OPTIMAL CONVERGENCE OF THE ORIGINAL DG METHOD FOR THE TRANSPORT-REACTION EQUATION ON SPECIAL MESHES

BERNARDO COCKBURN *, BO DONG †, AND JOHNNY GUZMÁN ‡

Abstract. We show that the approximation given by the original discontinuous Galerkin method for the transport-reaction equation in d space dimensions is optimal provided the meshes are suitably chosen: the L^2-norm of the error is of order $k+1$ when the method uses polynomials of degree k. These meshes are not necessarily conforming and do not satisfy any uniformity condition; they are only required to be made of simplexes each of which has a unique *outflow* face. We also find a new, element-by-element postprocessing of the derivative in the direction of the flow which superconverges with order $k+1$.

Key words. discontinuous Galerkin methods, transport-reaction equation, error estimates

AMS subject classifications. 65N30, 65M60

1. Introduction. We show that the original discontinuous Galerkin (DG) [16, 13], method can approximate in an *optimal* fashion the solution of the convection-reaction problem

$$\boldsymbol{\beta} \cdot \nabla u + c\, u = f \quad \text{in } \Omega, \tag{1.1a}$$

$$u = g \quad \text{on } \Gamma^-. \tag{1.1b}$$

Here $\Omega \subset R^d$ is a polyhedral domain, $\Gamma^- := \{x \in \partial\Omega : \boldsymbol{\beta} \cdot \boldsymbol{n}(x) < 0\}$, and $\boldsymbol{n}(x)$ is the outward unit normal at the point $x \in \partial\Omega$. The functions f and g are smooth, $\boldsymbol{\beta}$ is a non-zero constant unit vector and c is a bounded function. Indeed, if u_h denotes the approximation given by the DG method using polynomials of degree k, we prove that, for a *special* class of triangulations $\mathcal{T}_h$, we have

$$\| u - u_h \|_{L^2(\mathcal{T}_h)} + \| \mathsf{P}(\partial_{\boldsymbol{\beta}} u) - \partial_{\boldsymbol{\beta},h} u_h \|_{L^2(\mathcal{T}_h)} \le C\, | u |_{H^{k+1}(\mathcal{T}_h)}\, h^{k+1},$$

where $\partial_{\boldsymbol{\beta}} := \boldsymbol{\beta} \cdot \nabla$, P is the L^2-projection into the finite element space, and $\partial_{\boldsymbol{\beta},h} u_h$ is an approximation to $\partial_{\boldsymbol{\beta}}\, u$ obtained by using an element-by-element postprocessing of u_h. Note that the above approximation result is optimal for the quantity $\| u - u_h \|_{L^2(\mathcal{T}_h)}$ in the order of convergence in h as well as in the regularity of the exact solution; the estimate of the quantity $\| \mathsf{P}(\partial_{\boldsymbol{\beta}} u) - \partial_{\boldsymbol{\beta},h} u_h \|_{L^2(\mathcal{T}_h)}$ is clearly a superconvergence result. This has to be contrasted with the estimate for general triangulations

$$\| u - u_h \|_{L^2(\mathcal{T}_h)} + h^{1/2}\, \| \partial_{\boldsymbol{\beta}} u - \partial_{\boldsymbol{\beta}} u_h \|_{L^2(\mathcal{T}_h)} \le C\, | u |_{H^{k+1}(\mathcal{T}_h)}\, h^{k+1/2},$$

that follows from results obtained back in 1986 in [12].

The mechanisms that induce the loss of $h^{1/2}$ in the order of convergence of the L^2-norm of the error are not very well known yet. In 1988, it was shown [17] that, in the two-dimensional case, the estimate

$$\| u - u_h \|_{L^2(\mathcal{T}_h)} \le C\, | u |_{H^{k+2}(\mathcal{T}_h)}\, h^{k+1},$$

*School of Mathematics, University of Minnesota, Vincent Hall, Minneapolis, MN 55455, USA, email: cockburn@math.umn.edu. Supported in part by the National Science Foundation (Grant DMS-0411254) and by the University of Minnesota Supercomputing Institute.

†School of Mathematics, University of Minnesota, Vincent Hall, Minneapolis, MN 55455, USA, email: bdong@math.umn.edu.

‡School of Mathematics, University of Minnesota, Vincent Hall, Minneapolis, MN 55455, USA, email: guzma033@umn.edu. Supported by an NSF Mathematical Science Postdoctoral Research Fellowship (DMS-0503050).

holds for conforming triangulations obtained by using slabs of parallelograms divided into two triangles always in the same way. The triangles were chosen so that their sides are uniformly not aligned with the convection direction $\boldsymbol{\beta}$, that is, so that they satisfy what we could call the *transversality* condition

$$|\boldsymbol{\beta} \cdot \boldsymbol{n}_K| \geq \gamma > 0 \quad \text{on } \partial K, \quad \text{for all } K \in \mathcal{T}_h, \tag{1.2}$$

where $\boldsymbol{n}_K$ is the outward unit normal of the simplex K and γ is a fixed constant. In 1991, the rate of convergence of $h^{k+1/2}$ for $\| u - u_h \|_{L^2(\mathcal{T}_h)}$ was shown to be sharp in [14]; a rigorous proof was given for the case $k = 0$ and convincing numerical evidence was shown for the case $k = 1$, also in the two-dimensional case. As the reader might expect, in those numerical experiments, triangulations violating the transversality condition played a central role which lead the author to conjecture that this condition "*may be a natural condition under which to seek improved estimates*". However, a consequence of our main result is that improved estimates can be obtained even if the transversality condition is not satisfied in *any* triangle. Indeed, our improved estimates hold for triangulations $\mathcal{T}_h$ made of simplexes K satisfying the simple *flow* conditions with respect to $\boldsymbol{\beta}$

Each simplex K has a unique *outflow* face with respect to $\boldsymbol{\beta}$, e_K^+. (1.3a)

Each interior face e_K^+ is included in an *inflow* face with respect to $\boldsymbol{\beta}$ of another simplex . (1.3b)

We say that the face e of the simplex K is an outflow (inflow) face with respect to $\boldsymbol{\beta}$ if $\boldsymbol{\beta} \cdot \boldsymbol{n}_K|_e > \;(<)\; 0$. We say that a face is interior if it is not included in $\partial\Omega$. Note that the second condition allows the triangulation to be nonconforming. In two dimensions, this means that hanging nodes in a simplex K are allowed provided they are *not* in its outflow edge; an example of such a triangulation is given in Fig. 1.1. Note that this triangulation is not uniform or translation invariant. In the Appendix, we show how to construct triangulations satisfying the flow conditions in any number of space dimensions.

As usual, the families of triangulations we consider also satisfy the classical assumption of shape regularity, see [6], namely, there is a constant $\sigma > 0$ such that

$$\text{For each simplex } K \in \mathcal{T}_h : h_K/\rho_K \geq \sigma, \tag{1.4}$$

where h_K denotes the diameter of the simplex K and ρ_K the diameter of the biggest ball included in K.

The main idea behind the devising of these meshes, and the associated analysis giving rise to optimal error estimates, is a suitable adaptation to the convection-reaction equation under consideration of the projection $(\boldsymbol{\Pi}, \mathbb{P})$ recently introduced for the analysis of superconvergent discontinuous Galerkin methods for second-order elliptic problems in [7]. The first component of such projection, $\boldsymbol{\Pi}$, was previously used in the analysis of the so-called minimal-dissipation LDG method for convection-diffusion problems. Here, we use the second component, $\mathbb{P}$, to render the analysis of error in u not only extremely simple and optimal but also capable of handling non-conforming meshes that are not uniform in any way.

Let us emphasize that the role of the special meshes for the construction of the approximation $\partial_{\boldsymbol{\beta},h} u_h$ to the directional derivative $\partial_{\boldsymbol{\beta}} u$ is *not* essential for two reasons. The first is that such an approximation can be defined on any mesh of simplexes. The

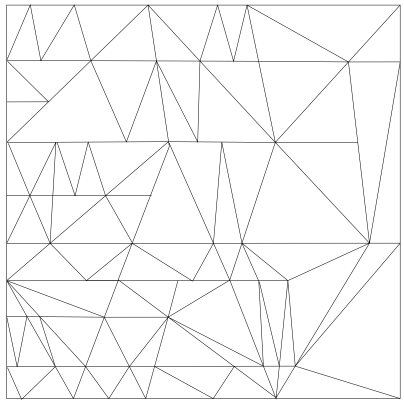

FIG. 1.1. *A triangulation satisfying the flow conditions with $\boldsymbol{\beta} = (1, 0)$.*

second is that, for the general meshes considered in [12], we can still obtain the estimate

$$\| \mathsf{P}(\partial_{\boldsymbol{\beta}} u) - \partial_{\boldsymbol{\beta},h} u_h \|_{L^2(\mathcal{T}_h)} \le C \,|\, u \,|_{H^{k+1}(\mathcal{T}_h)}\, h^{k+1/2},$$

which shows that even in this case, $\partial_{\boldsymbol{\beta},h} u_h$ is a better approximation than $\partial_{\boldsymbol{\beta}} u_h$. Postprocessings similar to the one giving rise to $\partial_{\boldsymbol{\beta},h} u_h$ have been used before in the context of flow through porous media, [4], in the context of the Navier-Stokes equations, [8], and in the context of linear elastic incompressible materials, [9]. In all the above-mentioned references, they have been used to construct exactly divergence-free approximations to the velocity or exactly divergence-free stresses.

It is interesting to note that, when the triangulations $\mathcal{T}_h$ satisfy the flow conditions (1.3) with respect to $\boldsymbol{\beta}$ *and* with respect to $-\boldsymbol{\beta}$, case in which the triangulation consists of tubes aligned with $\boldsymbol{\beta}$, we can obtain error estimates of the numerical trace $\widehat{u}_h$ in *each* outflow face e, namely,

$$\|\mathsf{P}_\partial u - \widehat{u}_h\|_{H^{-s}(e)} \le \mathsf{C}(c, s) h^{k+s+1} |\, u \,|_{H^{k+1}(\Omega_e)},$$

for $s \in [0, k]$, where P_∂ is the L^2-projection into the space of polynomials of degree at most k on each face, Ω_e is a suitably chosen subset of $\mathcal{T}_h$, and $c \in W^{s,\infty}(\Omega_e)$. In particular, this implies that the average of $\widehat{u}_h$ on the face e converges to the average of u on that face with order $2\,k+1$. This is the only result of this type in the current available scientific literature, to the knowledge of the authors.

Finally, we extend our results in two directions. First, we consider approximations that are polynomials of different degrees on different elements. With the condition that the degrees of polynomials are non-increasing in the direction of $\boldsymbol{\beta}$, we show that the estimates of approximations of u and $\partial_{\boldsymbol{\beta},h} u_h$ still hold. Then we consider the singularly perturbed problem in $\Omega \subset \mathbb{R}^2$

$$\begin{aligned} -\epsilon \Delta u + \boldsymbol{\beta} \cdot \nabla u + cu &= f \quad \text{in } \Omega,\\ u &= g \quad \text{on } \partial\Omega. \end{aligned}$$

where $0 < \epsilon \ll 1$ is a constant and prove optimal *local* error estimates for $\epsilon \leq h^2$ on quasi-uniform meshes. We also show that internal numerical layers have width $\max(h, \epsilon^{1/2})$ on triangulations $\mathcal{T}_h$ that satisfy the flow conditions (1.3) with respect to both $\boldsymbol{\beta}$ and $-\boldsymbol{\beta}$. This improves the best known result of $h^{1/2}$; see [11]. A similar result for the streamline-diffusion method is contained in [15].

The paper is organized as follows. In section 2 we state and prove our main results. In section 3, we extend our results to variable-degree versions of the method; we show the results for the singularly perturbed problem. Our theoretical results are verified in section 4 by numerical experiments. We end in Section 5 with some concluding remarks.

2. The main results.

2.1. The DG method. Suppose we have a family of triangulations $\{\mathcal{T}_h\}$ of Ω satisfying the flow conditions (1.3). To each triangulation $\mathcal{T}_h$, we associate the number $h = \sup_{K\in\mathcal{T}_h} h_K$, where $h_K = diam(K)$, and the finite-dimensional space V_h^k which is composed of functions that are polynomials of degree at most k on each simplex $K \in \mathcal{T}_h$.

The DG approximation $u_h \in V_h^k$ of the solution of (1.1) satisfies

$$B(u_h, v_h) = (f, v_h)_{\mathcal{T}_h} - \langle g, v_h\, \boldsymbol{\beta}\cdot\boldsymbol{n}\rangle_{\Gamma^-}, \quad \text{for all } v_h \in V_h^k, \tag{2.1a}$$

where

$$B(w, v) = -\,(w, \partial_{\boldsymbol{\beta}} v)_{\mathcal{T}_h} + \langle \widehat{w}, v\, \boldsymbol{\beta}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h\setminus\Gamma^-} + (c\, w, v)_{\mathcal{T}_h}, \tag{2.1b}$$

for any w, v in $H^1(\mathcal{T}_h)$. Note that we only need to define the numerical trace $\widehat{w}$ on faces that are not parallel to the direction $\boldsymbol{\beta}$ and do not belong to the inflow part of the boundary Γ^-. Therefore, the numerical trace of a function w on a simplex K for such faces e is given by

$$\widehat{w} := w^-, \tag{2.1c}$$

where $w^\pm(z) = \lim_{\delta\downarrow 0} w(z \pm \delta\boldsymbol{\beta})$ where $z \in e$. We are using the notation

$$(\boldsymbol{\sigma}, \boldsymbol{v})_{\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h} \int_K \boldsymbol{\sigma}(x)\cdot\boldsymbol{v}(x)\,dx, \qquad (\zeta, \omega)_{\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h} \int_K \zeta(x)\,\omega(x)\,dx,$$
$$\langle \zeta, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h} \int_{\partial K} \zeta(\gamma)\,\boldsymbol{v}(\gamma)\cdot\boldsymbol{n}\,d\gamma,$$

for any functions $\boldsymbol{\sigma}, \boldsymbol{v}$ in $\boldsymbol{H}^1(\mathcal{T}_h) := [H^1(\mathcal{T}_h)]^d$ and ζ, ω in $H^1(\mathcal{T}_h)$. The outward normal unit vector to ∂K is denoted by $\boldsymbol{n}$.

Notice that the exact solution u of (1.1) also satisfies the weak formulation (2.1), so we have the error equation

$$B(u - u_h, v_h) = 0 \qquad \text{for all } v_h \in V_h^k. \tag{2.2}$$

2.2. The approximation of u. To state our result on the approximation of u, we need to introduce a special projection, $\mathbb{P}$, defined on triangulations $\mathcal{T}_h$ satisfying the flow condition (1.3a). The function $\mathbb{P}u \in V_h^k$ restricted to $K \in \mathcal{T}_h$ is given by

$$(\mathbb{P}u - u, v)_K = 0, \qquad \text{for all } v \in \mathcal{P}^{k-1}(K) \tag{2.3a}$$
$$\langle \mathbb{P}u - u, w\rangle_{e_K^+} = 0, \qquad \text{for all } w \in \mathcal{P}^k(e_K^+), \tag{2.3b}$$

where $\mathcal{P}^\ell(D)$ stands for the space of polynomials of total degree at most ℓ defined on the set D.

The following lemma was proved in [7].

LEMMA 2.1. *If the triangulation $\mathcal{T}_h$ satisfies the flow condition* (1.3a)*, the projection $\mathbb{P}$ given by* (2.3) *is well defined. Moreover, if the triangulation $\mathcal{T}_h$ is shape-regular,* (1.4)*, then, on each simplex $K \in \mathcal{T}_h$ we have*

$$\|\mathbb{P}u - u\|_{L^2(K)} \le C h^{k+1} |\, u\,|_{H^{k+1}(K)}, \tag{2.4}$$

where C only depends on k and the shape regularity constant σ.

Now we can state our first error estimate.

THEOREM 2.2. *If $\mathcal{T}_h$ satisfies the flow conditions* (1.3) *and and the shape-regularity conditoin* (1.4)*, then $h^{1/2}\,\|\, c\,\|_{L^\infty(\Omega)}$ is small enough, the error between u_h given by the discontinuous Galerkin method* (2.1) *and the exact solution u of the equations* (1.1) *is bounded as follows:*

$$\|\mathbb{P}u - u_h\|_{L^2(\mathcal{T}_h)} \le C \|c(u - \mathbb{P}u)\|_{L^2(\mathcal{T}_h)},$$

where C depends on $\|\, c\,\|_{L^\infty(\Omega)}$ and the diameter of Ω. In particular, if $c \equiv 0$ then

$$u_h = \mathbb{P}u.$$

Note that, after a straightforward application of the triangle inequality, we get

$$\|u - u_h\|_{L^2(\mathcal{T}_h)} \le \|u - \mathbb{P}u\|_{L^2(\mathcal{T}_h)} + C\|c(u - \mathbb{P}u)\|_{L^2(\mathcal{T}_h)},$$

and, if we assume the shape-regularity condition (1.4) on the triangulation $\mathcal{T}_h$, by the approximation property of the projection $\mathbb{P}$, (2.4), we obtain that

$$\|\, u - u_h\,\|_{L^2(\mathcal{T}_h)} \le C\,|\, u\,|_{H^{k+1}(\mathcal{T}_h)}\, h^{k+1},$$

whenever $u \in H^{k+1}(\mathcal{T}_h)$, as claimed in the Introduction. Let us prove the above theorem.

Proof. Set $\mathbb{E} = u_h - \mathbb{P}u$. By the error equation (2.2), we have that for all $v \in V_h^k$,

$$B(\mathbb{E}, v) = B(u - \mathbb{P}u, v) = \sum_{i=1}^{3} T_i,$$

where, by definition of the bilinear form $B(\cdot,\cdot)$, (2.1b),

$$\begin{aligned}
T_1 :=& -(u - \mathbb{P}u, \partial_{\boldsymbol{\beta}} v)_{\mathcal{T}_h},\\
T_2 :=& \langle u - \widehat{\mathbb{P}u}, v\,\boldsymbol{\beta}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h\setminus\Gamma^-},\\
T_3 :=& (c\,(u - \mathbb{P}u), v)_{\mathcal{T}_h}.
\end{aligned}$$

Now, by the definition of the projection $\mathbb{P}$, (2.3a),

$$T_1 = 0.$$

Moreover,

$$\begin{aligned} T_2 =&\langle u - \widehat{\mathbb{P}u}, v\,\boldsymbol{\beta}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h\setminus\Gamma^-}\\ =&\sum_{K\in\mathcal{T}_h}\langle u - \widehat{\mathbb{P}u}, v\,\boldsymbol{\beta}\cdot\boldsymbol{n}\rangle_{\partial K\setminus\Gamma^-}\\ =&\sum_{K\in\mathcal{T}_h}\langle u - \widehat{\mathbb{P}u}, (v^- - v^+)\,\boldsymbol{\beta}\cdot\boldsymbol{n}\rangle_{e_K^+}\\ =&\sum_{K\in\mathcal{T}_h}\langle u - \mathbb{P}u, (v^- - v^+)\,\boldsymbol{\beta}\cdot\boldsymbol{n}\rangle_{e_K^+}. \end{aligned}$$

by the definition of the numerical trace $\widehat{\mathbb{P}\,u}$, (2.1c). But, since by the second flow condition (1.3b), $(v^- - v^+)|_{e_K^+}\in\mathcal{P}^k(e_K^+)$, we can conclude that

$$T_2 = 0,$$

by the definition of the projection $\mathbb{P}$, (2.3b). Therefore, we have that

$$B(\mathbb{E}, v) =(c(u-\mathbb{P}u), v)_{\mathcal{T}_h}\quad\text{for all } v\in V_h^k.$$

We claim that this implies

$$\|\mathbb{E}\|_{L^2(\mathcal{T}_h)} \le C\|c\,(u-\mathbb{P}u)\|_{L^2(\mathcal{T}_h)},$$

for $h^{1/2}\,\|\,c\,\|_{L^\infty(\Omega)}$ small enough. This claim is a straightforward consequence of the stability result Theorem 2.1 in [12], which implies that the DG approximation u_h defined by (2.1) satisfies the inequality

$$\|\,u_h\,\|_{L^2(\Omega)} \le C\left(\|\,f\,\|_{L^2(\Omega)} + \|\,\sqrt{|\boldsymbol{\beta}\cdot\boldsymbol{n}|}\;g\,\|_{L^2(\Gamma^-)}\right),$$

for $h^{1/2}\,\|\,c\,\|_{L^\infty(\Omega)}$ small enough. The claim follows by replacing u_h by $\mathbb{E}$, f by $c\,(u-\mathbb{P}u)$, and g by 0. Let us emphasize the fact that although this stability result was obtained for the two-dimensional case and conforming triangulations, it can be easily extended to multidimensions and non-conforming triangulations. This completes the proof. ◻

2.3. Post-processing: The approximation to $\partial_{\boldsymbol{\beta}}u$. Next we show how to post-process u_h in order to get a superconvergent approximation of $\partial_{\boldsymbol{\beta}}u$. To this end, for each simplex K we define $\boldsymbol{q}_h\in\boldsymbol{\mathcal{P}}^k(K)+\boldsymbol{x}\,\mathcal{P}^k(K)$ to be the solution of

$$(\boldsymbol{q}_h - \boldsymbol{\beta}u_h, \boldsymbol{v})_K = 0, \qquad \text{for all } \boldsymbol{v}\in\boldsymbol{\mathcal{P}}^{k-1}(K) \tag{2.5a}$$

$$\langle(\boldsymbol{q}_h - \boldsymbol{\beta}\lambda_h)\cdot\boldsymbol{n}, w\rangle_e = 0, \qquad \text{for all } w\in\mathcal{P}^k(e), \text{ for all faces } e \text{ of } K, \tag{2.5b}$$

where $\lambda_h = \mathsf{P}_\partial g$ on Γ^- and $\lambda_h = \widehat{u}_h$ otherwise; here $\boldsymbol{\mathcal{P}}^k(K) := [\mathcal{P}^k(K)]^d$. The existence and uniqueness of $\boldsymbol{q}_h$ is well known; see, for example, [5]. We then define

$$\partial_{\boldsymbol{\beta},h}u_h := \nabla\cdot\boldsymbol{q}_h \qquad \text{in } \mathcal{T}_h.$$

We can now state the error estimate between $\partial_{\boldsymbol{\beta},h}u_h$ and $\partial_{\boldsymbol{\beta}}u$.

THEOREM 2.3. *If* $\mathcal{T}_h$ *is an* arbitrary *triangulation of* Ω*, then*

$$\|\partial_{\boldsymbol{\beta},h} u_h - \mathsf{P}(\partial_{\boldsymbol{\beta}} u)\|_{L^2(\mathcal{T}_h)} \le C \|c(u-u_h)\|_{L^2(\mathcal{T}_h)}.$$

In particular, if $c \equiv 0$ *then*

$$\partial_{\boldsymbol{\beta},h} u_h = \mathsf{P}(\partial_{\boldsymbol{\beta}} u).$$

Here P *is the* L^2*-projection into* V_h^k*.*

We can thus see that, if we use the estimates for arbitrary triangulations $\mathcal{T}_h$ obtained in [12], which assume the shape-regularity condition (1.4), we obtain

$$\| \mathsf{P}(\partial_{\boldsymbol{\beta}} u) - \partial_{\boldsymbol{\beta},h} u_h \|_{L^2(\mathcal{T}_h)} \le C \, | \, u \, |_{H^{k+1}(\mathcal{T}_h)} \, h^{k+1/2},$$

and if, we assume that $\mathcal{T}_h$ satisfies the flow conditions (1.3), we obtain, by Theorem 2.2,

$$\| \mathsf{P}(\partial_{\boldsymbol{\beta}} u) - \partial_{\boldsymbol{\beta},h} u_h \|_{L^2(\mathcal{T}_h)} \le C \, | \, u \, |_{H^{k+1}(\mathcal{T}_h)} \, h^{k+1},$$

as claimed in the Introduction. This implies that, if u is smooth enough, the quantity $\| \partial_{\boldsymbol{\beta}} u - \partial_{\boldsymbol{\beta},h} u_h \|_{L^2(\mathcal{T}_h)}$ is of order $h^{k+1/2}$ for arbitrary meshes and of order h^{k+1} for the special meshes under consideration.

Proof. For $v \in V_h^k$ we have

$$\begin{aligned}
(\partial_{\boldsymbol{\beta},h} u_h - \mathsf{P}(\partial_{\boldsymbol{\beta}} u), v)_{\mathcal{T}_h} =& (\nabla \cdot \boldsymbol{q}_h - \partial_{\boldsymbol{\beta}} u, v)_{\mathcal{T}_h} \\
=& - (\boldsymbol{q}_h, \nabla v)_{\mathcal{T}_h} + \langle \boldsymbol{q}_h \cdot \boldsymbol{n}, v \rangle_{\partial \mathcal{T}_h} - (\partial_{\boldsymbol{\beta}} u, v)_{\mathcal{T}_h} \\
=& - (\boldsymbol{q}_h, \nabla v)_{\mathcal{T}_h} + \langle \boldsymbol{q}_h \cdot \boldsymbol{n}, v \rangle_{\partial \mathcal{T}_h} - (f, v)_{\mathcal{T}_h} + (c\, u, v)_{\mathcal{T}_h},
\end{aligned}$$

by the definition of the exact solution u of (1.1a). By the definition of $\boldsymbol{q}_h$, (2.5),

$$\begin{aligned}
(\partial_{\boldsymbol{\beta},h} u_h - \mathsf{P}(\partial_{\boldsymbol{\beta}} u), v)_{\mathcal{T}_h} =& - (u_h, \partial_{\boldsymbol{\beta}} v)_{\mathcal{T}_h} + \langle \widehat{u}_h, v \, \boldsymbol{\beta} \cdot \boldsymbol{n} \rangle_{\mathcal{T}_h \setminus \Gamma^-} + \langle g, v \boldsymbol{\beta} \cdot \boldsymbol{n} \rangle_{\Gamma^-} \\
& - (f, v)_{\mathcal{T}_h} + (c\, u, v)_{\mathcal{T}_h} \\
=& (c\, (u - u_h), v)_{\mathcal{T}_h},
\end{aligned}$$

by the definition of the approximate solution u_h, (2.1). The proof is complete once we take $v = \partial_{\boldsymbol{\beta},h} u_h - \mathsf{P}(\partial_{\boldsymbol{\beta}} u)$. ☐

2.4. The approximation of the numerical trace $\widehat{u}_h$ on outflow faces. To state the approximation result, let us introduce some notation. We are going to use the following negative-order norm

$$\| \, \gamma \, \|_{H^{-s}(e)} := \sup_{\varphi \in \mathcal{C}^\infty(e)} \frac{\langle \gamma, \varphi \rangle_e}{\| \, \varphi \, \|_{H^s(e)}}, \tag{2.6}$$

where e is the outflow face of the simplex $K \in \mathcal{T}_h$. To this face, we associate the subset of $\mathcal{T}_h$ defined by

$$\Omega_e := \{K' \in \mathcal{T}_h : \forall x \in K', \text{ such that } x + \boldsymbol{\beta} t \text{ lies on } e \text{ for some } t \ge 0\}.$$

An example is displayed in Fig. 2.1 below.

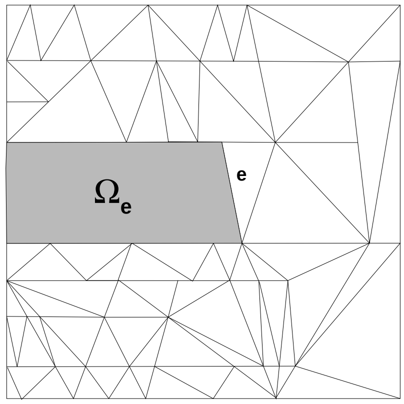

FIG. 2.1. *An example of the set* Ω_e *for* $\boldsymbol{\beta} = (1, 0)$.

THEOREM 2.4. *Let* $\mathcal{T}_h$ *be a triangulation of* Ω *satisfying the shape-regularity condition* (1.4) *and the flow conditions* (1.3)*, with respect to* $\boldsymbol{\beta}$ *and* $-\boldsymbol{\beta}$*. Then on any outflow face* e

$$\|\mathsf{P}_\partial u - \widehat{u}_h\|_{H^{-s}(e)} \leq \; \mathsf{C}(c, s)\; h^{k+s+1} |\, u\,|_{H^{k+1}(\Omega_e)},$$

for $s \in [0, k]$*, where* $\mathsf{C}(c, s) := C(\|c\|_{W^{\max\{s,0\},\infty}(\Omega_e),\Omega_e})\, \|c\|^2_{L^\infty(\Omega_e)} / |\, \boldsymbol{\beta} \cdot \boldsymbol{n}_e\,|$*. In particular, if* $c \equiv 0$ *then*

$$\mathsf{P}_\partial u = \widehat{u}_h.$$

Note that if the triangulation $\mathcal{T}_h$ satisfies the flow conditions (1.3) with respect to $\boldsymbol{\beta}$ and $-\boldsymbol{\beta}$, then each of its simplexes K has exactly one outflow face e^+_K and one inflow face e^-_K. Moreover, the outflow face e^+_K lies on $\partial\Omega \setminus \Gamma^-$ or coincides with an inflow face, and e^-_K lies on Γ^- or coincides with an outflow face.

This result implies, in particular, that the average of the numerical trace $\widehat{u}_h$ on each outflow face e converges to the average of the exact solution u on that face with order $2\,k + 1$ for general c and with order $2\,k + 2$ in the case $c = 0$. To see this, let us begin by noting that if $\Upsilon \in H^{k+1}(e)$, we have that

$$\begin{aligned}
\langle \widehat{u}_h - u, \Upsilon \rangle_e =& \langle \widehat{u}_h - \mathsf{P}_\partial u, \Upsilon \rangle_e + \langle \mathsf{P}_\partial u - u, \Upsilon \rangle_e \\
=& \langle \widehat{u}_h - \mathsf{P}_\partial u, \Upsilon \rangle_e + \langle \mathsf{P}_\partial u - u, \Upsilon - \mathsf{P}_\partial \Upsilon \rangle_e,
\end{aligned}$$

and so

$$\begin{aligned}
|\, \langle \widehat{u}_h - u, \Upsilon \rangle_e \,| \leq & \|\, \widehat{u}_h - \mathsf{P}_\partial u \,\|_{H^{-k}(e)} \,\|\, \Upsilon \,\|_{H^k(e)} \\
& + \|\, \mathsf{P}_\partial u - u \,\|_{L^2(e)} \,\|\, \Upsilon - \mathsf{P}_\partial \Upsilon \,\|_{L^2(e)} \\
\leq & \, C\, h^{2\,k+1},
\end{aligned}$$

where

$$C = \mathsf{C}(c, s)\, |\, u\,|_{H^k(\Omega_e)} \,\|\, \Upsilon \,\|_{H^{k+1}(e)} + C\, h\, |\, u\,|_{H^{k+1}(e)} \,|\, \Upsilon \,|_{H^{k+1}(e)}.$$

The claim follows by simply taking $\Upsilon = 1$.

Let us emphasize that the order of convergence of the approximation of linear functionals of the form $\langle u\,,\,\Upsilon\rangle_{\partial\Omega\setminus\Gamma^-}$ can be proven to be of order $2\,k+1$; see [1] for the case $c=0$ and $\Upsilon=1$. This result holds for functions Υ that are independent of the mesh; as a consequence, Υ cannot be taken to have support in a single element face. To the knowledge of the authors, Theorem 2.4 is the only result that allows this.

Next, we give a proof of the result. To do that, we need to introduce some notation. We begin by introducing so-called the corresponding dual problem, namely,

$$-\boldsymbol{\beta}\cdot\nabla\psi + c\,\psi = 0 \qquad \text{in } \Omega_e, \tag{2.7a}$$

$$\psi = \varphi/\boldsymbol{\beta}\cdot\boldsymbol{n} \qquad \text{on } e. \tag{2.7b}$$

We are also going to use two auxiliary projections. We set $\mathbb{P}^+ := \mathbb{P}$ which is defined in (2.3), and let $\mathbb{P}^-$ be the projection which satisfies (2.3a) and imposes (2.3b) on e_K^- instead of e_K^+.

We are now ready to prove Theorem 2.4.

Proof. To prove this result, we begin by noting that

$$\langle \mathsf{P}_\partial u - \widehat{u}_h, \varphi\rangle_e = \langle \mathsf{P}_\partial u - \widehat{u}_h, \psi\boldsymbol{\beta}\cdot\boldsymbol{n}\rangle_e,$$

by the boundary condition of the dual problem, (2.7b), and that

$$\langle \mathsf{P}_\partial u - \widehat{u}_h, \varphi\rangle_e = \langle \mathsf{P}_\partial u - \widehat{u}_h, \psi\boldsymbol{\beta}\cdot\boldsymbol{n}\rangle_{\partial\Omega_e\setminus\Gamma^-},$$

by our assumptions on the triangulation $\mathcal{T}_h$. Then

$$\begin{aligned}\langle \mathsf{P}_\partial u - \widehat{u}_h, \varphi\rangle_e =& \langle \mathsf{P}_\partial u - \widehat{u}_h, (\psi - \mathbb{P}^-\psi)\,\boldsymbol{\beta}\cdot\boldsymbol{n}\rangle_{\partial\Omega_e\setminus\Gamma^-}\\ &+ \langle \mathsf{P}_\partial u - \widehat{u}_h, \mathbb{P}^-\psi\,\boldsymbol{\beta}\cdot\boldsymbol{n}\rangle_{\partial\Omega_e\setminus\Gamma^-}.\end{aligned}$$

Since we can rewrite the error equation (2.2) in terms of the projections $\mathbb{P}^+$ and P_∂, as

$$-(\mathbb{P}^+u - u_h, \partial_{\boldsymbol{\beta}} v)_{\Omega_e} + \langle \mathsf{P}_\partial u - \widehat{u}_h, v\,\boldsymbol{\beta}\cdot\boldsymbol{n}\rangle_{\partial\Omega_e\setminus\Gamma^-} + (c\,(u-u_h), v)_{\Omega_e} = 0,$$

taking $v := \mathbb{P}^-\psi$, we get that

$$\begin{aligned}\langle \mathsf{P}_\partial u - \widehat{u}_h, \varphi\rangle_e =& \langle \mathsf{P}_\partial u - \widehat{u}_h, (\psi - \mathbb{P}^-\psi)\,\boldsymbol{\beta}\cdot\boldsymbol{n}\rangle_{\partial\Omega_e\setminus\Gamma^-}\\ &+ (\mathbb{P}^+u - u_h, \partial_{\boldsymbol{\beta}}\mathbb{P}^-\psi)_{\Omega_e}\\ &- (c\,(u-u_h), \mathbb{P}^-\psi)_{\Omega_e}\\ =& \langle \mathsf{P}_\partial u - \widehat{u}_h, (\psi - \mathbb{P}^-\psi)\,\boldsymbol{\beta}\cdot\boldsymbol{n}\rangle_{\partial\Omega_e\setminus\Gamma^-}\\ &+ (\mathbb{P}^+u - u_h, \partial_{\boldsymbol{\beta}}\mathbb{P}^-\psi)_{\Omega_e}\\ &- (c\,(\mathbb{P}^+u-u_h), \mathbb{P}^-\psi)_{\Omega_e}\\ &- (c\,(u-\mathbb{P}^+u), \mathbb{P}^-\psi)_{\Omega_e}.\end{aligned}$$

Hence, by the dual equation, (2.7a), we can write

$$\langle \mathsf{P}_\partial u - \widehat{u}_h, \varphi\rangle_e = \sum_{i=1}^{4} T_i,$$

where

$$\begin{aligned} T_1 :=&\ \langle \mathsf{P}_\partial u - \widehat{u}_h, (\psi - \mathbb{P}^- \psi)\, \boldsymbol{\beta}\cdot\boldsymbol{n}\rangle_{\partial\Omega_e\setminus\Gamma^-},\\ T_2 :=&\ (\mathbb{P}^+ u - u_h, \partial_{\boldsymbol{\beta}}(\mathbb{P}^-\psi - \psi))_{\Omega_e},\\ T_3 :=& -(c\,(\mathbb{P}^+ u - u_h), \mathbb{P}^-\psi - \psi)_{\Omega_e},\\ T_4 :=& -(c\,(u - \mathbb{P}^+ u), \mathbb{P}^-\psi)_{\Omega_e}. \end{aligned}$$

Let us estimate the terms $T_i, i = 1, 2, 3, 4$. We begin by showing that $T_1 + T_2 = 0$. After a simple integration by parts, we get that

$$\begin{aligned} T_2 =& -\langle \mathbb{P}^+ u - u_h, (\psi - \mathbb{P}^-\psi)\, \boldsymbol{\beta}\cdot\boldsymbol{n}\rangle_{\partial\Omega_e} + (\partial_{\boldsymbol{\beta}}(\mathbb{P}^+ u - u_h), \psi - \mathbb{P}^-\psi)_{\Omega_e}\\ =& -\langle \mathbb{P}^+ u - u_h, (\psi - \mathbb{P}^-\psi)\, \boldsymbol{\beta}\cdot\boldsymbol{n}\rangle_{\partial\Omega_e}, \end{aligned}$$

by the orthogonality property of the projection $\mathbb{P}^-$, (2.3a). Hence,

$$\begin{aligned} T_1 + T_2 =& \langle \mathsf{P}_\partial u - \widehat{u}_h, (\psi - \mathbb{P}^-\psi)\, \boldsymbol{\beta}\cdot\boldsymbol{n}\rangle_{\partial\Omega_e\setminus\Gamma^-} - \langle \mathbb{P}^+ u - u_h, (\psi - \mathbb{P}^-\psi)\, \boldsymbol{\beta}\cdot\boldsymbol{n}\rangle_{\partial\Omega_e}\\ =& 0 \end{aligned}$$

since, by the definition of $\mathbb{P}^+$ and $\widehat{u}_h$, we have that $\mathsf{P}_\partial u = \mathbb{P}^+ u$ and $\widehat{u}_h = u_h$ on any outflow face, and since, by the definition of $\mathbb{P}^-$, we have that

$$\langle \omega, (\psi - \mathbb{P}^-\psi)\, \boldsymbol{\beta}\cdot\boldsymbol{n}\rangle_{\tilde{e}} = 0 \quad \forall \omega \in \mathcal{P}^k(\tilde{e}),$$

on any inflow face $\tilde{e}$.

It remains to estimate T_3 and T_4. To do this, we are going to use the following stability estimate for the solution ψ of the dual problem (2.7),

$$|\psi|_{H^s(\Omega_e)} \le C\|\varphi\|_{H^s(e)}/\boldsymbol{\beta}\cdot\boldsymbol{n}, \tag{2.8}$$

where $C = C(\|c\|_{W^{\max\{s,0\},\infty}(\mathcal{T}_h)}, \Omega_e)$. It follows from the expression

$$\psi(x) = \varphi(x_0) e^{-\int_0^t\ c(x_0 - \tau\,\boldsymbol{\beta}))d\tau}/\boldsymbol{\beta}\cdot\boldsymbol{n}.$$

where $x = x_0 - \boldsymbol{\beta}\, t \in \Omega_e$ for some $t \ge 0$ and $x_0 \in e$, by successive differentiation.

The estimate of T_3 follows after a straightforward application of Cauchy-Schwarz inequality, the estimate of Theorem 2.2, the approximation property of the projection $\mathbb{P}^-$, (2.4), the stability estimate (2.8). Indeed, we easily get that

$$T_3 \le \mathsf{C}(c,s)\, h^{k+s+1} |\, u\,|_{H^{k+1}(\Omega_e)} \|\varphi\|_{H^s(e)},$$

for $s \in [0, k+1]$. Let us estimate T_4. If we denote by P^{k-1} be the L^2-projection onto V_h^{k-1}, we have that

$$\begin{aligned} T_4 =& (c\,(u - \mathbb{P}^+ u), \psi - \mathbb{P}^-\psi)_{\Omega_e} - (u - \mathbb{P}^+ u, c\psi)_{\Omega_e}\\ =& (c\,(u - \mathbb{P}^+ u), \psi - \mathbb{P}^-\psi)_{\Omega_e} - (u - \mathbb{P}^+ u, c\psi - \mathsf{P}^{k-1} c\psi)_{\Omega_e}, \end{aligned}$$

by the orthogonality property of the projection $\mathbb{P}^+$, (2.3a). Then we get

$$T_4 \le \mathsf{C}(c,s)\, h^{k+s+1} \|\varphi\|_{H^s(e)} |\, u\,|_{H^{k+1}(\Omega_e)},$$

for $s \in [0, k]$, by the approximation properties of the projections $\mathbb{P}^+$, $\mathbb{P}^-$, and P^{k-1}, and the stability estimate (2.8). This completes the proof. ☐

3. Extensions.

3.1. The variable-degree DG method. The variable-degree DG approximation u_h belongs to the finite dimensional space $V_h^{\boldsymbol{k}}$ which is given by

$$V_h^{\boldsymbol{k}} = \{v \in L^2(\Omega), v|_K \in \mathcal{P}^{k(K)}(K) \;\; \forall K \in \mathcal{T}_h\},$$

where $\boldsymbol{k} = \{k(K),\, \forall K \in \mathcal{T}_h\}$.

The definition of $\mathbb{P}$ can easily be generalized for the variable-degree space. We define $\mathbb{P}u \in V_h^{\boldsymbol{k}}$ using the definition of the previous section but with the space $\mathcal{P}^{k-1}(K)$ replaced with $\mathcal{P}^{k(K)-1}(K)$ and the space $\mathcal{P}^k(e_K^+)$ replaced with $\mathcal{P}^{k(K)}(e_K^+)$.

We need to impose a condition on the space $V_h^{\boldsymbol{k}}$:

$$k(K) \geq k(K') \text{ whenever } e_K^+ \subseteq e_{K'}^-. \tag{3.1}$$

The following result can be proven in the same way as Theorem 2.2.

THEOREM 3.1. *Let $u_h \in V_h^{\boldsymbol{k}}$ be the variable degree DG approximation given by* (2.1) *with V_h^k replaced with $V_h^{\boldsymbol{k}}$. If $\mathcal{T}_h$ satisfies the flow conditions* (1.3) *with respect to β and the shape-regularity condition* (1.4)*, and $V_h^{\boldsymbol{k}}$ satisfies the condition of the polynomial degrees* (3.1)*, then the results of Theorem 2.2 hold for the variable-degree DG method.*

3.2. Singularly Perturbed Problem. We present local error estimates of a DG method for the following singular perturbed problem in $\Omega \subset \mathbb{R}^2$

$$-\epsilon \triangle u + \boldsymbol{\beta} \cdot \nabla u + c\, u = f \quad \text{in } \Omega, \tag{3.2a}$$

$$u = g \quad \text{on } \partial\Omega. \tag{3.2b}$$

We use the interior penalty method to discretize the viscosity term $-\epsilon\triangle u$, see [2] and [10]. Any other DG method which is consistent and stable could be used to discretize this term, see [3].

Thus, the DG approximation $u_h \in V_h^k$, where $k \geq 1$, of (3.2) solves

$$\epsilon A(u_h, v_h) + B(u_h, v_h) = F(v_h) \quad \text{for all } v_h \in V_h^k, \tag{3.3}$$

where

$$A(\omega, v) = (\nabla\omega, \nabla v)_{\mathcal{T}_h} - \sum_{e \in \mathcal{E}_h} (\langle \{\!\{\nabla\omega\}\!\}, [\![v]\!]\rangle_e + \langle \{\!\{\nabla v\}\!\}, [\![\omega]\!]\rangle_e - \frac{\eta}{h}\langle [\![\omega]\!], [\![v]\!]\rangle_e),$$

$$F(v) = (f, v)_{\mathcal{T}_h} + \langle g, v\boldsymbol{\beta}\cdot\boldsymbol{n}\rangle_{\partial\Omega} + \epsilon\langle \nabla v \cdot \boldsymbol{n}, g\rangle_{\partial\Omega} - \epsilon\frac{\eta}{h}\langle g, v\rangle_{\partial\Omega}.$$

The set $\mathcal{E}_h$ is the collection of edges of the triangulation $\mathcal{T}_h$. The parameter η is large enough in order to ensure stability.

The average $\{\!\{\cdot\}\!\}$ and jump $[\![\cdot]\!]$ operators are defined as follows. For an interior edge e, we set

$$\{\!\{\boldsymbol{q}\}\!\} = \frac{1}{2}(\boldsymbol{q}_1 + \boldsymbol{q}_2), \qquad [\![\boldsymbol{q}]\!] = \boldsymbol{q}_1 \cdot \boldsymbol{n}_1 + \boldsymbol{q}_2 \cdot \boldsymbol{n}_2,$$

$$\{\!\{\varphi\}\!\} = \frac{1}{2}(\varphi_1 + \varphi_2), \qquad [\![\varphi]\!] = \varphi_1\boldsymbol{n}_1 + \varphi_2\boldsymbol{n}_2,$$

where $e = K_1 \cap K_2, \boldsymbol{q}_i = \boldsymbol{q}|_{K_i},\ \varphi_i = \varphi|_{K_i}$ and $\boldsymbol{n}_i$ is the exterior unit normal to $K_i,\ i = 1, 2$. For a boundary edge e, we set

$$\{\!\{\boldsymbol{q}\}\!\} = \boldsymbol{q}, \quad [\![\varphi]\!] = \varphi\boldsymbol{n}$$

where $\boldsymbol{n}$ is the outward unit normal. The quantities $[\![\boldsymbol{q}]\!]$ and $\{\!\{\varphi\}\!\}$ on boundary edges are not required, so they are left undefined.

In order to state the result we need to introduce a proper weight function. For simplicity we assume $\boldsymbol{\beta} = (1, 0)$. Accordingly, we set, for fixed x_0, y_1 and y_2,

$$\Omega_0 = \big((-\infty, x_0] \times [y_1, y_2]\big) \cap \Omega,$$

and construct a function ω satisfying

$$\begin{aligned}
C_1 \leq \omega(x, y) \leq C_2, \qquad & \text{for } (x, y) \in \Omega_0,\\
|\omega(x, y)| \leq C_2 e^{-(x-x_0)/M\rho}, \qquad & \text{for } x \geq x_0 + h,\\
|\omega(x, y)| \leq C_2 e^{-(y-y_2)/M\sigma}, \qquad & \text{for } y \geq y_2 + h,\\
|\omega(x, y)| \leq C_2 e^{-(y_1-y)/M\sigma}, \qquad & \text{for } y \leq y_1 - h.
\end{aligned}$$

Here $\rho \geq 0$, $\sigma \geq 0$ are parameters that will depend on the mesh size h and ϵ. We say that ρ is the size of the upwind layer and σ is the size of the crosswind layer. The positive constants C_1, C_2 and M are fixed.

Theorem 3.2. *Assume that $\epsilon \leq h$ and that the triangulation $\mathcal{T}_h$ is quasi-uniform, that is, assume that there is a parameter $\kappa > 0$ such that*

$$\max_{K \in \mathcal{T}_h} \{h_K\} \leq \kappa \min_{K \in \mathcal{T}_h} \{h_K\}.$$

If $\mathcal{T}_h$ satisfies the flow conditions (1.3) *with respect to $\boldsymbol{\beta}$. Then the error between u_h given by* (3.3) *and u given by* (3.2) *is*

$$\begin{aligned}
\|\omega(u - u_h)\|_{L^2(\mathcal{T}_h)} \leq C((1 + \frac{\epsilon^{1/2}}{h})\|\omega(u - \mathbb{P}u)\|_{L^2(\mathcal{T}_h)} + \epsilon^{1/2}\|\omega\nabla(u - \mathbb{P}u)\|_{L^2(\mathcal{T}_h)}\\
+ h\epsilon^{1/2}\|\omega D^2(u - \mathbb{P}u)\|_{L^2(\mathcal{T}_h)})
\end{aligned}$$

where ω is given above with $\rho = \log(\frac{1}{h})h$, $\sigma = h^{1/2}$ and M is a sufficiently large fixed constant. Moreover, if $\mathcal{T}_h$ also satisfies the flow conditions (1.3) *with respect to $-\boldsymbol{\beta}$, then we can choose $\sigma = \max(\epsilon^{1/2}, h)$.*

From this result, we immediately get optimal weighted error estimates if $\epsilon \leq h^2$. The size of the crosswind layer σ is typically $\sigma = h^{1/2}$ for general triangulations; see [11]. However, we see that the size of the crosswind layer is reduced to $\sigma = h$ if the flow conditions (1.3) for both $\boldsymbol{\beta}$ and $-\boldsymbol{\beta}$ are satisfied. This is exactly the result that was obtained for the streamline diffusion method in [15]. However, in [15] the following almost-uniform condition on the mesh was imposed: two adjacent elements K and K' that share an edge that is not aligned with $\boldsymbol{\beta}$ must satisfy

$$|h_{\boldsymbol{\beta},K} - h_{\boldsymbol{\beta},K'}| \leq C h_{\boldsymbol{\beta}}^2$$

where $h_{\boldsymbol{\beta},K}$ is the length of the edge of K that is parallel to $\boldsymbol{\beta}$ and $h_{\boldsymbol{\beta}} = \max_K h_{\boldsymbol{\beta},K}$; see the inequality (3.8) in [15]. In contrast, in our results, we only assume quasi-uniformity. Moreover, our results also hold for high-order elements.

The proof of Theorem 3.2 is very similar to the proof of the local estimates given in [11]. However, instead of using the L^2-projection one must use the projections $\mathbb{P} = \mathbb{P}^+$ and $\mathbb{P}^-$ used in this paper.

4. Numerical Results. In this section we present numerical experiments which validate our theoretical results. The domain is $\Omega = (-.5, .5) \times (-.5, .5)$. The coefficients are $\boldsymbol{\beta} \equiv (1, 0)$ and $c \equiv 1$. We choose the right-hand side f so that the solution is $u(x, y) = (x + 1/2)\sin(x)\sin(y)$.

We start with a uniform mesh of size h which satisfies the flow condition with respect to $\boldsymbol{\beta}$; see Fig. 4 left. We then perturb the coordinates of the interior nodes randomly by at most $\frac{2h}{5}$ in such a way that the resulting mesh is *no longer uniform but still satisfies the flow condition with respect to* $\boldsymbol{\beta}$; see Fig. 4 right. A mesh ℓ is a perturbation of a uniform mesh of size $h = \frac{1}{2^\ell}$.

In Table 4.1 we display the error and orders of convergence for approximate solutions using polynomials of degree $k = 0, 1, 2$. We see that order $k + 1$ is observed for $\|u - u_h\|_{L^2(\Omega)}$ and $\|\partial_{\boldsymbol{\beta}} u - \partial_{\boldsymbol{\beta},h} u_h\|_{L^2(\Omega)}$ as the theory predicts. Moreover, we see that $\max_{e \in \mathscr{E}_h} |\mathrm{avg}_e(u - \widehat{u}_h)|$ converges with order $2k + 1$ as expected.

TABLE 4.1
History of convergence

	mesh	$\|u - u_h\|_{L^2(\Omega)}$		$\|\partial_{\boldsymbol{\beta}} u - \partial_{\boldsymbol{\beta},h} u_h\|_{L^2(\Omega)}$		$\max_{e \in \mathscr{E}_h} \|\mathrm{avg}_e(u - \widehat{u}_h)\|$	
k	ℓ	error	order	error	order	error	order
0	1	.27e-1	-	.11e-0	-	.16e-1	-
	2	.19e-1	0.50	.54e-1	1.05	.88e-2	0.83
	3	.11e-1	0.80	.29e-1	0.93	.61e-2	0.54
	4	.58e-2	0.92	.15e-1	0.92	.35e-2	0.82
	5	.29e-2	1.01	.75e-2	1.02	.16e-2	1.11
	6	.15e-2	0.98	.38e-2	0.97	.82e-3	0.96
	7	.74e-3	0.99	.19e-2	1.00	.42e-3	0.98
1	1	.12e-1	-	.27e-1	-	.82e-3	-
	2	.34e-2	1.81	.69e-2	1.98	.92e-4	3.16
	3	.86e-3	1.97	.15e-2	2.17	.18e-4	2.36
	4	.23e-3	1.90	.43e-3	1.84	.25e-5	2.83
	5	.58e-4	1.99	.11e-3	2.01	.30e-6	3.08
	6	.15e-4	1.95	.28e-4	1.94	.44e-7	2.74
	7	.38e-5	1.99	.69e-5	2.00	.60e-8	2.90
2	1	.16e-2	-	.14e-2	-	.49e-5	-
	2	.19e-3	3.10	.17e-3	3.08	.53e-6	3.20
	3	.26e-4	2.88	.24e-4	2.82	.20e-7	4.74
	4	.33e-5	2.99	.30e-5	3.00	.66e-9	4.92
	5	.43e-6	2.92	.40e-6	2.91	.25e-10	4.74
	6	.54e-7	2.99	.51e-7	2.99	.93e-12	4.73

5. Concluding remarks. This paper contains the first instance in which the approximation error $\| u - u_h \|_{L^2(\mathcal{T}_h)}$, where u is the solution of the transport-reaction equation and u_h is given by the original DG method, is proven to be optimal in the mesh size h *and* in the regularity of the exact solution. Unexpectedly, this happens

Fig. 4.1. *Meshes satisfying the flow condition with respect to* $\boldsymbol{\beta} = (1,0)$*: Uniform mesh* $(h = \frac{1}{2^3})$ *(left) and its nonuniform perturbation* $(\ell = 3)$ *(right).*

with meshes whose main feature is to be, roughly speaking, *aligned* with the flow.

The fact that the direction of the flow $\boldsymbol{\beta}$ is a constant and the use of simplexes to define the triangulations $\mathcal{T}_h$ seem to play a major role in the result. The case of variable $\boldsymbol{\beta}$ and simplexes with curved boundaries is the subject of ongoing research.

Appendix.

The construction of triangulations satisfying the flow conditions. Let us show that it is always possible to construct a triangulation of the domain $\Omega \subset \mathbb{R}^d$ satisfying the flow conditions (1.3).

We can do this as follows. First, we triangulate the inflow border of Ω, Γ^-, by using $(d-1)$-dimensional simplexes T; let $\Gamma_h^- := \{T\}$ be such a triangulation. Next, for each T, we construct the $d-$dimensional *prism*

$$\mathcal{T}_T := (T \oplus \{\lambda\boldsymbol{\beta} : \lambda \in \mathbb{R}\}) \cap \Omega,$$

which we are going to triangulate by using conforming d-dimensional simplexes. Note that the triangulations of the prisms are completely independent of each other since no conformity between them is required thanks to the flow conditions 1.3.

To triangulate the prism $\mathcal{T}_T$, we only have to show that, if T and T' are two $(d-1)$-dimensional simplexes whose vertices are on the border of $\mathcal{T}_T$, we can triangulate the convex hull of T and T', $CH(T, T')$, by using exactly d d-dimensional simplexes each of which has one inflow and one outflow face. Those simplexes are $CH(T_{i-1}, T_i)$, $i = 1, \dots, d$, where the $(d-1)$-dimensional simplexes T_i, $i = 0, \dots, d$, are constructed in such a way that $T_0 := T$, $T_d := T'$, and, assuming that T is an inflow face, that T_{i-1} is the only inflow face of the simplex $CH(T_{i-1}, T_i)$ and T_i is the only outflow face of the simplex $CH(T_{i-1}, T_i)$.

If we identify the $(d-1)$-dimensional simplex T_i with its set of vertices, $\{\boldsymbol{x}_i^j\}_{j=1}^d$, it is not difficult to see that we can take

$$\begin{aligned}
T_0 \equiv& \{\boldsymbol{x}_0^1,\, \boldsymbol{x}_0^2, \dots, \boldsymbol{x}_0^{d-2},\, \boldsymbol{x}_0^{d-1},\, \boldsymbol{x}_0^d\},\\
T_1 \equiv& \{\boldsymbol{x}_0^1,\, \boldsymbol{x}_0^2, \dots, \boldsymbol{x}_0^{d-2},\, \boldsymbol{x}_0^{d-1},\, \boldsymbol{x}_d^d\},\\
T_2 \equiv& \{\boldsymbol{x}_0^1,\, \boldsymbol{x}_0^2, \dots, \boldsymbol{x}_0^{d-2},\, \boldsymbol{x}_d^{d-1},\, \boldsymbol{x}_d^d\},\\
T_3 \equiv& \{\boldsymbol{x}_0^1,\, \boldsymbol{x}_0^2, \dots, \boldsymbol{x}_d^{d-2},\, \boldsymbol{x}_d^{d-1},\, \boldsymbol{x}_d^d\},\\
&\qquad\dots\\
T_d \equiv& \{\boldsymbol{x}_d^1, \boldsymbol{x}_d^2, \dots, \boldsymbol{x}_d^{d-2}, \boldsymbol{x}_d^{d-1},\, \boldsymbol{x}_d^d\},
\end{aligned}$$

respectively. An example of this construction is illustrated in the Fig. 5.1.

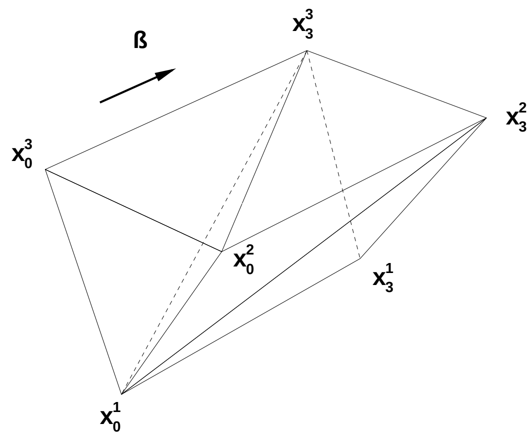

FIG. 5.1. *Detail of the construction of a 3D triangulation satisfying the flow conditions.*

REFERENCES

[1] S. Adjerid and T.C. Massey, *A posteriori discontinuous finite element error estimation for two-dimensional hyperbolic problems*, Comput. Methods Appl. Mech. Engrg. **191** (2002), 5877–5897.

[2] D.N. Arnold, *An interior penalty finite element method with discontinuous elements*, SIAM J. Numer. Anal., **19** (1982), 742-760.

[3] D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini, *Unified analysis of discontinuous Galerkin methods for elliptic problems*, SIAM J. Numer. Anal. **39** (2002), 1749–1779.

[4] P. Bastian and B. Rivière, *Superconvergence and $H(\mathrm{div})$ projection for discontinuous Galerkin methods*, Internat. J. Numer. Methods Fluids **42** (2003), 1043–1057.

[5] F. Brezzi and M. Fortin, *Mixed and hybrid finite element methods*, Springer Verlag, 1991.

[6] P. Ciarlet, *The finite element method for elliptic problems*, North-Holland, Armsterdam, 1978.

[7] B. Cockburn, B. Dong and J. Guzmán, *A superconvergent LDG-hybridizable Galerkin method for second-order elliptic problems*, Submitted to Math. Comp.

[8] B. Cockburn, G. Kanschat, and D. Schötzau, *A locally conservative LDG method for the incompressible Navier-Stokes equations*, Math. Comp. **74** (2005), 1067–1095.

[9] B. Cockburn, D. Schötzau, and J. Wang, *Discontinuous Galerkin methods for incompressible elastic materials*, Comput. Methods Appl. Mech. Engng. **195** (2006), 3184–3204, C. Dawson, Ed.

[10] J. Gopalakrishnan, G. Kanschat, *A multilevel discontinuous Galerkin method*, Numer. Math. 95 (2003), 527-550.

[11] J. Guzmán, *Local analysis of discontinuous Galerkin methods applied to singularly perturbed problems*, J. Numer. Math. **14** (2006), 41-56.

[12] C. Johnson and J. Pitkäranta, *An analysis of the discontinuous Galerkin method for a scalar hyperbolic equation*, Math. Comp., **46** (1986), 1-26.

[13] P. Lesaint and P.-A Raviart, *On a finite element method for solving the neutron transport equation*, In: Mathematical Aspects for Finite Elements in Partial Differential Equations (Ed. C. de Boor), Academic Press, New York, 1974.

[14] T.E. Peterson, *A note on the convergence of the discontinuous Galerkin method for a scalar hyperbolic equation*, SIAM J. Num. Anal., **28** (1991) , 133-140.

[15] G. Zhou and R. Rannacher, *Pointwise superconvergence of the streamline diffusion finite-element method*, Numer. Methods Partial Differential Equations, **12** (1996), 123-145

[16] W. Reed, T. Hill, *Triangular mesh methods for the neutron transport equation*, LA-UR-73-479, 1973.

[17] G.R. Richter, *An optimal-order error estimate for the discontinuous Galerkin method*. Math. Comp. **50** (1988), 75-88.