OPTIMAL CONVERGENCE OF THE ORIGINAL DG METHOD ON SPECIAL MESHES FOR VARIABLE TRANSPORT VELOCITY

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Abstract. We prove optimal convergence rates for the approximation provided by the original discontinuous Galerkin method for the transport-reaction problem. This is achieved in any dimension on meshes related in a suitable way to the possibly variable velocity carrying out the transport. Thus, if the method uses polynomials of degree k, the L^2-norm of the error is of order $k+1$. Moreover, we also show that, by means of an element-by-element postprocessing, a new approximate flux can be obtained which superconverges with order $k+1$.

Key words. discontinuous Galerkin methods, convection-reaction equation, error estimates

AMS subject classifications. 65N30, 65M60

1. Introduction. We prove *optimal* convergence properties of the original discontinuous Galerkin (DG) [12, 9] method for the convection-reaction problem

$$\boldsymbol{\beta} \cdot \nabla u + c\, u = f \quad \text{in } \Omega, \tag{1.1a}$$

$$u = g \quad \text{on } \Gamma^-. \tag{1.1b}$$

Here $\Omega \subset R^d$ is a bounded polyhedral domain, $\Gamma^- := \{x \in \partial\Omega : \boldsymbol{\beta} \cdot \boldsymbol{n}(x) < 0\}$, and $\boldsymbol{n}(x)$ is the outward unit normal at the point $x \in \partial\Omega$. The functions f and g are smooth, c is a bounded function and, more important, $\boldsymbol{\beta}$ is a *smooth, divergence-free* function.

Let us describe our result. It is well known that, for *constant* transport velocities $\boldsymbol{\beta}$, the DG method for the above problem provides approximations converging with *sub-optimal* rates on general meshes. This was shown for the first time in [10] for a particular type of two-dimensional mesh. The class of meshes for which this sub-optimal rate of convergence can be demonstrated was recently extended in [13] to include some two-dimensional smooth, periodically varying meshes. On the other hand, in a diametrically opposed effort, a class of special multi-dimensional meshes for which the *optimal* order of convergence is actually achieved was recently uncovered in [3]. Here, we continue this effort and prove that a similar result also holds for *variable* transport velocities $\boldsymbol{\beta}$.

Indeed, we show that, for a *special* class of triangulations $\mathcal{T}_h$,

$$\| u - u_h \|_{L^2(\mathcal{T}_h)} + \| \partial_{\boldsymbol{\beta}} u - \partial_{\boldsymbol{\beta},h} u_h \|_{L^2(\mathcal{T}_h)} \le C\, h^{k+1},$$

where u_h is the approximation given by the DG method using polynomials of degree k, and $\partial_{\boldsymbol{\beta},h} u_h$ is an approximation to $\partial_{\boldsymbol{\beta}} u = \boldsymbol{\beta} \cdot \nabla u$ obtained by using an element-by-element postprocessing of u_h. The constant C in the above estimate only depends

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on the H^{k+1}-seminorm of the exact solution u as well as on the regularity of the transport velocity $\boldsymbol{\beta}$.

The special triangulations $\mathcal{T}_h$ for which the above result holds are strongly related to the transport velocity as follows. They are made of simplexes K satisfying the following *flow* conditions:

$$\text{Each simplex } K \text{ has a unique } \textit{outflow} \text{ face with respect to } \boldsymbol{\beta},\ e_K^+. \tag{1.2a}$$

$$\begin{array}{l}\text{Each interior face } e_K^+ \text{ is included in an } \textit{inflow} \text{ face with respect to } \boldsymbol{\beta}\\ \text{of another simplex.}\end{array} \tag{1.2b}$$

On each face $e \neq e_K^+$ of the simplex K which is not an inflow face, we have

$$\frac{1}{|e|}|\,\langle \boldsymbol{\beta}\cdot\boldsymbol{n}, 1\rangle_e\,| \leq C_{\boldsymbol{\beta}}\, h_K, \tag{1.2c}$$

for some constant $C_{\boldsymbol{\beta}}$, where $h_K = diam(K)$.

As usual, we say that the face e of the simplex K is an outflow (inflow) face with respect to $\boldsymbol{\beta}$ if $\boldsymbol{\beta}\cdot\boldsymbol{n}_K|_e >\ (<)\ 0$. We say that a face is interior if it is not included in $\partial\Omega$.

Let us briefly discuss these conditions. The first two conditions define the special triangulations in the case in which the velocity $\boldsymbol{\beta}$ is constant; they were introduced in [3]. Note that in this case, the third condition is automatically satisfied with $C_{\boldsymbol{\beta}} = 0$ since $\boldsymbol{\beta}\cdot\boldsymbol{n} = 0$ on any face e which is not e_K^+ or an inflow face. When the velocity is not constant, our analysis shows that it is not necessary to request such a stringent condition. Instead, it is enough to require that the *average* on e of the normal component of the transport velocity be *small* enough. This condition is not difficult to satisfy. It holds, for example, when $\boldsymbol{\beta}\cdot\boldsymbol{n}(\boldsymbol{x}_0) = 0$ for *any* particular point $\boldsymbol{x}_0$ of the face e. Indeed, since

$$\langle \boldsymbol{\beta}\cdot\boldsymbol{n}, 1\rangle_e = \langle \boldsymbol{\beta}\cdot\boldsymbol{n} - \boldsymbol{\beta}\cdot\boldsymbol{n}(\boldsymbol{x}_0), 1\rangle_e,$$

we have that

$$\frac{1}{|e|}|\,\langle \boldsymbol{\beta}\cdot\boldsymbol{n}, 1\rangle_e\,| \leq |\,\boldsymbol{\beta}\,|_{\boldsymbol{W}^{1,\infty}(e)}\, h_K,$$

and the third condition is satisfied with $C_{\boldsymbol{\beta}} := |\,\boldsymbol{\beta}\,|_{\boldsymbol{W}^{1,\infty}(\Omega)}$.

Of course, the families of triangulations we consider also satisfy the classical assumption of shape regularity, see [2]. Thus, there is a constant $\sigma > 0$ such that

$$\text{For each simplex } K \in \mathcal{T}_h : h_K/\rho_K \geq \sigma, \tag{1.3}$$

where h_K denotes the diameter of the simplex K and ρ_K the diameter of the biggest ball included in K.

Finally, let us briefly discuss a technicality arising from the fact that the transport velocity $\boldsymbol{\beta}$ is variable. In order to prove the L^2-stability of the DG method in this case, we assume that

$$\frac{1}{2}\,\boldsymbol{\beta}(\boldsymbol{x})\cdot\boldsymbol{\beta}(\boldsymbol{x}) + c(\boldsymbol{x}) \geq \gamma > 0 \quad \text{for all } \boldsymbol{x}\in\Omega, \tag{1.4}$$

for a fixed $\gamma > 0$. A somewhat stronger assumption of this type was used in [6].

The paper is organized as follows. In Section 2 we state and prove our main results and, in Section 3, we carry out numerical experiments validating them. We end in Section 4 with some concluding remarks.

2. The main results.

2.1. The DG method. Let us introduce the original DG method for our problem (1.1). Suppose we have a family of triangulations $\{\mathcal{T}_h\}$ of Ω satisfying the flow conditions (1.2). To each triangulation $\mathcal{T}_h$, we associate the number $h = \sup_{K\in\mathcal{T}_h} h_K$, where $h_K = diam(K)$, and the finite-dimensional space V_h^k which is composed of functions that are polynomials of degree at most k on each simplex $K \in \mathcal{T}_h$. Then, the DG approximation $u_h \in V_h$ of the solution of (1.1) is defined as the solution of

$$B(u_h, v_h) =(f, v_h)_{\mathcal{T}_h} - \langle g, v_h\, \boldsymbol{\beta}\cdot\boldsymbol{n}\rangle_{\Gamma^-}, \quad \text{for all } v_h \in V_h, \tag{2.1a}$$

where

$$B(w, v) := -\,(w, \partial_{\boldsymbol{\beta}} v)_{\mathcal{T}_h} + \langle \widehat{w}, v\, \boldsymbol{\beta}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h\setminus\Gamma^-} + (c\, w, v)_{\mathcal{T}_h}, \tag{2.1b}$$

for any w, v in $H^1(\mathcal{T}_h)$. Here, the numerical trace of a function w on a point $z \in \partial K$ for a simplex $K \in \mathcal{T}_h$ is given by

$$\widehat{w} :=w^-, \tag{2.1c}$$

where $w^\pm(z) = \lim_{\delta\downarrow 0} w(z \pm \delta\boldsymbol{\beta}(z))$ where $z \in e$. We are using the notation

$$(\boldsymbol{\sigma}, \boldsymbol{v})_{\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h} \int_K \boldsymbol{\sigma}(x)\cdot\boldsymbol{v}(x)\, dx, \qquad (\zeta, \omega)_{\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h} \int_K \zeta(x)\, \omega(x)\, dx,$$
$$\langle \zeta, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h} \int_{\partial K} \zeta(\gamma)\, \boldsymbol{v}(\gamma)\cdot\boldsymbol{n}\, d\gamma,$$

for any functions $\boldsymbol{\sigma}, \boldsymbol{v}$ in $\boldsymbol{H}^1(\mathcal{T}_h) := [H^1(\mathcal{T}_h)]^d$ and ζ, ω in $H^1(\mathcal{T}_h)$. The outward normal unit vector to ∂K is denoted by $\boldsymbol{n}$.

Before we present our main result we need to state the following stability result for the DG method.

LEMMA 2.1. *Suppose that condition* (1.4) *holds and assume that w_h satisfies $B(w_h, v) = F(v)$ for all $v \in V_h$ where F is a linear form. Then, for h sufficiently small*

$$\|w_h\|_{L^2(\mathcal{T}_h)} \le C_s \max_{v\in V_h} \frac{|F(v)|}{\|v\|_{L^2(\mathcal{T}_h)}},$$

where C_s depends on the regularity of $\boldsymbol{\beta}$.

Of course, stability results like this are standard. For constant $\boldsymbol{\beta}$ the proof is in [7]. The case of variable $\boldsymbol{\beta}$, but imposing a stronger condition on the coefficients than (1.4), is contained in [6]. We sketch the proof of this Lemma in the appendix.

2.2. The approximation of u. Our main result is the following.

THEOREM 2.2. *Assume that the positivity condition* (1.4) *holds. Assume also that the triangulation $\mathcal{T}_h$ satisfies the flow conditions* (1.2) *and the shape-regularity condition* (1.3). *Then, if h is small enough, we have*

$$\|\, u - u_h\,\|_{L^2(\mathcal{T}_h)} \le \mathsf{C}\, h^{k+1},$$

where $\mathsf{C} = C\,(1 + |\,\boldsymbol{\beta}\,|_{\boldsymbol{W}^{1,\infty}(\Omega)} + C_{\boldsymbol{\beta}})\,|\,u\,|_{H^{k+1}(\mathcal{T}_h)}$ and C depends on C_s.

Note that, just as the corresponding result for the case of constant velocity in [3], this result is optimal in both the order of convergence as well as in the regularity of the exact solution. To prove it, we follow [3]. We proceed in several steps.

Step 1: The Projection $\mathbb{P}$. We begin by recalling the definition of the projection $\mathbb{P}$, defined on triangulations $\mathcal{T}_h$ satisfying the flow condition (1.2a); see [4, 3]. The function $\mathbb{P}u \in V_h$ restricted to $K \in \mathcal{T}_h$ is given by

$$(\mathbb{P}u - u, v)_K = 0, \qquad \text{for all } v \in \mathcal{P}^{k-1}(K) \text{ if } k > 0, \tag{2.2a}$$
$$\langle \mathbb{P}u - u, w\rangle_{e_K^+} = 0, \qquad \text{for all } w \in \mathcal{P}^{k}(e_K^+), \tag{2.2b}$$

where $\mathcal{P}^\ell(D)$ stands for the space of polynomials of total degree at most ℓ defined on the set D. The projection $\mathbb{P}$ has the following approximation property [3]

$$\|\mathbb{P}u - u\|_{L^2(K)} \le C h^{k+1} |\, u\,|_{H^{k+1}(K)}. \tag{2.3}$$

Since we have that

$$\|u - u_h\|_{L^2(\mathcal{T}_h)} \le \|u - \mathbb{P}u\|_{L^2(\mathcal{T}_h)} + \|\,\mathbb{E}\,\|_{L^2(\mathcal{T}_h)},$$

where $\mathbb{E} = u_h - \mathbb{P}u$, if we assume the shape-regularity condition (1.3) on the triangulation $\mathcal{T}_h$, by the approximation property of the projection $\mathbb{P}$ (2.3), we have that

$$\|\,u - u_h\,\|_{L^2(\mathcal{T}_h)} \le C\,|\,u\,|_{H^{k+1}(\mathcal{T}_h)}\, h^{k+1} + \|\,\mathbb{E}\,\|_{L^2(\mathcal{T}_h)},$$

whenever $u \in H^{k+1}(\mathcal{T}_h)$. It remains to estimate the projection of the error $\mathbb{E}$.

Step 2: Estimate of $\mathbb{E}$. By the error equation

$$B(u - u_h, v) = 0 \qquad \text{for all } v \in V_h,$$

and so, for all $v \in V_h$, we have that

$$B(\mathbb{E}, v) = B(u - \mathbb{P}u, v) = \sum_{i=1}^{3} T_i(v),$$

where, by definition of the bilinear form $B(\cdot,\cdot)$, (2.1b),

$$\begin{aligned} T_1(v) &:= -\,(u - \mathbb{P}u, \boldsymbol{\beta}\cdot\nabla v)_{\mathcal{T}_h},\\ T_2(v) &:= \langle u - \widehat{\mathbb{P}u}, v\,\boldsymbol{\beta}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h\setminus\Gamma^-},\\ T_3(v) &:= (c\,(u - \mathbb{P}u), v)_{\mathcal{T}_h}. \end{aligned}$$

Thus, by Lemma 2.1, we obtain

$$\|\mathbb{E}\|_{L^2(\mathcal{T}_h)} \le C_s \sum_{i=1}^{3} \sup_{v\in V_h} \frac{|\,T_i(v)\,|}{\|\,v\,\|_{L^2(\mathcal{T}_h)}},$$

for h small enough.

It remains to estimate the linear forms $T_i(v)$, $i = 1, 2, 3$. To do that, we are going to use the auxiliary vector-valued function $\boldsymbol{\beta}^o$ defined as follows. On each simplex $K \in \mathcal{T}_h$, $\boldsymbol{\beta}^o$ is a constant vector defined by

$$\langle(\boldsymbol{\beta} - \boldsymbol{\beta}^o)\cdot\boldsymbol{n}, 1\rangle_e = 0 \quad \text{for all faces } e \text{ of } K.$$

The vector $\boldsymbol{\beta}^o$ is nothing but the lowest-order Raviart-Thomas projection of $\boldsymbol{\beta}$. Note that, since $\nabla\cdot\boldsymbol{\beta}=0$, $\boldsymbol{\beta}^o$ is constant on K.

Step 3: Estimate of T_1. Let us estimate $T_1(v)$. We have

$$T_1(v) = -\,(u-\mathbb{P}u, (\boldsymbol{\beta}-\boldsymbol{\beta}^o)\cdot\nabla v)_{\mathcal{T}_h},$$

by the definition of the projection $\mathbb{P}$, (2.2a), and by the definition of $\boldsymbol{\beta}^o$. This implies that

$$|\,T_1(v)\,| \le \sum_{K\in\mathcal{T}_h} \|\,u-\mathbb{P}u\,\|_{L^2(K)}\,\|\,\boldsymbol{\beta}-\boldsymbol{\beta}^o\,\|_{\boldsymbol{L}^\infty(K)}\,\|\,\nabla v\,\|_{L^2(K)},$$

and, by a standard inverse inequality, that

$$\begin{aligned}
|\,T_1(v)\,| \le& \sum_{K\in\mathcal{T}_h} \|\,u-\mathbb{P}u\,\|_{L^2(K)}\,\|\,\boldsymbol{\beta}-\boldsymbol{\beta}^o\,\|_{\boldsymbol{L}^\infty(K)}\,C_K\,h_K^{-1}\|\,v\,\|_{L^2(K)}\\
\le& C\max_{K\in\mathcal{T}_h}\{\,h_K^{-1}\,\|\,\boldsymbol{\beta}-\boldsymbol{\beta}^o\,\|_{\boldsymbol{L}^\infty(K)}\}\,\|\,u-\mathbb{P}u\,\|_{L^2(\mathcal{T}_h)}\,\|\,v\,\|_{L^2(\mathcal{T}_h)}\\
\le& C\,|\,\boldsymbol{\beta}\,|_{\boldsymbol{W}^{1,\infty}(\mathcal{T}_h)}\,\|\,u-\mathbb{P}u\,\|_{L^2(\mathcal{T}_h)}\,\|\,v\,\|_{L^2(\mathcal{T}_h)}\\
\le& C\,|\,\boldsymbol{\beta}\,|_{\boldsymbol{W}^{1,\infty}(\mathcal{T}_h)}\,h^{k+1}\,|\,u\,|_{H^{k+1}(\mathcal{T}_h)}\,\|\,v\,\|_{L^2(\mathcal{T}_h)},
\end{aligned}$$

by the approximation properties of $\boldsymbol{\beta}^0$, the shape-regularity assumption on the mesh (1.3), and by the approximation properties of $\mathbb{P}$ [3].

Step 4: Estimate of T_2. Let us now estimate $T_2(v)$. We begin by writing $T_2(v)$ as

$$T_2(v) = U_1(v) + U_2(v),$$

where

$$\begin{aligned}
U_1(v) =& \sum_{K\in\mathcal{T}_h} \langle u-\widehat{\mathbb{P}u}, v\,(\boldsymbol{\beta}-\boldsymbol{\beta}^o)\cdot\boldsymbol{n}\rangle_{\partial K\setminus\Gamma^-}\\
U_2(v) =& \sum_{K\in\mathcal{T}_h} \langle u-\widehat{\mathbb{P}u}, v\,\boldsymbol{\beta}^o\cdot\boldsymbol{n}\rangle_{\partial K\setminus\Gamma^-}.
\end{aligned}$$

Next, let us estimate $U_1(v)$. We have

$$\begin{aligned}
|\,U_1(v)\,| \le& \sum_{K\in\mathcal{T}_h} \|\,u-\widehat{\mathbb{P}}u\,\|_{L^2(\partial K)}\,\|\,v\,\|_{L^2(\partial K)}\,\|\,(\boldsymbol{\beta}-\boldsymbol{\beta}^o)\cdot\boldsymbol{n}\,\|_{L^\infty(\partial K)}\\
\le& \sum_{K\in\mathcal{T}_h} C_K\,h_K^{k+1/2}|\,u\,|_{H^{k+1}(K)}\,\|\,v\,\|_{L^2(\partial K)}\,h_K\,|\,\boldsymbol{\beta}\,|_{\boldsymbol{W}^{1,\infty}(\partial K)},
\end{aligned}$$

by the approximation properties of $\mathbb{P}$ and $\boldsymbol{\beta}^o$. Then, after applying a simple inverse inequality, we get

$$|\,U_1(v)\,| \le C\,|\,\boldsymbol{\beta}\,|_{\boldsymbol{W}^{1,\infty}(\Omega)}\,h^{k+1}\,|\,u\,|_{H^{k+1}(\mathcal{T}_h)}\,\|\,v\,\|_{L^2(\mathcal{T}_h)}.$$

Let us estimate $U_2(v)$. To do that, we rewrite $U_2(v)$ as

$$U_2(v) = S_1(v) + S_2(v)$$

where

$$\begin{aligned} S_1(v) &= \sum_{K\in\mathcal{T}_h} \langle u - \widehat{\mathbb{P}u}, v\,\boldsymbol{\beta}^o\cdot\boldsymbol{n}\rangle_{(\partial K\setminus e_K^o)\setminus\Gamma^-},\\ S_2(v) &= \sum_{K\in\mathcal{T}_h} \langle u - \widehat{\mathbb{P}u}, v\,\boldsymbol{\beta}^o\cdot\boldsymbol{n}\rangle_{e_K^o\setminus\Gamma^-}. \end{aligned}$$

Where e_K^o is the collection of faces of K that are neither inflow nor outflow faces. By the first flow condition on the mesh (1.2a), it is easily follows that

$$S_1(v) = \sum_{K\in\mathcal{T}_h} \langle u - \widehat{\mathbb{P}u}, (v^- - v^+)\,\boldsymbol{\beta}^o\cdot\boldsymbol{n}\rangle_{e_K^+},$$

and, by the definition of the numerical trace $\widehat{\mathbb{P}\,u}$, (2.1c), that

$$S_1(v) = \sum_{K\in\mathcal{T}_h} \langle u - \mathbb{P}u, (v^- - v^+)\,\boldsymbol{\beta}^o\cdot\boldsymbol{n}\rangle_{e_K^+}.$$

But, since, by the second flow condition on the mesh (1.2b), $(v^- - v^+)\boldsymbol{\beta}^o\cdot\boldsymbol{n}|_{e_K^+} \in \mathcal{P}^k(e_K^+)$, we can conclude that, by the definition of the projection $\mathbb{P}$, (2.2b),

$$S_1(v) = 0.$$

It remains to estimate $S_2(v)$. We have

$$|\,S_2(v)\,| \le \sum_{K\in\mathcal{T}_h} \|\,u - \widehat{\mathbb{P}u}\,\|_{L^2(\partial K)}\,\|\,v\,\|_{L^2(\partial K)}\,\|\,\boldsymbol{\beta}^o\cdot\boldsymbol{n}\,\|_{L^\infty(e_K^o\setminus\Gamma^-)},$$

and by the third flow condition on the mesh (1.2c),

$$\begin{aligned} |\,S_2(v)\,| &\le \sum_{K\in\mathcal{T}_h} \|\,u - \widehat{\mathbb{P}u}\,\|_{L^2(\partial K)}\,\|\,v\,\|_{L^2(\partial K)}\,C_{\boldsymbol{\beta}}\,h_K\\ &\le C\,C_{\boldsymbol{\beta}}\,h^{k+1}\,|\,u\,|_{H^{k+1}(\mathcal{T}_h)}\,\|\,v\,\|_{L^2(\mathcal{T}_h)}, \end{aligned}$$

proceeding as before. This implies that

$$|\,T_2(v)\,| \le C(C_{\boldsymbol{\beta}} + |\,\boldsymbol{\beta}\,|_{\boldsymbol{W}^{1,\infty}(\Omega)})\,|\,u\,|_{H^{k+1}(\mathcal{T}_h)}\,\|\,v\,\|_{L^2(\mathcal{T}_h)}.$$

Step 5: Estimate of T_3. A simple application of the Cauchy-Schwarz inequality gives

$$\begin{aligned} |\,T_3(v)\,| &\le \|\,c\,(u - \mathbb{P}u)\,\|\,\|\,v\,\|_{L^2(\mathcal{T}_h)}\\ &\le C\,|\,u\,|_{H^{k+1}(\mathcal{T}_h)}\,h^{k+1}\,\|\,v\,\|_{L^2(\mathcal{T}_h)}, \end{aligned}$$

by the approximation properties of $\mathbb{P}$.

Step 6: Conclusion. We have

$$\begin{aligned}\| u - u_h \|_{L^2(\mathcal{T}_h)} &\leq C\, | u |_{H^{k+1}(\mathcal{T}_h)}\, h^{k+1} + \| \mathbb{E} \|_{L^2(\mathcal{T}_h)}, && \text{by Step 1,}\\ &\leq C\, | u |_{H^{k+1}(\mathcal{T}_h)}\, h^{k+1} + C_s \sum_{i=1}^{3} \sup_{v \in V_h} \frac{| T_i(v) |}{\| v \|_{L^2(\mathcal{T}_h)}}, && \text{by Step 2,}\\ &\leq C\,(1 + | \boldsymbol{\beta} |_{\boldsymbol{W}^{1,\infty}(\Omega)} + C_{\boldsymbol{\beta}})\, h^{k+1}\, | u |_{H^{k+1}(\mathcal{T}_h)},\end{aligned}$$

by Steps 3, 4 and 5. This completes the proof of Theorem 2.2.

2.3. Post-processing: The approximation to $\partial_{\boldsymbol{\beta}} u$. Next we postprocess u_h to get a superconvergent approximation of $\partial_{\boldsymbol{\beta}} u$. We follow [3] and, for each simplex K we define $\boldsymbol{q}_h \in \boldsymbol{\mathcal{P}}^k(K) + \boldsymbol{x}\, \mathcal{P}^k(K)$ to be the solution of

$$(\boldsymbol{q}_h - \boldsymbol{\beta} u_h, \boldsymbol{v})_K = 0, \qquad \text{for all } \boldsymbol{v} \in \boldsymbol{\mathcal{P}}^{k-1}(K) \text{ if } k > 0, \tag{2.4a}$$

$$\langle (\boldsymbol{q}_h - \boldsymbol{\beta}\lambda_h) \cdot \boldsymbol{n}, w \rangle_e = 0, \qquad \text{for all } w \in \mathcal{P}^k(e), \text{ for all faces } e \text{ of } K, \tag{2.4b}$$

where $\lambda_h = \mathsf{P}_\partial g$ on Γ^- and $\lambda_h = \widehat{u}_h$ otherwise; here $\boldsymbol{\mathcal{P}}^k(K) := [\mathcal{P}^k(K)]^d$. The existence and uniqueness of $\boldsymbol{q}_h$ is well known; see, for example, [1]. We then define

$$\partial_{\boldsymbol{\beta},h} u_h := \nabla \cdot \boldsymbol{q}_h \qquad \text{in } \mathcal{T}_h.$$

We can now state the error estimate between $\partial_{\boldsymbol{\beta},h} u_h$ and $\partial_{\boldsymbol{\beta}} u$.

THEOREM 2.3. *Assume that the positivity condition* (1.4) *is satisfied. Assume also that the triangulation $\mathcal{T}_h$ satisfies the flow conditions* (1.2) *and the shape-regularity condition* (1.3). *Then, if h is small enough, we have*

$$\|\partial_{\boldsymbol{\beta},h} u_h - \mathsf{P}(\partial_{\boldsymbol{\beta}} u)\|_{L^2(\mathcal{T}_h)} \leq C\, \mathsf{C}\ h^{k+1}.$$

Proof. Following exactly the same argument use in [3], we can show that, if $\mathcal{T}_h$ is an *arbitrary*, shape-regular triangulation of Ω, we have

$$\|\partial_{\boldsymbol{\beta},h} u_h - \mathsf{P}(\partial_{\boldsymbol{\beta}} u)\|_{L^2(\mathcal{T}_h)} \leq C\, \| c\,(u - u_h) \|_{L^2(\mathcal{T}_h)}.$$

Then, for the special triangulation $\mathcal{T}_h$ under consideration, we get that

$$\|\partial_{\boldsymbol{\beta},h} u_h - \mathsf{P}(\partial_{\boldsymbol{\beta}} u)\|_{L^2(\mathcal{T}_h)} \leq C\, \mathsf{C}\, h^{k+1},$$

by Theorem 2.2. This completes the proof. □

3. Numerical Results. In this section we present two numerical experiments which validate our theoretical results. Let us motivate them.

Physically, the model equation (1.1) describes transport-reaction phenomena including neutron transport and radiative transfer. We take two examples from geometrical optics and quantum mechanics where the model equation is the so-called Liouville equation.

In geometrical optics for wave propagation, we need to solve Liouville equations efficiently and accurately to be able to compute the multivalued traveltime and the amplitude, see [11, 5, 8]. In the case of acoustic waves, the Hamiltonian defining the Liouville operator has the form $H(x,y) = xy$ and the resulting velocity $\boldsymbol{\beta}(x,y) = (H_y, -H_x) = (x, -y)$. This is our first test problem. In semi-classical approximation for quantum mechanics, we also need to solve the Liouville equation to compute multivalued phases and densities to construct semi-classical solutions to the Schrödinger equation. In the case of harmonic oscillators, the Hamiltonian defining the Liouville operator has the form $H(x,y) = \frac{1}{2}(x^2 + y^2)$ and the resulting velocity is $\boldsymbol{\beta}(x,y) = (H_y, -H_x) = (y, -x)$. This is our second test problem. Therefore, the results presented here lay down theoretical foundation for developing fast and accurate algorithms for solving Liouville equations.

In both numerical experiments, the domain is $\Omega = (1,2) \times (1,2)$, and the reaction coefficient is $c(x,y) = y$. We choose the right-hand side f so that the solution is $u(x,y) = (x + 1/2)^3 \sin(y)$.

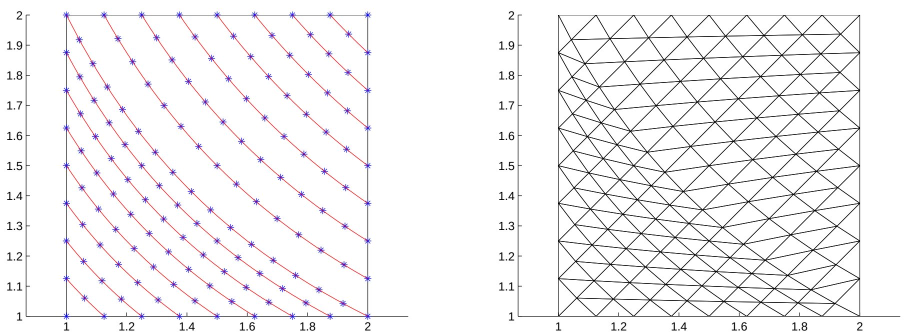

Fig. 3.1. *Meshes satisfying the flow condition with respect to $\boldsymbol{\beta} = (x, -y)$: The streamlines of β and the nodes of the mesh (left) and the actual mesh ($\ell = 3$) (right).*

Table 3.1
History of convergence for the acoustic waves problem.

	mesh	$\|u - u_h\|_{L^2(\Omega)}$		$\|\partial_{\boldsymbol{\beta}} u - \partial_{\boldsymbol{\beta},h} u_h\|_{L^2(\Omega)}$	
k	ℓ	error	order	error	order
	1	.15e+1	-	.37e+1	-
	2	.81e-0	0.90	.19e+1	0.93
0	3	.42e-0	0.96	.98e-0	0.96
	4	.21e-0	0.98	.50e-0	0.98
	5	.11e-0	0.99	.25e-0	0.99
	1	.86e-1	-	.23e-0	-
	2	.23e-1	1.90	.62e-1	1.88
1	3	.59e-2	1.96	.16e-1	1.95
	4	.15e-2	1.98	.41e-2	1.98
	5	.38e-3	1.99	.10e-2	1.99
	1	.26e-2	-	.49e-2	-
	2	.31e-3	3.08	.66e-3	2.89
2	3	.37e-4	3.08	.85e-4	2.96
	4	.44e-5	3.05	.11e-4	2.98
	5	.54e-6	3.03	.15e-5	2.89

In the first numerical experiment, the convection coefficients is $\boldsymbol{\beta}(x,y)=(x,-y)$. We choose the nodes of the meshes so that all of them lie on streamlines of $\boldsymbol{\beta}$ as seen in Fig. 3.1 left. In this way, the three conditions on the mesh (1.2) are satisfied; see Fig. 3.1 right. A mesh ℓ means the mesh size is $h=\frac{1}{2^\ell}$.

In Table 3.1, we display the history of convergence for the approximate solution using polynomials of degree $k=0,1,2$. We see that the order $k+1$ is observed for both $\|u-u_h\|_{L^2(\Omega)}$ and $\|\partial_{\boldsymbol{\beta}} u-\partial_{\boldsymbol{\beta},h} u_h\|_{L^2(\Omega)}$ just as the theory predicts.

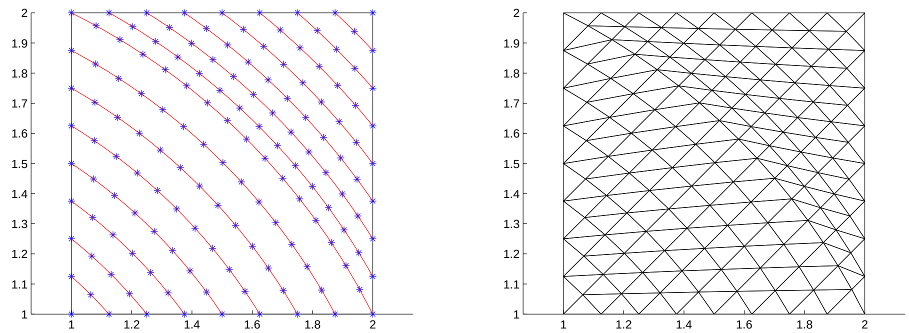

Fig. 3.2. *Meshes satisfying the flow condition with respect to* $\boldsymbol{\beta}=(y,-x)$*: The streamlines of* β *and the nodes of the mesh (left) and the actual mesh (*$\ell=3$*) (right).*

Table 3.2
History of convergence for the harmonic oscillator problem.

	mesh	$\|u-u_h\|_{L^2(\Omega)}$		$\|\partial_{\boldsymbol{\beta}} u-\partial_{\boldsymbol{\beta},h} u_h\|_{L^2(\Omega)}$	
k	ℓ	error	order	error	order
	1	.14e+1	-	.31e+1	-
	2	.75e-0	0.92	.16e+1	1.02
0	3	.39e-0	0.97	.77e-0	1.01
	4	.19e-0	0.98	.38e-0	1.01
	5	.98e-1	0.99	.19e-0	1.00
	1	.84e-1	-	.84e-1	-
	2	.22e-1	1.94	.22e-1	1.94
1	3	.56e-2	1.98	.56e-2	1.98
	4	.14e-2	1.99	.14e-2	1.99
	5	.35e-3	2.00	.35e-3	2.00
	1	.22e-2	-	.76e-2	-
	2	.27e-3	3.04	.93e-3	3.03
2	3	.33e-4	3.02	.11e-3	3.02
	4	.41e-5	3.01	.14e-4	3.01
	5	.51e-6	3.01	.18e-5	2.95

In the second numerical experiment, we take the convection coefficient to be $\boldsymbol{\beta}(x,y)=(y,-x)$. Similarly, We choose the nodes of the meshes so that all of them lie on streamlines of $\boldsymbol{\beta}$; see Fig. 3.2 left. Again, the three conditions on the mesh (1.2) are satisfied; see Fig. 3.1 right.

In Table 3.2, we display the history of convergence for the approximate solution using polynomials of degree $k = 0, 1, 2$. We see that order $k+1$ is observed for $\|u - u_h\|_{L^2(\Omega)}$ and $\|\partial_{\boldsymbol\beta} u - \partial_{\boldsymbol\beta,h} u_h\|_{L^2(\Omega)}$ just as the theory predicts.

4. Concluding remarks. In this paper, we have uncovered a class of meshes for which the DG method converges in an optimal way thus extending the results in [3] for constant transport velocities $\boldsymbol\beta$ to divergence-free, variable ones. The extension of these results to more general transport velocities $\boldsymbol\beta$ and to curved-boundary domains Ω constitutes the subject of ongoing work.

5. Appendix: Sketch of Proof of Lemma 2.1. We modify the proof of [7] to allow for a variable velocity $\boldsymbol\beta$. To this end, we set $\psi(\boldsymbol{x}) = e^{-(\boldsymbol{x}-\boldsymbol{x}_0)\cdot\boldsymbol\beta(\boldsymbol{x})}$ where $\boldsymbol{x}_0 \in \partial\Omega$ is fixed. Then, by using integration by parts, we can easily show that

$$\begin{aligned} B(w_h, \psi w_h) =&(w_h, (\frac{1}{2}\boldsymbol\beta\cdot\boldsymbol\beta + c)\,\psi\, w_h)_{\mathcal{T}_h} \\ &+ \frac{1}{2}\sum_{e\in\mathscr{E}_h^0} \|\sqrt{\psi\,|\boldsymbol\beta\cdot\boldsymbol{n}|}\,(w_h^+ - w_h^-)\|^2_{L^2(e)} + \frac{1}{2}\sum_{e\in\mathscr{E}_h^\partial} \|\sqrt{\psi\,|\boldsymbol\beta\cdot\boldsymbol{n}|}\, w_h\|^2_{L^2(e)}, \end{aligned}$$

where $\mathscr{E}_h^0$ is the collection of interior edges and $\mathscr{E}_h^\partial$ is the collection of boundary edges. Next we use the fact that there exists a constant $\delta > 0$ such that $|\psi(\boldsymbol{x})| \geq \delta$ for all $\boldsymbol{x} \in \Omega$ and the positivity condition (1.4) to obtain that

$$\delta\,\min\{\frac{1}{2}, \gamma\}\|w_h\|_h^2 \leq B(w_h, \psi w_h) \tag{5.1}$$

where

$$\|w_h\|_h^2 :=\|w_h\|^2_{L^2(\mathcal{T}_h)} + \frac{1}{2}\sum_{e\in\mathscr{E}_h^0} \|\sqrt{|\boldsymbol\beta\cdot\boldsymbol{n}|}\,(w_h^+ - w_h^-)\|^2_{L^2(e)} + \frac{1}{2}\sum_{e\in\mathscr{E}_h^\partial} \|\sqrt{|\boldsymbol\beta\cdot\boldsymbol{n}|} w_h\|^2_{L^2(e)}.$$

Since ψw_h is not in the finite element space we consider its L^2-projection, $\mathsf{P}(\psi w_h)$, to get

$$\begin{aligned} B(w_h, \psi w_h) =&B(w_h, \mathsf{P}(\psi w_h)) + B(w_h, \psi w_h - \mathsf{P}(\psi w_h)) \\ =&F(\mathsf{P}(\psi w_h)) + B(w_h, \psi w_h - \mathsf{P}(\psi w_h)). \end{aligned} \tag{5.2}$$

It remains to bound the last two terms of the right-hand side. Clearly,

$$\begin{aligned} F(\mathsf{P}(\psi w_h)) &\leq \|\mathsf{P}(\psi w_h)\|_{L^2(\mathcal{T}_h)} \max_{v\in V_h} \frac{|F(v)|}{\|v\|_{L^2(\mathcal{T}_h)}} \\ &\leq \|\psi w_h\|_{L^2(\mathcal{T}_h)} \max_{v\in V_h} \frac{|F(v)|}{\|v\|_{L^2(\mathcal{T}_h)}} \\ &\leq C\|w_h\|_{L^2(\mathcal{T}_h)} \max_{v\in V_h} \frac{|F(v)|}{\|v\|_{L^2(\mathcal{T}_h)}}. \end{aligned} \tag{5.3}$$

Following the argument giving in [7], where we use here that $\boldsymbol\beta$ is smooth, we can show that

$$B(w_h, \psi w_h - \mathsf{P}(\psi w_h)) \leq \epsilon\|w_h\|_h^2 + \frac{C\,h}{\epsilon}\|w_h\|^2_{L^2(\mathcal{T}_h)}$$

where $\epsilon > 0$ is arbitrary.

Choosing $\epsilon = \frac{\delta}{4}\min\{\frac{1}{2},\gamma\}$ and assuming h is sufficiently small, we get that

$$B(w_h, \psi w_h - \mathsf{P}(\psi w_h)) \le \frac{\delta}{2}\min\{\frac{1}{2},\gamma)\|w_h\|_h^2. \tag{5.4}$$

Finally, we conclude the proof by combining (5.1), (5.2), (5.3) and (5.4).

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