A NOTE ON THE LADYŽENSKAJA-BABUŠKA-BREZZI CONDITION

JOHNNY GUZMÁN, ABNER J. SALGADO, AND FRANCISCO-JAVIER SAYAS

ABSTRACT. The analysis of finite-element-like Galerkin discretization techniques for the stationary Stokes problem relies on the so-called LBB condition. In this work we discuss equivalent formulations of the LBB condition.

1. INTRODUCTION

The well known Ladyženskaja-Babuška-Brezzi (LBB) condition is a particular instance of the so-called discrete inf–sup condition which is necessary and sufficient for the well-posedness of discrete saddle point problems arising from discretization via Galerkin methods. If $\mathbf{X}_h$ denotes the discrete velocity space and M_h the discrete pressure space, then the LBB condition for the Stokes problem states that there is a constant c independent of the discretization parameter h such that

$$c\|q_h\|_{L^2} \leq \sup_{v_h \in \mathbf{X}_h} \frac{\int_\Omega (\nabla \cdot v_h)\, q_h}{\|v_h\|_{\mathbf{H}^1}}, \quad \forall q_h \in M_h. \tag{LBB}$$

The reader is referred to [6] for the basic theory on saddle point problems on Banach spaces and their numerical analysis. Simply put, this condition sets a structural restriction on the discrete spaces so that the continuous level property that the divergence operator is closed and surjective, see [1, 4], is preserved uniformly with respect to the discretization parameter.

In the literature the following condition, which we shall denote the generalized LBB condition, is also assumed

$$c\|\nabla q_h\|_{\mathbf{L}^2} \leq \sup_{v_h \in \mathbf{X}_h} \frac{\int_\Omega (\nabla \cdot v_h)\, q_h}{\|v_h\|_{\mathbf{L}^2}}, \quad \forall q_h \in M_h, \tag{GLBB}$$

here and throughout we assume $M_h \subset H^1(\Omega)$. By properly defining a discrete gradient operator, the case of discontinuous pressure spaces can be analyzed with similar arguments to those that we shall present. Condition (GLBB), for example, was used by Guermond ([8, 9]) to show that approximate solutions to the three-dimensional Navier Stokes equations constructed using the Faedo-Galerkin method converge to a suitable, in the sense of Scheffer, weak solution. On the basis of (GLBB), the same author has also built ([10]) an $\mathbf{H}^s$-approximation theory for the Stokes problem, $0 \leq s \leq 1$. Olshanskiĭ, in [12], under the assumption that the spaces satisfy (GLBB) carries out a multigrid analysis for the Stokes problem. Finally, Mardal et al., [11], use a weighted inf–sup condition to analyze preconditioning techniques for singularly perturbed Stokes problems (see Section 5 below).

Date: Submitted to Journal of Scientific Computing on March 8, 2012.
2010 *Mathematics Subject Classification.* 76D07, 65M60, 65M12.
Key words and phrases. Mixed problems; Stokes problem; inf–sup condition; approximation; LBB.
J. Guzmán is supported by NSF grant DMS-0914596.
AJS is supported by NSF grants CBET-0754983 and DMS-0807811 and an AMS-Simons Travel Grant.

It is not difficult to show that, on quasi-uniform meshes, (GLBB) implies (LBB), see [8]. We include the proof of this result below for completeness. The question that naturally arises is whether the converse holds. Recall that a well-known result of Fortin [2] shows that the inf–sup condition (LBB) is equivalent to the existence of a so-called Fortin projection that is stable in $\mathbf{H}_0^1(\Omega)$. In this work, under the assumption that the mesh is shape regular and quasi-uniform, we will show that (GLBB) is equivalent to the existence of a Fortin projection that has $\mathbf{L}^2$-approximation properties. Moreover, when the domain is such that the solution to the Stokes problem possesses $\mathbf{H}^2$-regularity, we will prove that (GLBB) is in fact equivalent to (LBB), again on quasi-uniform meshes.

The work by Girault and Scott ([7]) must be mentioned when dealing with the construction of Fortin projection operators with $\mathbf{L}^2$-approximation properties. They have constructed such operators for many commonly used inf–sup stable spaces, one notable exception being the lowest order Taylor-Hood element in three dimensions. However, (GLBB) has been shown to hold for the lowest order Taylor-Hood element directly [8]. Our results then can be applied to show that, (GLBB) is satisfied by almost all inf–sup stable finite element spaces, regardless of the smoothness of the domain.

This work is organized as follows. Section 2 introduces the notation and assumptions we shall work with. Condition (GLBB) is discussed in Section 3. In Section 4 we actually show the equivalence of conditions (LBB) and (GLBB), provided the domain is smooth enough. A weighted inf–sup condition related to uniform preconditioning of the time-dependent Stokes problem is presented in Section 5, where we show that (GLBB) implies it. Some concluding remarks are provided in Section 6.

2. Preliminaries

Throughout this work, we will denote by $\Omega \subset \mathbb{R}^d$ with $d = 2$ or 3 an open bounded domain with Lipschitz boundary. If additional smoothness of the domain is needed, it will be specified explicitly. $L^2(\Omega)$, $H^1(\Omega)$ and $H_0^1(\Omega)$ denote, respectively, the usual Lebesgue and Sobolev spaces. We denote by $L^2_{\int=0}(\Omega)$ the set of functions in $L^2(\Omega)$ with mean zero. Vector valued spaces will be denoted by bold characters.

We introduce a conforming triangulation $\mathcal{T}_h$ of Ω which we assume shape-regular and quasi-uniform in the sense of [2]. The size of the cells in the triangulation is characterized by $h > 0$. We introduce finite dimensional spaces $\mathbf{X}_h \subset \mathbf{H}_0^1(\Omega)$ and $M_h \subset L^2_{\int=0}(\Omega) \cap H^1(\Omega)$ which are constructed, for instance using finite elements, on the triangulation $\mathcal{T}_h$. For these spaces, the inverse inequalities

$$\|v_h\|_{\mathbf{H}^1} \leq ch^{-1}\|v_h\|_{\mathbf{L}^2}, \quad \forall v_h \in \mathbf{X}_h, \tag{2.1}$$

and

$$\|q_h\|_{H^1} \leq ch^{-1}\|q_h\|_{L^2}, \quad \forall q_h \in M_h, \tag{2.2}$$

hold, see [2]. Here and in what follows we denote by c will a constant that is independent of h.

We shall denote by $\mathcal{C}_h : \mathbf{H}_0^1(\Omega) \to \mathbf{X}_h$ the so-called Scott-Zhang interpolation operator ([13]) onto the velocity space and we recall that

$$\|v - \mathcal{C}_h v\|_{\mathbf{L}^2} + h\|\mathcal{C}_h v\|_{\mathbf{H}^1} \leq ch\|v\|_{\mathbf{H}^1}, \quad \forall v \in \mathbf{H}_0^1(\Omega). \tag{2.3}$$

and

$$\|v - \mathcal{C}_h v\|_{\mathbf{H}^1} \leq ch\|v\|_{\mathbf{H}^2}, \quad \forall v \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega) \tag{2.4}$$

The Scott-Zhang interpolation operator onto the pressure space $\mathcal{I}_h : L^2_{\int=0}(\Omega) \to M_h$ can be defined analogously and satisfies similar stability and approximation properties. We shall denote by π_h :

$\mathbf{L}^2(\Omega) \to \mathbf{X}_h$ the $\mathbf{L}^2$-projection onto $\mathbf{X}_h$ and by $\Pi_0 : L^2(\Omega) \to L^2(\Omega)$ the L^2-projection operator onto the space of piecewise constant functions, i.e.,

$$\Pi_0 q = \sum_{T\in\mathcal{T}_h} \frac{1}{|T|}\left(\int_T q\right)\chi_T, \quad \forall q \in L^2(\Omega).$$

For one result below we shall require full $\mathbf{H}^2$-regularity of the solution to the Stokes problem:

Assumption 1. *The domain Ω is such that for any $f \in \mathbf{L}^2(\Omega)$, the solution $(\psi,\theta) \in \mathbf{H}^1_0(\Omega) \times L^2_{\int=0}(\Omega)$ to the Stokes problem*

$$\begin{cases} -\Delta\psi + \nabla\theta = f, & \text{in } \Omega,\\ \nabla\cdot\psi = 0, & \text{in } \Omega,\\ \psi = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.5}$$

satisfies the following estimate:

$$\|\psi\|_{\mathbf{H}^2} + \|\theta\|_{H^1} \le c\|f\|_{\mathbf{L}^2}. \tag{2.6}$$

Assumption 1 is known to hold in two and three dimensions ($d = 2, 3$) whenever Ω is convex or of class $\mathcal{C}^{1,1}$, see [3, Theorem 6.3].

By suitably defining a discrete gradient operator acting on the pressure space, the proofs for discontinuous pressure spaces can be carried out with similar arguments.

We introduce the definition of a Fortin projection.

Definition 2.7. An operator $\mathcal{F}_h : \mathbf{H}^1_0(\Omega) \to \mathbf{X}_h$ is called a Fortin projection if $\mathcal{F}_h^2 = \mathcal{F}_h$ and

$$\int_\Omega \nabla\cdot(v - \mathcal{F}_h v)q_h = 0, \quad \forall v \in \mathbf{H}^1_0(\Omega), \quad \forall q_h \in M_h. \tag{2.8}$$

We shall be interested in Fortin projections $\mathcal{F}_h$ that satisfy the condition:

$$\|\mathcal{F}_h v\|_{\mathbf{H}^1} \le c\|v\|_{\mathbf{H}^1}, \quad \forall v \in \mathbf{H}^1_0(\Omega), \tag{FH1}$$

or

$$\|v - \mathcal{F}_h v\|_{\mathbf{L}^2} \le ch\|v\|_{\mathbf{H}^1}, \quad \forall v \in \mathbf{H}^1_0(\Omega). \tag{FL2}$$

Let us remark that the approximation property (FL2) implies $\mathbf{H}^1$-stability.

Lemma 2.9. If an operator $\mathcal{F}_h : \mathbf{H}^1_0(\Omega) \to \mathbf{X}_h$ satisfies (FL2) then it is $\mathbf{H}^1$-stable, i.e., (FH1) is satisfied.

Proof. The proof relies on the stability and approximation properties (2.3) of the Scott-Zhang operator and on the inverse estimate (2.1), for if $v \in \mathbf{H}^1_0(\Omega)$,

$$\begin{aligned} \|\mathcal{F}_h v\|_{\mathbf{H}^1} &\le \|\mathcal{F}_h v - \mathcal{C}_h v\|_{\mathbf{H}^1} + c\|v\|_{\mathbf{H}^1} \le ch^{-1}\|\mathcal{F}_h v - \mathcal{C}_h v\|_{\mathbf{L}^2} + c\|v\|_{\mathbf{H}^1}\\ &\le ch^{-1}\|v - \mathcal{F}_h v\|_{\mathbf{L}^2} + ch^{-1}\|v - \mathcal{C}_h v\|_{\mathbf{L}^2} + c\|v\|_{\mathbf{H}^1}. \end{aligned}$$

Conclude using the $\mathbf{L}^2$-approximation properties of the operators $\mathcal{F}_h$ and $\mathcal{C}_h$. □

Remark 2.10. Girault and Scott, [7], explicitly constructed a Fortin projection that satisfies (FH1) and (FL2) for many commonly used spaces. In fact, they showed that the approximation is local, i.e.,

$$\|\mathcal{F}_h v - v\|_{\mathbf{L}^2(T)} + h_T\|\mathcal{F}_h v - v\|_{\mathbf{H}^1(T)} \le ch_T\|v\|_{\mathbf{H}^1(\mathcal{N}(T))}, \quad \forall v \in \mathbf{H}^1_0(\Omega) \text{ and } \forall T \in \mathcal{T}_h,$$

where $\mathcal{N}(T)$ is a patch containing T. In particular, they have shown the existence of this projection for the Taylor-Hood elements in two dimensions. In three dimensions they proved this result for all the Taylor-Hood elements except the lowest order case.

In this work we shall prove the implications

$$\begin{array}{ccc}
\text{(LBB)} \Longleftrightarrow \exists \mathcal{F}_h \text{ s.t. (2.8) and (FH1)} & & \\
\Uparrow & & \\
\text{(GLBB)} \Longleftrightarrow \exists \mathcal{F}_h \text{ s.t. (2.8) and (FL2)} \Longleftarrow \text{(LBB) and Assumption 1} & &
\end{array}$$

thus showing that, in our setting, all these conditions are indeed equivalent. The top equivalence is well-known, see [2, 6, 5]. The left implication is also known (see [8]), for completeness we show this in Theorem 3.3. The bottom implications, although simple to prove, seem to be new.

3. The Generalized LBB Condition

Let us begin by noticing that the generalized LBB condition (GLBB) is actually a statement about coercivity of the $\mathbf{L}^2$-projection on gradients of functions in the pressure space. Namely, (GLBB) is equivalent to

$$\|\pi_h \nabla q_h\|_{\mathbf{L}^2} \geq c\|\nabla q_h\|_{\mathbf{L}^2}, \quad \forall q_h \in M_h. \tag{3.1}$$

It is well known that (GLBB) implies (LBB). For completeness we present the proof. We begin with a perturbation result.

Lemma 3.2. There exists a constant c independent of h such that, for all $q_h \in M_h$, the following holds:

$$c\|q_h\|_{L^2} \leq \sup_{v_h \in \mathbf{X}_h} \frac{\int_\Omega (\nabla \cdot v_h)\, q_h}{\|\nabla v_h\|_{\mathbf{L}^2}} + h\|\nabla q_h\|_{\mathbf{L}^2}.$$

Proof. The proof relies on the properties (2.3) of the Scott-Zhang interpolation operator $\mathcal{C}_h$,

$$\begin{aligned}
c\|q_h\|_{L^2} &\leq \sup_{v \in \mathbf{H}^1_0(\Omega)} \frac{\int_\Omega (\nabla \cdot v)\ q_h}{\|\nabla v\|_{\mathbf{L}^2}} \leq \sup_{v \in \mathbf{H}^1_0(\Omega)} \frac{\int_\Omega (\nabla \cdot \mathcal{C}_h v)\, q_h}{\|\nabla (\mathcal{C}_h v)\|_{\mathbf{L}^2}} + \sup_{v \in \mathbf{H}^1_0(\Omega)} \frac{\int_\Omega \big(\nabla \cdot (v - \mathcal{C}_h v)\big) q_h}{\|\nabla v\|_{\mathbf{L}^2}} \\
&\leq \sup_{v_h \in \mathbf{X}_h} \frac{\int_\Omega (\nabla \cdot v_h)\, q_h}{\|\nabla v_h\|_{\mathbf{L}^2}} + \sup_{v \in \mathbf{H}^1_0(\Omega)} \frac{\int_\Omega (v - \mathcal{C}_h v) \cdot \nabla q_h}{\|\nabla v\|_{\mathbf{L}^2}},
\end{aligned}$$

conclude using (2.3). □

On the basis of Lemma 3.2 we can readily show that (GLBB) implies (LBB). Again, this result is not new and we only include the proof for completeness.

Theorem 3.3. (GLBB) *implies* (LBB).

Proof. Since we assumed that $M_h \subset L^2_{\int=0}(\Omega) \cap H^1(\Omega)$, the proof is straightforward:

$$\sup_{v_h \in \mathbf{X}_h} \frac{\int_\Omega (\nabla \cdot v_h)\, q_h}{\|\nabla v_h\|_{\mathbf{L}^2}} = \sup_{v_h \in \mathbf{X}_h} \frac{\int_\Omega v_h \cdot \nabla q_h}{\|\nabla v_h\|_{\mathbf{L}^2}} \geq \frac{\int_\Omega \pi_h \nabla q_h \cdot \nabla q_h}{\|\nabla \pi_h \nabla q_h\|_{\mathbf{L}^2}} = \frac{\|\pi_h \nabla q_h\|^2_{\mathbf{L}^2}}{\|\nabla \pi_h \nabla q_h\|_{\mathbf{L}^2}} \geq ch\|\pi_h \nabla q_h\|_{\mathbf{L}^2}$$

where, in the last step, we used the inverse inequality (2.1). This, in conjunction with Lemma 3.2 and the characterization (3.1), implies the result. □

Let us now show that the generalized LBB condition (GLBB) is equivalent to the existence of a Fortin operator satisfying (FL2). We begin with a modification of a classical result.

Lemma 3.4. For all $p \in H^1(\Omega)$ there is $v \in \mathbf{H}^1_0(\Omega)$ such that

$$\nabla \cdot v = p - \Pi_0 p, \qquad v|_{\partial T} = 0 \quad \forall T \in \mathcal{T}_h,$$

and

$$\|v\|_{\mathbf{L}^2} \leq c \left(\sum_{T \in \mathcal{T}_h} h_T^4 \|\nabla p\|^2_{\mathbf{L}^2(T)} \right)^{1/2}.$$

Proof. Let $p \in H^1(\Omega)$ and $T \in \mathcal{T}_h$. Clearly,

$$\int_T p - \Pi_0 p = 0.$$

A classical result ([1, 14, 6, 4]) implies that there is a $v_T \in \mathbf{H}^1_0(T)$ with $\nabla \cdot v_T = p - \Pi_0 p$ in T and

$$\|\nabla v_T\|_{\mathbf{L}^2(T)} \leq c \|p - \Pi_0 p\|_{L^2(T)}. \tag{3.5}$$

Given that the mesh is assumed to be shape regular, by mapping to the reference element it is seen that the constant in the last inequality does not depend on $T \in \mathcal{T}_h$.

Let $v \in \mathbf{H}^1_0(\Omega)$ be defined as $v|_T = v_T$ for all T in $\mathcal{T}_h$. By construction,

$$\nabla \cdot v = p - \Pi_0 p, \quad \text{a.e. in } \Omega.$$

Moreover,

$$\|v\|^2_{\mathbf{L}^2} = \sum_{T \in \mathcal{T}_h} \|v\|^2_{\mathbf{L}^2(T)} \leq c \sum_{T \in \mathcal{T}_h} h_T^2 \|\nabla v\|^2_{\mathbf{L}^2(T)} \leq c \sum_{T \in \mathcal{T}_h} h_T^2 \|p - \Pi_0 p\|^2_{L^2(T)} \leq c \sum_{T \in \mathcal{T}_h} h_T^4 \|\nabla p\|^2_{\mathbf{L}^2(T)}.$$

The first equality is by definition; then we applied the Poincaré-Friedrichs inequality (since $v|_T = v_T \in \mathbf{H}^1_0(T)$); next we used the properties of the function v_T and the approximation properties of the projector Π_0. □

With this result at hand we can prove the following.

Theorem 3.6. *If there exists a Fortin operator $\mathcal{F}_h$ that satisfies* (FL2)*, then* (GLBB) *holds.*

Proof. Let $q_h \in M_h$. Using the properties of the operator Π_0 and the local analogue of the inverse inequality (2.2), we get

$$\|\nabla q_h\|^2_{\mathbf{L}^2} = \sum_{T \in \mathcal{T}_h} \|\nabla (q_h - \Pi_0 q_h)\|^2_{\mathbf{L}^2(T)} \leq \sum_{T \in \mathcal{T}_h} \frac{1}{h_T^2} \|q_h - \Pi_0 q_h\|^2_{\mathbf{L}^2(T)} \leq \frac{c}{h^2} \|q_h - \Pi_0 q_h\|^2_{\mathbf{L}^2}.$$

From Lemma 3.4 we know there exists $v \in \mathbf{H}^1_0(\Omega)$ with $\nabla \cdot v = q_h - \Pi_0 q_h$ and

$$\|v\|_{\mathbf{L}^2} \leq c h^2 \|\nabla q_h\|_{\mathbf{L}^2},$$

hence

$$\|\nabla q_h\|^2_{\mathbf{L}^2} \leq \frac{c}{h^2} \|q_h - \Pi_0 q_h\|^2_{L^2} = \frac{c}{h^2} \int_\Omega (\nabla \cdot v)(q_h - \Pi_0 q_h) = \frac{c}{h^2} \int_\Omega (\nabla \cdot v)\, q_h,$$

where the last inequality follows from integration by parts over each T and using the fact that $v|_{\partial T} = 0$ (see Lemma 3.4).

Using the existence of the operator $\mathcal{F}_h$,

$$\|\nabla q_h\|^2_{\mathbf{L}^2} \leq \frac{c}{h^2} \int_\Omega (\nabla \cdot \mathcal{F}_h v) q_h \leq \left(\sup_{w_h \in \mathbf{X}_h} \frac{\int_\Omega (\nabla \cdot w_h)\, q_h}{\|w_h\|_{\mathbf{L}^2}} \right) \frac{c}{h^2} \|\mathcal{F}_h v\|_{\mathbf{L}^2}.$$

It remains to show that

$$\|\mathcal{F}_h v\|_{\mathbf{L}^2} \leq ch^2 \|\nabla q_h\|_{\mathbf{L}^2}.$$

For this purpose, we use the approximation property (FL2) and Lemma 3.4

$$\|\mathcal{F}_h v\|_{\mathbf{L}^2} \leq \|\mathcal{F}_h v - v\|_{\mathbf{L}^2} + \|v\|_{\mathbf{L}^2} \leq ch\|\nabla v\|_{\mathbf{L}^2} + ch^2\|\nabla q_h\|_{\mathbf{L}^2} \leq ch^2\|\nabla q_h\|_{\mathbf{L}^2},$$

where the last inequality holds because of (3.5). □

The converse of Theorem 3.6 is given in the following.

Theorem 3.7. *If* (GLBB) *holds, then there exists a Fortin projector $\mathcal{F}_h$ that satisfies* (FL2)*.*

Proof. Let $v \in \mathbf{H}_0^1(\Omega)$. Define $(z_h, p_h) \in \mathbf{X}_h \times M_h$ as the solution of

$$\begin{cases} \int_\Omega z_h \cdot w_h - \int_\Omega p_h \nabla\cdot w_h = \int_\Omega v \cdot w_h, & \forall w_h \in \mathbf{X}_h, \\ \int_\Omega q_h \nabla\cdot z_h = \int_\Omega q_h \nabla\cdot v, & \forall q_h \in M_h. \end{cases} \tag{3.8}$$

Notice that (GLBB) provides precisely necessary and sufficient conditions for this problem to have a unique solution.

Define $\mathcal{F}_h v := z_h$ we claim that this is indeed a Fortin projection that satisfies (FL2). By construction, (2.8) holds (see the second equation in (3.8)). To show that this is indeed a projection, assume that $v = v_h \in \mathbf{X}_h$ in (3.8), setting $w_h = z_h - v_h$ we readily obtain that

$$\|z_h - v_h\|_{\mathbf{L}^2}^2 = 0.$$

It remains to show the approximation properties of this operator. We begin by noticing that (GLBB) implies

$$c\|\nabla p_h\|_{\mathbf{L}^2} \leq \sup_{w_h \in \mathbf{X}_h} \frac{\int_\Omega p_h \nabla\cdot w_h}{\|w_h\|_{\mathbf{L}^2}} \leq \sup_{w_h \in \mathbf{X}_h} \frac{\int_\Omega (v - \mathcal{F}_h v)\cdot w_h}{\|w_h\|_{\mathbf{L}^2}} \leq \|v - \mathcal{F}_h v\|_{\mathbf{L}^2}, \tag{3.9}$$

where we used (3.8). To obtain the approximation property (FL2) we use the Scott-Zhang interpolation operator $\mathcal{C}_h$,

$$\begin{aligned} \|\mathcal{F}_h v - v\|_{\mathbf{L}^2}^2 &= \int_\Omega (\mathcal{C}_h v - v)\cdot(\mathcal{F}_h v - v) + \int_\Omega (\mathcal{F}_h v - \mathcal{C}_h v)\cdot(\mathcal{F}_h v - v) \\ &\leq \|\mathcal{C}_h v - v\|_{\mathbf{L}^2}\|\mathcal{F}_h v - v\|_{\mathbf{L}^2} + \int_\Omega (\mathcal{F}_h v - \mathcal{C}_h v)\cdot(\mathcal{F}_h v - v). \end{aligned}$$

We bound the first term using the approximation property (2.3) of $\mathcal{C}_h$. To bound the second term we use problem (3.8) with $w_h = \mathcal{F}_h v - \mathcal{C}_h v$, then

$$\int_\Omega (\mathcal{F}_h v - \mathcal{C}_h v)\cdot(\mathcal{F}_h v - v) = \int_\Omega p_h \nabla\cdot(\mathcal{F}_h v - \mathcal{C}_h v) = \int_\Omega p_h \nabla\cdot(v - \mathcal{C}_h v) = -\int_\Omega \nabla p_h \cdot (v - \mathcal{C}_h v),$$

we conclude applying the Cauchy-Schwarz inequality and using (3.9). □

4. Smooth Domains

Here we show that, provided (LBB) holds and, moreover, the domain Ω is such that Assumption 1 is satisfied, then (FL2) holds and hence (GLBB) holds as well. This is shown in the following.

Theorem 4.1. *Assume the domain Ω is such that the solution to* (2.5) *possesses $\mathbf{H}^2$-elliptic regularity, i.e., Assumption 1 holds. Then* (LBB) *implies that there is a Fortin operator $\mathcal{F}_h$ that satisfies* (FL2).

Proof. Let $v \in \mathbf{H}^1_0(\Omega)$. Define $(z_h, p_h) \in \mathbf{X}_h \times M_h$ as the solution to the discrete Stokes problem

$$
\begin{cases} \int_\Omega \nabla z_h : \nabla w_h - \int_\Omega p_h \nabla\cdot w_h = \int_\Omega \nabla v : \nabla w_h, & \forall w_h \in \mathbf{X}_h, \\ \int_\Omega q_h \nabla\cdot z_h = \int_\Omega q_h \nabla\cdot v, & \forall q_h \in M_h, \end{cases} \tag{4.2}
$$

where, in (4.2), the colon is used to denote the tensor product of matrices. Notice that (LBB) implies that this problem always has a unique solution.

Set $\mathcal{F}_h v := z_h$. Proceeding as in the proof of Theorem 3.7 we see that this is indeed a projection. Moreover, (2.8) holds by construction. It remains to show that (FL2) is satisfied. To this end, analogously to the proof of Theorem 3.7, we notice that (LBB) implies

$$
\|p_h\|_{L^2} \leq c \|\nabla(\mathcal{F}_h v - v)\|_{\mathbf{L}^2}.
$$

We now argue by duality. Let ψ and ϕ solve (2.5) with $f = \mathcal{F}_h v - v$. Assumption (2.6) then implies

$$
\begin{aligned}
\|\mathcal{F}_h v - v\|^2_{\mathbf{L}^2} &= \int_\Omega (\mathcal{F}_h v - v)\cdot(-\Delta\psi + \nabla\theta) \\
&= \int_\Omega \nabla(\mathcal{F}_h v - v) : \nabla(\psi - \mathcal{C}_h\psi) - \int_\Omega (\theta - \mathcal{I}_h\theta)\, \nabla\cdot(\mathcal{F}_h v - v) \\
&\quad + \int_\Omega \nabla(\mathcal{F}_h v - v) : \nabla(\mathcal{C}_h\psi) - \int_\Omega (\mathcal{I}_h\theta)\, \nabla\cdot(\mathcal{F}_h v - v)
\end{aligned}
$$

Notice that since $\mathcal{I}_h\theta \in M_h$, $\int_\Omega (\mathcal{I}_h\theta)\, \nabla\cdot(\mathcal{F}_h v - v) = 0$. Since $\nabla\cdot\psi = 0$, using (4.2), the estimate for p_h, (2.4) and (2.6),

$$
\int_\Omega \nabla(\mathcal{F}_h v - v) : \nabla(\mathcal{C}_h\psi) = \int_\Omega p_h \nabla\cdot(\mathcal{C}_h\psi - \psi) \leq ch\|v - \mathcal{F}_h v\|_{\mathbf{H}^1} \|v - \mathcal{F}_h v\|_{\mathbf{L}^2}.
$$

A direct application of of (2.4), (2.3) and (2.6) allows us to obtain the following estimates:

$$
\int_\Omega (\theta - \mathcal{I}_h\theta)\, \nabla\cdot(\mathcal{F}_h v - v) + \int_\Omega \nabla(\mathcal{F}_h v - v) : \nabla(\psi - \mathcal{C}_h\psi) \leq ch\|\mathcal{F}_h v - v\|_{\mathbf{L}^2} \|v\|_{\mathbf{H}^1}
$$

We conclude using a stability estimate for (4.2)

$$
\|\mathcal{F}_h v - v\|_{\mathbf{L}^2} \leq ch\|\mathcal{F}_h v - v\|_{\mathbf{H}^1} \leq ch\|v\|_{\mathbf{H}^1},
$$

which, given (LBB), is uniform in h. ☐

5. The Weighted LBB condition

In relation to the construction of uniform preconditioners for discretizations of the time dependent Stokes problem, Mardal, Schöberl and Winther, [11], consider the following inf–sup condition,

$$c\|q_h\|_{H^1+\epsilon^{-1}L^2} \le \sup_{v_h\in\mathbf{X}_h} \frac{\int_\Omega \nabla\cdot v_h q_h}{\|v_h\|_{\mathbf{L}^2\cap\epsilon\mathbf{H}^1}}, \quad \forall q_h \in M_h. \tag{5.1}$$

where

$$\|q\|^2_{H^1+\epsilon^{-1}L^2} = \inf_{q_1+q_2=q} \left(\|q_1\|^2_{H^1} + \epsilon^{-2}\|q_2\|^2_{L^2}\right),$$

and

$$\|v\|^2_{\mathbf{L}^2\cap\epsilon\mathbf{H}^1} = \|v\|^2_{\mathbf{L}^2} + \epsilon^2\|v\|^2_{\mathbf{H}^1}.$$

By constructing a Fortin projection operator that is $\mathbf{L}^2$-bounded they have showed, on quasi-uniform meshes, that the inf–sup condition (5.1) holds for the lowest order Taylor-Hood element in two dimension. In addition, they proved the same result, on shape regular meshes, for the mini-element. Here, we show that (5.1) holds if we assume (GLBB). A simple consequence of this result is that, on quasi-uniform meshes, (5.1) holds for any order Taylor-Hood elements in two and three dimensions.

Theorem 5.2. *Let Ω be star shaped with respect to ball. If the spaces $\mathbf{X}_h$ and M_h are such that* (GLBB) *is satisfied, then the inf–sup condition* (5.1) *holds with a constant that does not depend on ϵ or h.*

Proof. We consider two cases: $\epsilon \ge h$ and $\epsilon < h$.

Given that the domain Ω is star shaped with respect to a ball, we can conclude ([11]) that the following continuous inf–sup condition holds,

$$c\|q\|_{H^1+\epsilon^{-1}L^2} \le \sup_{v\in\mathbf{H}^1_0(\Omega)} \frac{\int_\Omega q\,\nabla\cdot v}{\|v\|_{\mathbf{L}^2\cap\epsilon\mathbf{H}^1}}, \quad \forall q \in L^2_{\int=0}(\Omega), \tag{5.3}$$

with a constant c independent of ϵ.

We first assume that $\epsilon \ge h$. Using (5.3) for $q_h \in M_h$ we have,

$$\begin{aligned} c\|q_h\|_{H^1+\epsilon^{-1}L^2} &\le \sup_{v\in\mathbf{H}^1_0(\Omega)} \frac{\int q_h\,\nabla\cdot v}{\|v\|_{\mathbf{L}^2\cap\epsilon\mathbf{H}^1}} = \sup_{v\in\mathbf{H}^1_0(\Omega)} \frac{\int_\Omega q_h\,\nabla\cdot(\mathcal{F}_h v)}{\|\mathcal{F}_h v\|_{\mathbf{L}^2\cap\epsilon\mathbf{H}^1}} \frac{\|\mathcal{F}_h v\|_{\mathbf{L}^2\cap\epsilon\mathbf{H}^1}}{\|v\|_{\mathbf{L}^2\cap\epsilon\mathbf{H}^1}} \\ &\le \sup_{v_h\in\mathbf{X}_h} \frac{\int_\Omega q_h\,\nabla\cdot v_h}{\|v_h\|_{\mathbf{L}^2\cap\epsilon\mathbf{H}^1}} \sup_{v\in\mathbf{H}^1_0(\Omega)} \frac{\|\mathcal{F}_h v\|_{\mathbf{L}^2\cap\epsilon\mathbf{H}^1}}{\|v\|_{\mathbf{L}^2\cap\epsilon\mathbf{H}^1}}, \end{aligned}$$

where we used that, since (GLBB) holds, Theorem 3.7 shows that there exists a Fortin operator $\mathcal{F}_h$ that satisfies (2.8). By Lemma 2.9 and the approximation properties (FL2) of the Fortin operator,

$$\begin{aligned} \|\mathcal{F}_h v\|_{\mathbf{L}^2\cap\epsilon\mathbf{H}^1} &\le c\left(\|\mathcal{F}_h v\|_{\mathbf{L}^2} + \epsilon\|\mathcal{F}_h v\|_{\mathbf{H}^1}\right) \le c\left(\|v\|_{\mathbf{L}^2} + \|v-\mathcal{F}_h v\|_{\mathbf{L}^2} + \epsilon\|v\|_{\mathbf{H}^1}\right) \\ &\le c\left(\|v\|_{\mathbf{L}^2} + (\epsilon+h)\|v\|_{\mathbf{H}^1}\right) \le c\left(\|v\|_{\mathbf{L}^2} + 2\epsilon\|v\|_{\mathbf{H}^1}\right) \le c\|v\|_{\mathbf{L}^2\cap\epsilon\mathbf{H}^1}, \end{aligned}$$

where we used that $h \le \epsilon$.

On the other hand, if $\epsilon < h$ we use $q_1 = q_h$ and $q_2 = 0$ in the definition of the weighted norm for the pressure space. Condition (GLBB) then implies

$$\|q_h\|_{H^1+\epsilon^{-1}L^2} \le c\|\nabla q_h\|_{\mathbf{L}^2} \le c \sup_{v_h\in\mathbf{X}_h} \frac{\int_\Omega q_h\,\nabla\cdot v_h}{\|v_h\|_{\mathbf{L}^2}} \le c \sup_{v_h\in\mathbf{X}_h} \frac{\int_\Omega q_h\,\nabla\cdot v_h}{\|v_h\|_{\mathbf{L}^2\cap\epsilon\mathbf{H}^1}} \sup_{v_h\in\mathbf{X}_h} \frac{\|v_h\|_{\mathbf{L}^2\cap\epsilon\mathbf{H}^1}}{\|v_h\|_{\mathbf{L}^2}}.$$

By the inverse inequality (2.1),

$$\frac{\|v_h\|_{\mathbf{L}^2\cap\epsilon\mathbf{H}^1}}{\|v_h\|_{\mathbf{L}^2}} \leq c\left(1+\epsilon h^{-1}\right).$$

Conclude using that $\epsilon < h$. □

6. Concluding Remarks

There seems to be one main drawback to our methods of proof. Namely, all our results rely heavily on the fact that we have a quasi-uniform mesh. However, at the present moment we do not know whether this condition can be removed. Finally, it will be interesting to see if (LBB) is in fact equivalent to (GLBB) on domains that do not satisfy the regularity assumption (2.6) (e.g. non convex polyhedral domains).

On the other hand, it seems to us that condition (GLBB) must be regarded as the most important one. Our results show that, under the sole assumption that the mesh is quasi-uniform, this condition implies the classical condition (LBB) (Theorem 3.3). Moreover, as shown in Theorem 5.2, this condition implies the weighted inf–sup condition (5.1) on quasi-uniform meshes.

References

[1] M.E. Bogovskiĭ. Solutions of some problems of vector analysis, associated with the operators div and grad. In *Theory of cubature formulas and the application of functional analysis to problems of mathematical physics*, volume 1980 of *Trudy Sem. S. L. Soboleva, No. 1*, pages 5–40, 149. Akad. Nauk SSSR Sibirsk. Otdel. Inst. Mat., Novosibirsk, Russia, 1980.

[2] F. Brezzi and M. Fortin. *Mixed and Hybrid Finite Element Methods*. Springer-Verlag, New York, NY, 1991.

[3] M. Dauge. Stationary Stokes and Navier-Stokes systems on two- or three-dimensional domains with corners. I. Linearized equations. *SIAM J. Math. Anal.*, 20(1):74–97, 1989.

[4] R.G. Durán and M.A. Muschietti. An explicit right inverse of the divergence operator which is continuous in weighted norms. *Studia Math.*, 148(3):207–219, 2001.

[5] A. Ern and J.-L. Guermond. *Theory and practice of finite elements*, volume 159 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2004.

[6] V. Girault and P.-A. Raviart. *Finite Element Methods for Navier-Stokes Equations. Theory and Algorithms*. Springer Series in Computational Mathematics. Springer-Verlag, Berlin, Germany, 1986.

[7] V. Girault and L.R. Scott. A quasi-local interpolation operator preserving the discrete divergence. *Calcolo*, 40(1):1–19, 2003.

[8] J.-L. Guermond. Finite-element-based Faedo-Galerkin weak solutions to the Navier-Stokes equations in the three-dimensional torus are suitable. *J. Math. Pures Appl. (9)*, 85(3):451–464, 2006.

[9] J.-L. Guermond. Faedo-Galerkin weak solutions of the Navier-Stokes equations with Dirichlet boundary conditions are suitable. *J. Math. Pures Appl. (9)*, 88(1):87–106, 2007.

[10] J.-L. Guermond. The LBB condition in fractional Sobolev spaces and applications. *IMA J. Numer. Anal.*, 29(3):790–805, 2009.

[11] K.-A. Mardal, J. Schöberl, and R. Winther. A uniform inf–sup condition with applications to preconditioning. arXiv:1201.1513v1, 2012.

[12] M.A. Olshanskiĭ. Multigrid analysis for the time dependent Stokes problem. *Math. Comp.*, 81(277):57–79, 2012.

[13] L.R. Scott and S. Zhang. Finite element interpolation of nonsmooth functions satisfying boundary conditions. *Math. Comp.*, 54(190):483–493, 1990.

[14] R. Temam. *Navier-Stokes equations*. AMS Chelsea Publishing, Providence, RI, 2001. Reprint of the 1984 edition.

(J. Guzmán) Division of Applied Mathematics, Brown University, Providence, RI 02912
E-mail address: johnny_guzman@brown.edu

(A.J. Salgado) Department of Mathematics, University of Maryland, College Park, MD 20742
E-mail address: abnersg@math.umd.edu

(F.-J. Sayas) Department of Mathematical Sciences, University of Delaware, Newark, DE 19716
E-mail address: fjsayas@math.udel.edu