(\Omega)}). \tag{4.3}$$

The Green's matrix for the problem (4.2) $\vec{G}_j = (G_{1,j}, G_{2,j}, G_{3,j})^T$ and the functions $G_{4,j}$ for $j = 1, 2, 3, 4$ are solutions of the problem

$$-\Delta_x\vec{G}_j(x,\xi) + \nabla_x G_{4,j}(x,\xi) = \delta(x-\xi)(\delta_{1,j}, \delta_{2,j}, \delta_{3,j})^T \quad \text{for } x, \xi \in \Omega \tag{4.4a}$$

$$\nabla_x \cdot \vec{G}_j(x,\xi) = (\delta(x-\xi) - \phi(x))\delta_{4,j} \quad \text{for } x, \xi \in \Omega \tag{4.4b}$$

$$\vec{G}_j(x,\xi)(= \vec{0} \qquad \text{for } x \in \partial\Omega,\ \xi \in \Omega. \tag{4.4c}$$

and $G_{4,j}$ satisfies the condition

$$\int_{\Omega} G_{4,j}(x,\xi)\phi(x)dx \ = \ 0, \quad \text{for } \xi \in \Omega, \quad j = 1, 2, 3, 4. \tag{4.5}$$

Here, $\delta(x)$ is the delta function, and $\delta_{i,j}$ is the Kronecker delta symbol. In addition,

$$G_{i,j}(x,\xi) \ = \ G_{j,i}(\xi, x) \quad \text{for } x, \xi \in \Omega, \quad i, j = 1, 2, 3, 4/$$

The following Theorem, (cf. [7], [8]) gives us the existence and uniqueness of such a matrix.

Theorem 3. *There exists a uniquely determined Green's matrix $G(x,\xi)$ such that the vector functions*

$$x \to \zeta(x,\xi)(\vec{G}_j(x,\xi),\ G_{4,j}(x,\xi))$$

belong to the space $[H_0^1(\Omega)]^3 \times L^2(\Omega)$ for each $\xi \in \Omega$ and for every infinitely differentiable function $\zeta(\cdot,\xi)$ equal zero in a neighborhood of the point $x = \xi$.

Then, we have the following representation (cf. [12]) of the solution of problem 4.2 in terms of the Green's matrix

$$u_i(x) = \sum_{j=1}^{3} \int_\Omega G_{i,j}(x,\xi) f_j(\xi) d\xi + \int_\Omega G_{i,4}(x,\xi) q(\xi) d\xi \qquad i = 1,2,3. \tag{4.6a}$$

$$p(x) = \sum_{j=1}^{3} \int_\Omega G_{4,j}(x,\xi) f_j(\xi) d\xi + \int_\Omega G_{4,4}(x,\xi) q(\xi) d\xi \tag{4.6b}$$

The following estimates were established in papers of [9, 7, 8, 11] (see also [10] Sec. 11.5).

Theorem 4. *Let $\Omega \subset \mathbb{R}^3$ be a convex domain of polyhedral type. Then there exists a constant C such that*

$$|\partial_x^\alpha \partial_\xi^\beta G_{i,j}(x,\xi)| \ \le \ C|x-\xi|^{-1-|\alpha|-|\beta|-\delta_{i,4}-\delta_{j,4}}, \tag{4.7}$$

for $|\alpha| \le 1-\delta_{i,4}$, $|\beta| \le 1-\delta_{j,4}$, $x,\xi \in \Omega$, $x \neq \xi$, and multi-indices $0 \le |\alpha|, |\beta| \le 1$. Moreover, for polyhedral domain the Green's matrix satisfies the Hölder type estimate

(4.8)
$$\frac{|\partial_x^\alpha \partial_\xi^\beta G_{i,j}(x,\xi) - \partial_y^\alpha \partial_\xi^\beta G_{i,j}(y,\xi)|}{|x-y|^\sigma} \le C(|x-\xi|^{-1-\sigma-|\alpha|-|\beta|-\delta_{i,4}-\delta_{j,4}} + |y-\xi|^{-1-\sigma-|\alpha|-|\beta|-\delta_{i,4}-\delta_{j,4}}),$$

for $|\alpha| \le 1-\delta_{i,4}$, $|\beta| \le 1-\delta_{j,4}$. Here σ is a sufficiently small positive number which depends on the geometry of the domain.

4.2. **Preliminary results.** Let z be an arbitrary point of $\overline{\Omega}$ and let $T_z \in \mathcal{T}_h$ be the element containing z. Our aim is to estimate $|\partial_{x_j}(\vec{u}_h)_i(z)|$ and $|p_h(z)|$, where $1 \le i,j \le 3$ are arbitrary. We will start representing them in terms of the smooth Green's function. Then after some manipulations the problem is reduced to estimate the error of the Green's function in $L^1(\Omega)$ norm, that estimate is presented in this section and we leave the rest of the proof for section 4.3. Preliminarily, we define the smooth delta function. Let $\delta_h^z(x) = \delta_h \in C_0^1(T_z)$ be a smooth function such that

$$r(z) = (r, \delta_h)_{T_z}, \qquad \forall r \in P^l(T_z), \tag{4.9}$$

where $P^l(T_z)$ is the space of polynomials of degree at most l defined on T_z, with the following property

$$\|\delta_h\|_{W_q^k(T_z)} \ \le \ Ch^{-k-3(1-1/q)}, \qquad 1 \le q \le \infty, \ h = 0,1.$$

We highlight that, in particular,

$$\|\delta_h\|_{L^1(T_z)} \ \le \ C \tag{4.10a}$$

$$\|\delta_h\|_{L^2(T_z)} \ \le \ Ch^{-3/2}. \tag{4.10b}$$

The explicit construction of a such function is given in [13]. Next, we define the approximate Green's function $(\vec{g},\lambda) \in [H_0^1(\Omega)]^3 \times L_\phi^2(\Omega)$ to be the solution of the following equation:

$$\Delta\vec{g} + \nabla\lambda = a(\partial_{x_j}\delta_h)\vec{\mathbf{e}}_i \quad \text{in } \Omega \tag{4.11a}$$

$$\nabla\cdot\vec{g} = b(\delta_h - \phi) \quad \text{in } \Omega \tag{4.11b}$$

$$\vec{g} = \vec{0} \quad \text{on } \partial\Omega. \tag{4.11c}$$

Here $\vec{\mathbf{e}}_i$ denote the i-th standard basis vector in $\mathbb{R}^3$ and will be fixed throughout the paper and $a, b \in \mathbb{R}$. Note that (2.3) implies that $\int_\Omega(\delta_h(x) - \phi(x))dx = 0$. Again, λ is unique up to a constant. In the course of the proof we will need estimates $\vec{g}$ and λ in certain Hölder norms on subdomains away from the singular point z. The next lemma is almost identical to Lemma 5.1 in [5]. We include the proof for completeness.

Lemma 4.1. *Let $D \subset \Omega$ be such that $dist(D, z) \geq d \geq 2h$. Then there exists a constant C independent of d and D such that*

$$\|\vec{g}\|_{C^{1+\sigma}(D)} + \|\lambda\|_{C^\sigma(D)} \;\leq\; Cd^{-3-\sigma}.$$

Proof. We use the Green's function representation presented in Section 4.1

$$\begin{aligned}(\vec{g})_k(x) = g_k(x) &= a\int_\Omega G_{k,i}(x,\xi)(\partial_\xi\delta_h(\xi))d\xi + b\int_\Omega G_{i,4}(x,\xi)\delta_h(\xi)d\xi \\ \lambda(x) &= a\int_\Omega G_{4,i}(x,\xi)(\partial_\xi\delta_h(\xi))d\xi + b\int_\Omega G_{4.4}(x,\xi)\delta_h(\xi)d\xi\end{aligned}$$

for $k = 1, 2, 3$ and i fixed. Then, we have

$$\begin{aligned}\partial_x g_k(x) - \partial_y g_k(y) &= a\int_\Omega(\partial_x G_{k,i}(x,\xi) - \partial_y G_{k,i}(y,\xi))(\partial_\xi\delta_h(\xi))d\xi \\ &\quad + b\int_\Omega(\partial_x G_{i,4}(x,\xi) - \partial_y G_{i,4}(y,\xi))\delta_h(\xi)d\xi \\ &= -a\int_\Omega \partial_\xi(\partial_x G_{k,i}(x,\xi) - \partial_y G_{k,i}(y,\xi))\delta_h(\xi)d\xi \\ &\quad + b\int_\Omega(\partial_x G_{i,4}(x,\xi) - \partial_y G_{i,4}(y,\xi))\delta_h(\xi)d\xi.\end{aligned}$$

Let $x, y \in D$, $x \neq y$, then using that $1 \leq i \leq 3$ by (4.8), we have

$$\begin{aligned}\frac{|\partial_x g_k(z) - \partial_y g_k(y)|}{|x-y|^\sigma} &\leq a\max_{\xi\in T_z}\frac{|\partial_\xi\partial_x G_{k,i}(x,\xi) - \partial_\xi\partial_y G_{k,i}(y,\xi))|}{|x-y|^\sigma}\|\delta_h\|_{L^1(T_z)} \\ &\quad + b\max_{\xi\in T_z}\frac{|\partial_x G_{k,i}(x,\xi) - \partial_y G_{k,i}(y,\xi)|}{|x-y|^\sigma}\|\delta_h\|_{L^1(T_z)} \\ &\leq 2C\max\{a,b\}\max_{\xi\in T_z}(|x-\xi|^{-3-\sigma} + |y-\xi|^{-3-\sigma}) \;\leq\; C\max\{a,b\}d^{-3-\sigma}\end{aligned}$$

The last inequality is due to that for any $\xi \in T_z$, $|x-\xi|, |y-\xi| \geq d/2$, and $\|\delta_h\|_{L^1(T_z)} \leq C$. Therefore, taking supremum over k we conclude

$$\sum_{x,y\in D}\frac{|\nabla\vec{g}(x) - \nabla\vec{g}(y)|}{|x-y|^\sigma} \;\leq\; C\max\{a,b\}d^{-3-\sigma}.$$

Similarly, for λ we have

$$\lambda(x)-\lambda(y) \;=\; -a\int_\Omega (\partial_\xi G_{4,i}(x,\xi)-\partial_\xi G_{4,i}(y,\xi))\delta_h(\xi)d\xi \\ +b\int_\Omega (G_{4.4}(x,\xi)-G_{4.4}(y,\xi))\delta_h(\xi)d\xi$$

Then, for $x, y\in D$, $x\neq y$,

$$\begin{aligned}\frac{|\lambda(x)-\lambda(y)|}{|x-y|^\sigma} &\le a\max_{\xi\in T_z}\frac{|\partial_\xi G_{4,i}(x,\xi)-\partial_\xi G_{4,i}(y,\xi)|}{|x-y|^\sigma}\|\delta_h\|_{L^1(T_z)} \\ &\quad +b\max_{\xi\in T_z}\frac{|G_{4.4}(x,\xi)-G_{4.4}(y,\xi)|}{|x-y|^\sigma}\|\delta_h\|_{L^1(T_z)} \\ &\le 2C\max\{a,b\}\max_{\xi\in T_z}(|x-\xi|^{-3-\sigma}+|y-\xi|^{-3-\sigma}) \;\le\; C\max\{a,b\}d^{-3-\sigma}\end{aligned}$$

This completes the proof after taking the supremum. □

Let $(\vec{g}_h,\lambda_h)\in\vec{V}_h\times M_h$ be the corresponding finite element solution, i.e., the unique solution that satisfies

$$(\nabla(\vec{g}-\vec{g}_h),\nabla\vec{\chi})-(\lambda-\lambda_h,\nabla\cdot\vec{\chi}) \;=\; 0, \qquad \forall\vec{\chi}\in\vec{V}_h \tag{4.12a}$$

$$(w,\nabla\cdot(\vec{g}-\vec{g}_h)) \;=\; 0 \qquad \forall w\in M_h \tag{4.12b}$$

and $\lambda_h\in L^2_\phi(\Omega)$. The next lemma is the analogue to lemma 5.2 in [5]. In this case we use the local inf-sup condition instead of the quasi-local Fortin projection to achieve the result.

Lemma 4.2. *There exists a constant C, independent of h and $\vec{g}$, such that*

$$\|\nabla(\vec{g}-\vec{g}_h)\|_{L^1(\Omega)}\le C. \tag{4.13}$$

Proof. At this point we introduce some notations. Let $\vec{e}_{\vec{g}}=\vec{g}-\vec{g}_h$, $\vec{\eta}_{\vec{g}}=\vec{g}-\mathbf{P}\vec{g}$ and $\vec{\xi}_{\vec{g}}=\mathbf{P}\vec{g}-\vec{g}_h$, clearly $\vec{e}_{\vec{g}}=\vec{\eta}_{\vec{g}}+\vec{\xi}_{\vec{g}}$. Similarly, for the scalar variables $e_\lambda=\lambda-\lambda_h$, $\eta_\lambda=\lambda-\mathbf{R}\lambda$ and $\xi_\lambda=\mathbf{R}\lambda-\lambda_h$. The proof is broken down, as the proof of Lemma 5.2 in [5], into four steps.

Step 1 (Dyadic decomposition). We assume without loss of generality that $|\Omega|\le 1$. Define $d_j=2^{-j}$ and J be the integer such that $2^{-(J+1)}\le Kh\le 2^{-J}$ where K is a large enough constant to be chosen later. Then, consider the following decomposition of Ω

$$\Omega \;=\; \Omega^*\cup\bigcup_{j=0}^{J}\Omega_j \tag{4.14}$$

where $\Omega^*=\{x\in\Omega: |x-z|\le Kh\}$, $\quad\Omega_j=\{x\in\Omega: d_{j+1}\le|x-z|\le d_j\}$.

Henceforth, we will denote by C the generic constants not depending on K or h.
We break (4.13) using the dyadic decomposition (4.14) and then applying the Cauchy-Schwartz (C-S.) inequality we obtain

$$\|\nabla\vec{e}_{\vec{g}}\|_{L^1(\Omega)}\le CK^{3/2}h^{3/2}\|\nabla\vec{e}_{\vec{g}}\|_{L^1(\Omega^*)}+C\sum_{j=0}^{J}d_j^{3/2}\|\nabla\vec{e}_{\vec{g}}\|_{L^1(\Omega_j)}.$$

Firstly, we estimate the term involving the set Ω^*

$$\begin{aligned} h^{3/2}\|\nabla \vec{e}_{\vec{g}}\|_{L^2(\Omega^*)} &\leq h^{3/2}\|\nabla \vec{e}_{\vec{g}}\|_{L^2(\Omega)} &&\leq Ch^{5/2}(\|\vec{g}\|_{H^2(\Omega)} + \|\lambda\|_{H^1(\Omega)}) \\ &\leq Ch^{5/2}\|\nabla \delta_h\|_{L^2(T)} &&\leq C \end{aligned}$$

Defining $M_j = d_j^{3/2}\|\nabla \vec{e}_{\vec{g}}\|_{L^2(\Omega_j)}$, it follows that

$$\|\nabla \vec{e}_{\vec{g}}\|_{L^1(\Omega)} \leq CK^{3/2} + \sum_{j=0}^{J} M_j. \tag{4.15}$$

Step 2 (Initial Estimate for M_j). Let us define the following sets:

$$\begin{aligned} \Omega_j' &= \{x \in \Omega : d_{j+2} \leq |x - z| \leq d_{j-1}\} \\ \Omega_j'' &= \{x \in \Omega : d_{j+3} \leq |x - z| \leq d_{j-2}\} \\ \Omega_j''' &= \{x \in \Omega : d_{j+4} \leq |x - z| \leq d_{j-3}\} \\ \Omega_j'''' &= \{x \in \Omega : d_{j+5} \leq |x - z| \leq d_{j-4}\} \end{aligned}$$

We apply the local energy estimate proved in Theorem 2 to $A_1 = \Omega_j$ and $A_2 = \Omega_j'$ $(d = d_j)$, and any $0 < \varepsilon < 1$,

$$\begin{aligned} \|\nabla \vec{e}_{\vec{g}}\|_{L^2(\Omega_j)} &\leq C\left(\varepsilon^{-1}\|\nabla \vec{\eta}_{\vec{g}}\|_{L^2(\Omega_j')} + (\varepsilon d_j)^{-1}\|\vec{\eta}_{\vec{g}}\|_{L^2(\Omega_j')} + \|\eta_\lambda\|_{L^2(\Omega_j')}\right) \\ &\quad + \varepsilon\|\nabla \vec{e}_{\vec{g}}\|_{L^2(\Omega_j')} + \frac{C}{\varepsilon d_j}\|\vec{e}_{\vec{g}}\|_{L^2(\Omega_j')} \end{aligned} \tag{4.16}$$

$$= CI + \varepsilon\|\nabla \vec{e}_{\vec{g}}\|_{L^2(\Omega_j')} + \frac{C}{\varepsilon d_j}\|\vec{e}_{\vec{g}}\|_{L^2(\Omega_j')}. \tag{4.17}$$

We start treating the first three terms on the right-hand side.

$$\begin{aligned} I &\leq Cd_j^{3/2}\left(\varepsilon^{-1}\|\nabla \vec{\eta}_{\vec{g}}\|_{L^\infty(\Omega_j')} + (\varepsilon d_j)^{-1}\|\vec{\eta}_{\vec{g}}\|_{L^\infty(\Omega_j')} + \|\eta_\lambda\|_{L^\infty(\Omega_j')}\right) && \text{(by C-S. ineq.)} \\ &\leq Cd_j^{3/2}h^\sigma\left((\varepsilon^{-1} + \varepsilon^{-1}\frac{h}{d_j})\|\vec{g}\|_{C^{1+\sigma}(\Omega_j'')} + \|\lambda\|_{C^\sigma(\Omega_j'')}\right) && \text{(by } \mathbf{A2}\text{)} \\ &\leq Cd_j^{3/2}h^\sigma\left((\varepsilon^{-1} + \varepsilon^{-1}\frac{h}{d_j})d_j^{-3-\sigma} + d_j^{-3-\sigma}\right) && \text{(by Lemma 4.1)} \\ &\leq Cd_j^{-3/2}\left(\frac{h}{d_j}\right)^\sigma\left(\varepsilon^{-1} + \varepsilon^{-1}\frac{h}{d_j} + 1\right) \\ &\leq Cd_j^{-3/2}\left(\frac{h}{d_j}\right)^\sigma\varepsilon^{-1}\left(1 + \frac{h}{d_j}\right) \end{aligned}$$

Summarizing, we obtain the following estimate for M_j

$$M_j \leq C\left(\frac{h}{d_j}\right)^\sigma\varepsilon^{-1}\left(1 + \frac{h}{d_j}\right) + \varepsilon d_j^{3/2}\|\nabla \vec{e}_{\vec{g}}\|_{L^2(\Omega_j')} + Cd_j^{1/2}\varepsilon^{-1}\|\vec{e}_{\vec{g}}\|_{L^2(\Omega_j')}$$

In Step 3 below we present a duality argument to estimate the last term on the right-hand side.

Step 3 (Duality argument). We use the following duality representation of the L^2 norm.

$$\|\vec{e}_g\|_{L^2(\Omega_j')} = \sup_{\substack{\vec{v}\in C_c^\infty(\Omega_j') \\ \|\vec{v}\|_{L^2(\Omega_j')\leq 1}}} (\vec{e}_{\vec{g}}, \vec{v}).$$

Now, for each $\vec{v} \in C_c^\infty(\Omega_j')$ with $\|\vec{v}\|_{L^2(\Omega_j')} \leq 1$, let $\vec{w}$, φ be the solution of the problem:

$$\begin{aligned} -\Delta\vec{w} + \nabla\varphi &= \vec{v} \quad \text{in } \Omega \\ \nabla\cdot\vec{w} &= 0 \quad \text{in } \Omega \\ \vec{w} &= \vec{0} \quad \text{on } \partial\Omega. \end{aligned}$$

Now, we test the variational problem associated with $\vec{g} - \vec{g}_h$, i.e.

$$\begin{aligned} (\vec{e}_{\vec{g}}, \vec{v}) &= (\nabla\vec{e}_{\vec{g}}, \nabla\vec{w}) - (\varphi, \nabla\cdot\vec{e}_{\vec{g}}) \\ &= (\nabla\vec{e}_g, \nabla(\vec{w} - \mathbf{P}\vec{w})) + (\nabla\vec{e}_g, \nabla\mathbf{P}\vec{w}) - (\varphi - \mathbf{R}\varphi, \nabla\cdot\vec{e}_g) \\ &= (\nabla\vec{e}_g, \nabla\vec{\eta}_w) - (e_\lambda, \nabla\cdot\mathbf{P}\vec{w}) - (\eta_\varphi, \nabla\cdot\vec{e}_g) \\ &= (\nabla\vec{e}_g, \nabla\vec{\eta}_w) - (e_\lambda, \nabla\cdot\vec{\eta}_w) - (\eta_\varphi, \nabla\cdot\vec{e}_g) \\ &= (\nabla\vec{e}_g, \nabla\vec{\eta}_w) - (\eta_\lambda, \nabla\cdot\vec{\eta}_w) - (\xi_\lambda, \nabla\cdot\vec{\eta}_w) - (\eta_\varphi, \nabla\cdot\vec{e}_g) \\ &= (\nabla\vec{e}_g, \nabla\vec{\eta}_w) - (\eta_\lambda, \nabla\cdot\vec{\eta}_w) + (\nabla\xi_\lambda, \vec{\eta}_w) - (\eta_\varphi, \nabla\cdot\vec{e}_g) \\ &=: J_1 + J_2 + J_3 + J_4 \end{aligned}$$

In order to make the estimates for J_1, J_2, J_3, J_4 clearer, we establish the following results.

Proposition 4.1. *There exists $C > 0$ independent of h such that*

(i) $\|\nabla\vec{\eta}_{\vec{w}}\|_{L^2(\Omega)} + \|\eta_\varphi\|_{L^2(\Omega)} \leq Ch$

(ii) $\|\nabla\vec{\eta}_{\vec{w}}\|_{L^\infty(\Omega\setminus\Omega_j'')} + \|\eta_\varphi\|_{L^\infty(\Omega\setminus\Omega_j'')} \leq C\left(\dfrac{h}{d_j}\right)^\sigma d_j^{-1/2}$

(iii) $\|\eta_\lambda\|_{L^2(\Omega_j'''')} \leq Cd_j^{-3/2}$

(iv) $\|\eta_\lambda\|_{L^1(\Omega)} \leq C.$

Next, we split J_i, into two terms as follows $J_i = J_i|_{\Omega_j'''} + J_i|_{\Omega\setminus\Omega_j'''}$, for $i = 1, 2, 3, 4$. For example $J_1 = J_1|_{\Omega_j'''} + J_1|_{\Omega\setminus\Omega_j'''} = (\nabla\vec{e}_g, \nabla\vec{\eta}_w)_{\Omega_j'''} + (\nabla\vec{e}_g, \nabla\vec{\eta}_w)_{\Omega\setminus\Omega_j'''}$ and estimate them using Cauchy-Schwartz inequality, in L^2 norm in Ω_j''' and in $L^1 - L^\infty$ norms in $\Omega\setminus\Omega_j'''$.

We start estimating J_1, and J_4 using Proposition 4.1 (i) and (ii)

$$\begin{aligned} J_1|_{\Omega_j'''} &\leq \|\nabla\vec{e}_{\vec{g}}\|_{L^2(\Omega_j''')}\|\nabla\vec{\eta}_{\vec{w}}\|_{L^2(\Omega)} &&\leq Ch\|\nabla\vec{e}_{\vec{g}}\|_{L^2(\Omega_j''')}, \\ J_1|_{\Omega\setminus\Omega_j'''} &\leq \|\nabla\vec{e}_{\vec{g}}\|_{L^1(\Omega)}\|\nabla\vec{\eta}_{\vec{w}}\|_{L^\infty(\Omega\setminus\Omega_j'')} &&\leq Cd_j^{-1/2}\left(\frac{h}{d_j}\right)^\sigma \|\nabla\vec{e}_{\vec{g}}\|_{L^1(\Omega)}, \\ J_4|_{\Omega_j'''} &\leq \|\eta_\varphi\|_{L^2(\Omega)}\|\nabla\vec{e}_{\vec{g}}\|_{L^2(\Omega_j''')} &&\leq Ch\|\nabla\vec{e}_{\vec{g}}\|_{L^2(\Omega_j''')}, \\ J_4|_{\Omega\setminus\Omega_j'''} &\leq \|\eta_\varphi\|_{L^\infty(\Omega\setminus\Omega_j'')}\|\nabla\vec{e}_{\vec{g}}\|_{L^1(\Omega)} &&\leq Cd_j^{-1/2}\left(\frac{h}{d_j}\right)^\sigma \|\nabla\vec{e}_{\vec{g}}\|_{L^1(\Omega)}. \end{aligned}$$

Hence

$$J_1 + J_4 \;\leq\; Ch\|\nabla \vec{e}_{\vec{g}}\|_{L^2(\Omega_j''')} + Cd_j^{-1/2}\left(\frac{h}{d_j}\right)^{\sigma}\|\nabla \vec{e}_g\|_{L^1(\Omega)} \tag{4.18}$$

To estimate J_2 we apply Proposition 4.1 (i) and (ii) as before and then apply (iii) and (iv)

$$\begin{aligned} J_2|_{\Omega_j'''} &\leq \|\eta_\lambda\|_{L^2(\Omega_j''')}\|\nabla \vec{\eta}_{\vec{w}}\|_{L^2(\Omega)} \;\leq\; \|\eta_\lambda\|_{L^2(\Omega_j''')}Ch \;\leq\; Chd_j^{-3/2} \\ J_2|_{\Omega\setminus\Omega_j'''} &\leq \|\eta_\lambda\|_{L^1(\Omega)}\|\nabla \vec{\eta}_{\vec{w}}\|_{L^\infty(\Omega\setminus\Omega_j''')} \;\leq\; \|\eta_\lambda\|_{L^1(\Omega)}C\left(\frac{h}{d_j}\right)^{\sigma} d_j^{-1/2} \;\leq\; C\left(\frac{h}{d_j}\right)^{\sigma} d_j^{-1/2}. \end{aligned}$$

Then

$$J_2 \;\leq\; C(hd_j^{-3/2} + \left(\frac{h}{d_j}\right)^{\sigma} d_j^{-1/2}) \tag{4.19}$$

It remains to estimate J_3. We first estimate $J_3|_{\Omega_j'''}$. Applying C-S. inequality and Prop. 4.1 (i), we get

$$J_3|_{\Omega_j'''} \;=\; (\nabla\xi_\lambda, \vec{\eta}_{\vec{w}})_{\tilde{\Omega}_j'''} \;\leq\; \|\nabla\xi_\lambda\|_{L^2(\Omega_j''')}\|\vec{\eta}_{\vec{w}}\|_{L^2(\Omega)} \;\leq\; Ch^2\|\nabla\xi_\lambda\|_{L^2(\Omega_j''')}$$

To estimate the term in the right-hand side we use the local inf-sup condition **A5**, the identity $e_\lambda = \eta_\lambda + \xi_\lambda$, integration by parts, (4.12a), C-S. inequality and Prop. 4.1 (iii), obtaining

$$\begin{aligned} \beta h\|\nabla\xi_\lambda\|_{L^2(\Omega_j''')} &\leq \sup_{\substack{\vec{z}\in\vec{V}_h \\ \mathrm{supp}(\vec{z})\subseteq\tilde{\Omega}_j'''}} \frac{(\xi_\lambda, \nabla\cdot\vec{z})}{\|\vec{z}\|_{H^1(\tilde{\Omega}_j''')}} \;\leq\; \sup_{\substack{\vec{z}\in\vec{V}_h \\ \mathrm{supp}(\vec{z})\subseteq\tilde{\Omega}_j'''}} \frac{(e_\lambda - \eta_\lambda, \nabla\cdot\vec{z})}{\|\vec{z}\|_{H^1(\tilde{\Omega}_j''')}} \\ &\leq \|\eta_\lambda\|_{L^2(\tilde{\Omega}_j''')} + \sup_{\substack{\vec{z}\in\vec{V}_h \\ \mathrm{supp}(\vec{z})\subseteq\tilde{\Omega}_j'''}} \frac{(e_\lambda, \nabla\cdot\vec{z})}{\|\vec{z}\|_{H^1(\tilde{\Omega}_j''')}} \\ &\leq \|\eta_\lambda\|_{L^2(\tilde{\Omega}_j''')} + \sup_{\substack{\vec{z}\in\vec{V}_h \\ \mathrm{supp}(\vec{z})\subseteq\tilde{\Omega}_j'''}} \frac{(\nabla\vec{e}_{\vec{g}}, \vec{z})}{\|\vec{z}\|_{H^1(\tilde{\Omega}_j''')}} \\ &\leq \|\eta_\lambda\|_{L^2(\tilde{\Omega}_j''')} + \|\nabla\vec{e}_{\vec{g}}\|_{L^2(\tilde{\Omega}_j''')} \;\leq\; Cd_j^{-3/2} + \|\nabla\vec{e}_{\vec{g}}\|_{L^2(\tilde{\Omega}_j''')}, \end{aligned}$$

where $\tilde{\Omega}_j''' \supseteq \Omega_j'''$ with $dist(\tilde{\Omega}_j''', \Omega_j''') \leq lh$. Observe that $\Omega_j''' \subseteq \tilde{\Omega}_j''' \subset \Omega_j''''$
Hence,

$$J_3|_{\Omega_j'''} \;\leq\; Ch(Cd_j^{-3/2} + \|\nabla\vec{e}_{\vec{g}}\|_{L^2(\tilde{\Omega}_j''')}) \tag{4.20}$$

For $J_3|_{\Omega\setminus\Omega_j'''}$, C-S. inequality and Prop. 4.1 (ii) yield to

$$J_3|_{\Omega\setminus\Omega_j'''} \;\leq\; \|\nabla\xi_\lambda\|_{L^1(\Omega)}\|\vec{\eta}_{\vec{w}}\|_{L^\infty(\Omega\setminus\Omega_j''')} \;\leq\; C\left(\frac{h}{d_j}\right)^{\sigma} d_j^{-1/2}h\|\nabla\xi_\lambda\|_{L^1(\Omega)}$$

To estimate the term in the right-hand side we use the L^1 inf-sup condition (**A 6**), the identity $e_\lambda = \eta_\lambda + \xi_\lambda$, integration by parts, (4.12a), C-S. inequality and Prop. 4.1 (iv), obtaining

$$\begin{aligned}
\gamma\|\nabla\xi_\lambda\|_{L^1(\Omega)} &\leq \sup_{\vec{z}\in\vec{V}_h}\frac{(\xi_\lambda,\nabla\cdot\vec{z})}{\|\vec{z}\|_{W^1_\infty(\Omega)}} = \sup_{\vec{z}\in\vec{V}_h}\frac{(e_\lambda-\eta_\lambda,\nabla\cdot\vec{z})}{\|\vec{z}\|_{W^1_\infty(\Omega)}}\\
&\leq \|\eta_\lambda\|_{L^1(\Omega)} + \sup_{\vec{z}\in\vec{V}_h}\frac{(e_\lambda,\nabla\cdot\vec{z})}{\|\vec{z}\|_{W^1_\infty(\Omega)}}\\
&= \|\eta_\lambda\|_{L^1(\Omega)} + \sup_{\vec{z}\in\vec{V}_h}\frac{-(\nabla e_\lambda,\vec{z})}{\|\vec{z}\|_{W^1_\infty(\Omega)}}\\
&\leq \|\eta_\lambda\|_{L^1(\Omega)} + \sup_{\vec{z}\in\vec{V}_h}\frac{(\nabla\vec{e}_{\vec{g}},\nabla\vec{z})}{\|\vec{z}\|_{W^1_\infty(\Omega)}}\\
&\leq \|\eta_\lambda\|_{L^1(\Omega)} + \|\nabla\vec{e}_{\vec{g}}\|_{L^1(\Omega)} \leq C + \|\nabla\vec{e}_{\vec{g}}\|_{L^1(\Omega)}.
\end{aligned}$$

Then

$$J_3|_{\Omega\setminus\Omega_j'''} \leq C\left(\frac{h}{d_j}\right)^\sigma d_j^{-1/2}(C+\|\nabla\vec{e}_{\vec{g}}\|_{L^1(\Omega)}) \tag{4.21}$$

It follows from (4.20) and (4.21) that

$$J_3 \leq Ch\|\nabla\vec{e}_{\vec{g}}\|_{L^2(\tilde{\Omega}_j''')} + C\left(\frac{h}{d_j}\right)^\sigma d_j^{-1/2}\|\nabla\vec{e}_{\vec{g}}\|_{L^1(\Omega)} + Cd_j^{-1/2}(hd_j^{-1}+\left(\frac{h}{d_j}\right)^\sigma). \tag{4.22}$$

Therefore, estimate for J_1+J_4, J_2 and J_3, (4.18),(4.19) and (4.22), respectively, give

$$\begin{aligned}
d_j^{1/2}\|\vec{e}_{\vec{g}}\|_{L^2(\Omega_j')} &\leq Chd_j^{-1} + C\left(\frac{h}{d_j}\right)^\sigma + C\left(\frac{h}{d_j}\right)^\sigma\|\nabla\vec{e}_{\vec{g}}\|_{L^1(\Omega)}\\
&\quad + Chd_j^{1/2}(\|\nabla\vec{e}_{\vec{g}}\|_{L^2(\Omega_j''')} + \|\nabla\vec{e}_{\vec{g}}\|_{L^2(\tilde{\Omega}_j''')}).
\end{aligned}$$

To summarize,

$$M_j \leq C\left(\frac{h}{d_j}\right)^\sigma(1+\frac{1}{\varepsilon}) + C\frac{h}{d_j\varepsilon} + C(\frac{hd_j^{1/2}}{\varepsilon}+\varepsilon d_j^{3/2})\|\nabla\vec{e}_{\vec{g}}\|_{L^2(\Omega_j'''')} + \frac{C}{\varepsilon}\left(\frac{h}{d_j}\right)^\sigma\|\nabla\vec{e}_{\vec{g}}\|_{L^1(\Omega)}$$

Step 4 (Double kick-back argument). We sum over j in the last expression obtaining

$$\begin{aligned}
\sum_{j=0}^J M_j &\leq \sum_{j=0}^J\left\{C\left(\frac{h}{d_j}\right)^\sigma(1+\frac{1}{\varepsilon}) + C\frac{h}{d_j\varepsilon} + \frac{C}{\varepsilon}\left(\frac{h}{d_j}\right)^\sigma\|\nabla\vec{e}_{\vec{g}}\|_{L^1(\Omega)}\right\}\\
&\quad + C(\frac{h}{d_J\varepsilon}+\varepsilon)\sum_{j=0}^J d_j^{3/2}\|\nabla\vec{e}_{\vec{g}}\|_{L^2(\Omega_j'''')}
\end{aligned}$$

Observe that

$$\sum_{j=0}^J\left(\frac{h}{d_j}\right)^\sigma = h^\sigma\sum_{j=0}^J(2^j)^\sigma = h^\sigma\frac{(2^\sigma)^{J+1}-1}{2^\sigma-1} \leq \left(\frac{h}{d_J}\right)^\sigma\frac{2^\sigma}{2^\sigma-1} \leq CK^{-\sigma}$$

in the last expression C depends on σ which is fixed. Then,

$$\sum_{j=0}^{J} M_j \;\leq\; C\frac{(1+\varepsilon^{-1})}{K^\sigma} + C\frac{1}{\varepsilon K} + \frac{C}{\varepsilon K^\sigma}\|\nabla\vec{e}_{\vec{g}}\|_{L^1(\Omega)} + C(\frac{1}{\varepsilon K}+\varepsilon)\sum_{j=0}^{J} d_j^{3/2}\|\nabla\vec{e}_{\vec{g}}\|_{L^2(\Omega_j'''')}$$

Observing that $\Omega_j'''' \subset \Omega^* \cup \bigcup_{s\in S}\Omega_s$, for some finite number S, we can bound the last term in the right-hand side as follows

$$\sum_{j=0}^{J} d_j^{3/2}\|\nabla\vec{e}_{\vec{g}}\|_{L^2(\Omega_j'''')} \;\leq\; C\sum_{j=0}^{J} M_j + C(Kh)^{3/2}\|\nabla\vec{e}_{\vec{g}}\|_{L^2(\Omega^*)} \;\leq\; C\sum_{j=0}^{J} M_j + CK^{3/2}.$$

Choosing K large enough and a sufficiently small ε we have

$$\sum_{j=0}^{J} M_j \;\leq\; C_{K,\varepsilon} + \frac{C}{\varepsilon K^\sigma}\|\nabla\vec{e}_{\vec{g}}\|_{L^1(\Omega)}$$

This result allows us to conclude in (4.15) that

$$\|\nabla\vec{e}_{\vec{g}}\|_{L^1(\Omega)} \;\leq\; C_{K,\varepsilon} + \frac{C}{K^\sigma\varepsilon}\|\nabla\vec{e}_{\vec{g}}\|_{L^1(\Omega)}$$

which, by means of a large enough choice of K, implies the desired result

$$\|\nabla\vec{e}_{\vec{g}}\|_{L^1(\Omega)} \;\leq\; C_{K,\varepsilon}.$$

This completes the proof. □

Proof. (Proposition 4.1)

(i) By H^2-regularity and property of $\mathbf{R}$ we have

$$\|\nabla\vec{\eta}_{\vec{w}}\|_{L^2(\Omega)} + \|\eta_\varphi\|_{L^2(\Omega)} \;\leq\; Ch(\|w\|_{H^2(\Omega)} + \|\nabla\varphi\|_{L^2(\Omega)}) \;\leq\; Ch$$

the last inequality is due to $\|v\|_{L^2(\Omega_j')} \leq 1$

(ii) We observe that by Hölder inequality $\|\vec{\eta}_w\|_{L^\infty(\Omega\backslash\Omega_j''')} \;\leq\; Ch^\sigma\|\vec{w}\|_{C^{1+\sigma}(\Omega\backslash\Omega_j'')}$
Then, since $\Omega\backslash\Omega_j''$ is separated from Ω_j' by at least d_j, for $x, y \in \Omega\backslash\Omega_j''$, using (4.6a) and (4.8) ,we have

$$\begin{aligned}\frac{|\partial_x w_k(x) - \partial_y w_k(y)|}{|x-y|^\sigma} \;&\leq\; \sum_{i=1}^{3}\int_{\Omega_j'}\frac{\partial_x G_{k,i}(x,\xi) - \partial_y G_{k,i}(y,\xi)|}{|x-y|^\sigma}|\vec{v}(\xi)|d\xi\\ &\leq\; C\max_{\xi\in\Omega_j'}(|x-\xi|+|y-\xi|)^{-2-\sigma}\int_{\Omega_j'}|\vec{v}(\xi)|d\xi\\ &\leq\; Cd_j^{-2-\sigma}d_j^{3/2}\|\vec{v}\|_{L^2(\Omega_j')} \;\leq\; Cd_j^{-1/2-\sigma}, \qquad \text{for } k=1,2,3.\end{aligned}$$

It follows that

$$\|\vec{\eta}_w\|_{L^\infty(\Omega\backslash\Omega_j''')} \;\leq\; C\left(\frac{h}{d_j}\right)^\sigma d_j^{-1/2}.$$

Similarly, for $x, y \in \Omega\backslash\Omega_j''$, using (4.6b) and (4.8) ,we have

$$
\begin{aligned}
\frac{|\varphi(x)-\varphi(y)|}{|x-y|^\sigma} &\le \sum_{i=1}^{3}\int_{\Omega_j'} \frac{\partial_x G_{4,i}(x,\xi)-\partial_y G_{4,i}(y,\xi)|}{|x-y|^\sigma}|\vec{v}(\xi)|d\xi \\
&\le C\max_{\xi\in\Omega_j'}(|x-\xi|+|y-\xi|)^{-2-\sigma}\int_{\Omega_j'}|\vec{v}(\xi)|d\xi \\
&\le Cd_j^{-2-\sigma}d_j^{3/2}\|\vec{v}\|_{L^2(\Omega_j')} \le Cd_j^{-1/2-\sigma}, \qquad \text{for } k=1,2,3.
\end{aligned}
$$

Then, by **A3** we have

$$
\|\eta_\varphi\|_{L^\infty(\Omega\backslash\Omega_j''')} \le Ch^\sigma\|\varphi\|_{C^\sigma(\Omega\backslash\Omega_j')} \le \left(\frac{h}{d_j}\right)^\sigma d_j^{-1/2}.
$$

(iii) Using (4.6b), (4.7) and $dist(\Omega_j'''', T_z) = O(d_j)$ we have

$$
\begin{aligned}
\lambda(x) &= \sum_{k=1}^{3}\int_{T_z} G_{4,k}(x,\xi)(\partial_\xi\delta_h(\xi))\delta_{i,k}d\xi \\
&= -\int_{T_z}\partial_\xi G_{4,i}(x,\xi)\delta_h(\xi)d\xi \le Cd_j^{-3}\|\delta_h\|_{L^1(T_z)} \le Cd_j^{-3}.
\end{aligned}
$$

Thus, $\|\eta_\lambda\|_{L^2(\Omega_j'''')} \le C\|\lambda\|_{L^2(\Omega_j'''')} \le Cd_j^{-3/2}$.

(iv) Using the dyadic decomposition (4.14) and C-S. inequality, we have

$$
\|\eta_\lambda\|_{L^1(\Omega)} \le CK^{3/2}h^{3/2}\|\eta_\lambda\|_{L^2(\Omega^*)} + C\sum_{j=0}^{J} d_j^{3/2}\|\eta_\lambda\|_{L^2(\Omega_j)}.
$$

Approximation property of **R A2**, H^2-regularity and (4.10b) imply that

$$
h^{3/2}\|\eta_\lambda\|_{L^2(\Omega^*)} \le Ch^{3/2+1}\|\nabla\lambda\|_{L^2(\Omega)} \le Ch^{5/2}\|\nabla\delta_h\|_{L^2(T)} \le C.
$$

Finally, using (iii) we conclude that

$$
\|\eta_\lambda\|_{L^1(\Omega)} \le CK^{3/2} + C\sum_{j=0}^{J}\left(\frac{h}{d_j}\right)^\sigma \le C_K.
$$

□

4.3. **Proof of Theorem 1.** We start this section with the L^∞ estimate for the velocity. Consider the problem (4.11) with $a = 1$ and $b = 0$. We will estimate $|\partial_{x_j}(\vec{u})_i(z)|$, where $1 \le i, j \le 3$ are arbitrary and arbitrary $z \in \bar{\Omega}$. We start the estimate using the definition of the delta function, then we have

$$\begin{aligned}
-\partial(\vec{u}_h)_i(z) &= (\vec{u}_h, (\partial_{x_j}\delta_h)\vec{e}_i) \\
&= (\vec{u}_h, -\Delta\vec{g} + \nabla\lambda \\
&= (\nabla\vec{u}_h, \nabla\vec{g}) + (\vec{u}_h, \nabla\lambda) \\
&= (\nabla\vec{u}_h, \nabla\vec{g}) + (\vec{u}_h, \nabla\lambda_h) + (\nabla\vec{u}_h, \nabla(\vec{g}_h - \vec{g})) \\
&= (\nabla\vec{u}_h, \nabla\vec{g}_h) \\
&= (\nabla\vec{u}, \nabla\vec{g}_h) + (\nabla(p - p_h), \vec{g}_h) \\
&= (\nabla\vec{u}, \nabla\vec{g}_h) + (\nabla p, \vec{g}_h) \\
&= (\nabla\vec{u}, \nabla\vec{g}_h) + (\nabla\vec{u}, \nabla\vec{g}) + (\nabla p, \vec{g}_h - \vec{g}) + (\vec{u}, \nabla\lambda) \\
&= (\nabla\vec{u}, \nabla\vec{g}_h) + (\vec{u}, -\Delta\vec{g} + \nabla\lambda) + (\vec{g} - \vec{g}_h, \nabla p) \\
&= (\nabla\vec{u}, \nabla\vec{g}_h) - (\tfrac{\partial(\vec{u})_i}{\partial x_j}, \delta_h) - (\nabla\cdot(\vec{g} - \vec{g}_h), p).
\end{aligned}$$

We take supremum over all partial derivatives in both sides of the equation, and taking into account that $\|\delta_h\|_{L^1(\Omega)} \le C$, then we can conclude that

$$\|\nabla\vec{u}_h\|_{L^\infty(\Omega)} \le (C + \|\nabla(\vec{g} - \vec{g}_h)\|_{L^1(\Omega)})(\|\nabla\vec{u}\|_{L^\infty(\Omega)} + \|p\|_{L^\infty(\Omega)}). \tag{4.23}$$

The result (4.23) is completed by Lemma 4.2.

Next, we prove the stability of the pressure in the maximum norm.

Let $z \in T_z$ and consider the problem (4.11) with $a = 0$ and $b = 1$. Then, using the definition of the delta function we have

$$p_h(z) = (p_h, \delta_h) = (p_h, \delta_h - \phi) + (p_h, \phi).$$

We estimate the second term in the right hand side using C-S. inequality and the a priori error estimate as follows

$$\begin{aligned}
(p_h, \phi) &= (p_h - p, \phi) + (p, \phi) \\
&\le C(\|p - p_h\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)})\|\phi\|_{L^2(\Omega)} \\
&\le C(\|\nabla\vec{u}\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)}) \\
&\le C(\|\nabla\vec{u}\|_{L^\infty(\Omega)} + \|p\|_{L^\infty(\Omega)}).
\end{aligned}$$

Now, to estimate $(p_h, \delta_h - \phi)$ we use (4.11b)

$$\begin{aligned}
(p_h, \delta_h - \phi) &= (p_h, \nabla\cdot\vec{g}) = (p_h, \nabla\cdot\vec{g}_h) = (p, \nabla\cdot\vec{g}_h) + (p_h - p, \nabla\cdot\vec{g}_h) \\
&= (p, \nabla\cdot\vec{g}) + (p, \nabla\cdot(\vec{g}_h - \vec{g})) + (\nabla(u - u_h), \nabla\vec{g}_h) \\
&= (p, \nabla\cdot\vec{g}) + (p, \nabla\cdot(\vec{g}_h - \vec{g})) + (\nabla(u - u_h), \nabla(\vec{g}_h - \vec{g})) + (\nabla(u - u_h), \nabla\vec{g}) \\
&= (p, \nabla\cdot\vec{g}) + (p, \nabla\cdot(\vec{g}_h - \vec{g})) + (\nabla(u - u_h), \nabla(\vec{g}_h - \vec{g})) + (\nabla\cdot(u - u_h), \lambda) \\
&= (p, \delta_h - \phi) + (p, \nabla\cdot(\vec{g}_h - \vec{g})) + (\nabla(u - u_h), \nabla(\vec{g}_h - \vec{g})) \\
&\quad + (\nabla\cdot(u - u_h), \lambda - \mathbf{R}\lambda) \\
&\le (\|\nabla(u - u_h)\|_{L^\infty(\Omega)} + \|p\|_{L^\infty(\Omega)})(\|\delta_h\|_{L^1(\Omega)} + \|\phi\|_{L^1(\Omega)} \\
&\quad + \|\nabla(\vec{g}_h - \vec{g})\|_{L^1(\Omega)} + \|\lambda - \mathbf{R}\lambda\|_{L^1(\Omega)}) \\
&\le (\|\nabla(u - u_h)\|_{L^\infty(\Omega)} + \|p\|_{L^\infty(\Omega)})(C + \|\nabla(\vec{g}_h - \vec{g})\|_{L^1(\Omega)} + \|\lambda - \mathbf{R}\lambda\|_{L^1(\Omega)})
\end{aligned}$$

The result (4.23) is completed by Lemma 4.2, Proposition 4.1 and the previous estimate for the velocity in the L^∞ norm.

5. Taylor-Hood elements

We consider the Taylor-Hood elements of degree 2 in three dimension ($d = 3$), i.e.

$$\vec{V}_h = \{\vec{v} \in [C^0(\bar{\Omega})]^3 : \vec{v}|_T \in [\mathbb{P}_2]^3, \forall T \in \mathcal{T}_H, \vec{v}|_{\partial\Omega} = \vec{0}\} \tag{5.1}$$

$$M_h = \{q \in C^0(\bar{\Omega}) : q|_T \in \mathbb{P}_1, \forall T \in \mathcal{T}_h\} \cap L^2_0(\Omega). \tag{5.2}$$

Assumptions **A1-A3** hold for example by choosing **P** and **R** to be the Scott-Zhang [17] interpolants onto $\vec{V}_h$ and M_h, respectively (see [14] and [3]). It is clear that the **A4** assumption holds in this case. We will prove assumptions **A5** and **A6** also hold.

We start with the local inf-sup condition **A5**.

Definition 1. *Let $\vec{b}$ be a vertex of $\mathcal{T}_h$. We define $\sigma(\vec{b})$, the patch associated to the vertex $\vec{b}$, as the set of all elements containing $\vec{b}$, i.e.*

$$\sigma(\vec{b}) := \{T \in \mathcal{T}_h | \quad \vec{b} \in T\}$$

Lemma 5.1. *Assume that every mesh element has at least 3 edges in $int(\Omega)$. Let $B \subset \Omega$. Then, there exists a constant c and a set $B_h \subset \mathcal{T}_h$ which contains B and $dist(B, \partial B_h \backslash \Omega) \leq 2h$ such that the following inequality holds*

$$\sup_{\substack{\vec{v} \in \vec{V}_h \\ \text{supp}(\vec{v}) \subset B_h}} \frac{\int_\Omega q \nabla \cdot \vec{v}}{\|\vec{v}\|_{H^1(B_h)}} \geq c \left(\sum_{T \in B_h} h_T^2 |q|^2_{H^1(T)} \right)^{1/2} \geq ch^2 |q_h|_{H^1(B)}.$$

for all $q \in M_h$.

Proof. (We follow the proof in [4] section 4.2.5., see also [15])

Define the set of vertices

$$\vec{X} := \{\vec{x} \in int(\Omega) : \vec{x} \text{ is a vertex of an element } T \in \mathcal{T}_h \text{ such that } T \cap B \neq \emptyset\}$$

Then, we define the set

$$B_h := \bigcup_{\vec{x} \in \vec{X}} \sigma(\vec{x}),$$

Note that, the assumption that every mesh element has at least d edges in $int(\Omega)$ implies that $B \subset B_h$, and $dist(B, B_h) \leq 2h$. We claim that every element of B_h has at most one face on ∂B_h. In fact, let $T \in B_h$, by definition T belongs to the patch of an interior vertex. Then, the claim follows from the observation that all the elements of an interior patch has at most one face on the boundary of the patch.

Let $N_{ed}^{i,h}$ be the number of interior edges in B_h. For the edge i, with $1 \leq i \leq N_{ed}^{i,h}$, denote by $\vec{d}_i$ and $\vec{f}_i$ its two extremities and by $\vec{m}_i$ its midpoint. Set $l_i = \|\vec{f}_i - \vec{d}_i\|_3$ and $\vec{\tau} = \frac{\vec{f}_i - \vec{d}_i}{\|\vec{f}_i - \vec{d}_i\|_3}$, the length and the unit vector.
Then, for $q \in M_h$ we define $\vec{v} \in \vec{V}_h$ for all $T \in \mathcal{T}_h$ as follows

$$\begin{cases} \vec{v} = 0, & \text{if } T \in \mathcal{T}_h \setminus int(B_h) \\ \vec{v} = 0, & \text{at the vertices of } T, \text{ if } T \in B_h \\ \vec{v}(\vec{m}_i) = -l_i^2 \vec{\tau}_i \text{sgn}(\partial_{\vec{\tau}_i} q) |\partial_{\vec{\tau}_i} q|, & \text{for all the interior edges } i \text{ of } T, \text{ if } T \in B_h \end{cases}$$

Then, it is clear that $\text{supp}(\vec{v}) = B_h$ and $\vec{v} \in \vec{V}_h$.

Using the following quadrature formula,

$$\int_T \phi(x)dx = \left(\sum_{\vec{m}} \frac{\phi(\vec{m})}{5} - \sum_{\vec{n}} \frac{\phi(\vec{n})}{20}\right)|T| \qquad ,\forall\phi \in \mathbb{P}^2(T)$$

where $\vec{m}$ spans the set of the edge midpoint of T and $\vec{n}$ the set of nodes of T, we infer

$$\begin{aligned}
\int_\Omega q_h \nabla \cdot \vec{v} dx &= -\int_\Omega \vec{v} \cdot \nabla q dx \\
&= -\sum_{T \in B_h} \int_T \vec{v} \cdot \nabla q dx \\
&= -\sum_{T \in B_h} \left(\sum_{\vec{m} \in T} \frac{\vec{v}(\vec{m}) \cdot \nabla \vec{q}(\vec{m})}{5} - \sum_{\vec{n} \in T} \frac{\vec{v}(\vec{n}) \cdot \nabla q(\vec{n})}{20}\right)|T| \\
&= -\sum_{T \in B_h} \sum_{\vec{m}_i \in T} \frac{\vec{v}(\vec{m}_i) \cdot \nabla q(\vec{m}_i)}{5}|T| \\
&= \sum_{T \in B_h} \sum_{i:\vec{m}_i \in T} l_i^2 |\nabla q \cdot \vec{\tau}_i|^2 \frac{|T|}{5} \\
&\geq c \sum_{T \in B_h} h_T^2 |q|^2_{H^1(T)}.
\end{aligned}$$

We observe that the last step ($\sum_{i:\vec{m}_i \in T} |\nabla q \cdot \vec{\tau}_i|^2 \geq |\nabla q|^2$) is only possible if every element of B_h has at least 3 edges on $int(B_h)$, which is satisfied by our construction of B_h and hypothesis on the mesh (every element has at least 3 edges in Ω). Furthermore, for $T \in B_h$ we have that

$$\|\vec{v}\|^2_{H^1(T)} \leq c h_T^2 |q|^2_{H^1(T)}$$

then,

$$\|\vec{v}\|_{H^1(B_h)} = \left(\sum_{T \in B_h} \|\vec{v}\|^2_{H^1(T)}\right)^{1/2} \leq \left(\sum_{T \in B_h} c h_T^2 |q|^2_{H^1(T)}\right)^{1/2}$$

Therefore

$$\begin{aligned}
\sup_{\substack{\vec{v} \in \vec{V}_h \\ \text{supp}(\vec{v}) \subseteq B_h}} \frac{\int_\Omega q \nabla \cdot \vec{v}}{\|\vec{v}\|_{H^1(B_h)}} &\geq C \sum_{T \in B_h} h_T^2 |q|^2_{H^1(T)} \left(\sum_{T \in B_h} h_T^2 |q|^2_{H^1(T)}\right)^{-1/2} \\
&= C \left(\sum_{T \in B_h} h_T^2 |q|^2_{H^1(T)}\right)^{1/2} \\
&\geq C h^2 |q|_{H^1(B)}.
\end{aligned}$$

□

Finally, using the same arguments we prove the assumption **A6**.

Lemma 5.2. *Assume that every mesh element has at least* 3 *edges in* $int(\Omega)$*. There exists a constant* $c > 0$ *independent of* h *such that*

$$\sup_{\vec{v}\in\vec{V}_h\setminus\{\vec{0}\}} \frac{(q, \nabla\cdot\vec{v})}{\|\vec{v}\|_{W^1_\infty(\Omega)}} \geq ch\|\nabla q\|_{L^1(\Omega)}, \qquad \forall q \in M_h.$$

Proof. Similarly to the previous proof we define the number of internal edges N^i_{ed}. For edge i, with $1 \leq i \leq N^i_{ed}$ denote by d_i, f_i and $\vec{m}_i$ as before. Define $\vec{v} \in \vec{V}_h$ for $q \in M_h$ and for all $T \in \mathcal{T}_h$ as follows

$$\begin{cases} \vec{v} = 0, & \text{at the vertices of } T \\ \vec{v}(\vec{m}_i) = -l_i\vec{\tau}_i\mathrm{sgn}(\partial_{\vec{\tau}_i} q), & \text{for all the interior edges } i \text{ of } T \end{cases}$$

Then, it is clear that $\vec{v} \in \vec{V}_h$ and

$$\begin{aligned} \int_\Omega q\nabla\cdot\vec{v}dx &= -\int_\Omega \vec{v}\cdot\nabla q dx \\ &= -\sum_{T\in\mathcal{T}_h}\int_T \vec{v}\cdot\nabla q dx \\ &= -\sum_{T\in\mathcal{T}_h}\left(\sum_{\vec{m}\in T}\frac{\vec{v}(\vec{m})\cdot\nabla q(\vec{m})}{5} - \sum_{\vec{n}\in T}\frac{\vec{v}(\vec{n})\cdot\nabla q(\vec{n})}{20}\right)|T| \\ &= -\sum_{T\in\mathcal{T}_h}\sum_{\vec{m}_i\in T}\frac{\vec{v}(\vec{m}_i)\cdot\nabla\vec{q}(\vec{m}_i)}{5}|T| \\ &= \sum_{T\in\mathcal{T}_h}\sum_{m_i\in T}|\partial_{\tau_i} q| l_i\frac{|T|}{5} \\ &\geq c\sum_{T\in\mathcal{T}_h} h_T\|\nabla q\|_{L^1(T)}. \end{aligned}$$

Recalling again that the inequality $|\nabla q\cdot\tau_i| \leq |\nabla q|$ is possible thanks to that every element has at least 3 internal edges. Furthermore, using the definition of $\vec{v}$ and its local shape function representation we have

$$\|\vec{v}\|_{W^\infty_1(\Omega)} \leq Ch^{-1}\max_{T\in\mathcal{T}_h}\max_{\vec{m}_i\in T}|\vec{v}(\vec{m}_i)| \leq C.$$

This completes the proof. ☐

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