LOCAL ENERGY ESTIMATES FOR THE FINITE ELEMENT
METHOD ON SHARPLY VARYING GRIDS

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ABSTRACT. Local energy error estimates for the finite element method for elliptic problems were originally proved in 1974 by Nitsche and Schatz. These estimates show that the local energy error may be bounded by a local approximation term, plus a global “pollution” term that measures the influence of solution quality from outside the domain of interest and is heuristically of higher order. However, the original analysis of Nitsche and Schatz is restricted to quasi-uniform grids. We present local a priori energy estimates that are valid on shape regular grids, an assumption which allows for highly graded meshes and which much more closely matches the typical practical situation. Our chief technical innovation is an improved superapproximation result.

1. Introduction

In this note we prove local energy error estimates for the finite element method for second-order linear elliptic problems on highly refined triangulations. Most a priori error analyses for the finite element method in norms other than the global energy norm place severe restrictions on the mesh. In particular, such error analyses are most often carried out under the assumption that the grid is quasi uniform, that is, all simplices in the mesh are required to have diameter equivalent to some fixed parameter $h$. The typical practical situation is rather different. Many (especially adaptive) finite element codes enforce only shape regularity of elements, meaning that all elements in the mesh must have bounded aspect ratio. Though it places a weak restriction upon the rate with which the diameters of elements in the mesh may change, shape regularity allows for the locally refined meshes that are needed to resolve the singularities and other sharp local variations of the solution that occur in the majority of practical applications.

In the work [NS74] of Nitsche and Schatz, local energy error estimates were established for interior subdomains under the assumption that the finite element grid is quasi-uniform. Such local energy estimates are helpful in understanding basic error behavior, especially “pollution effects” of global solution properties on local approximation quality, and they also provide an important technical tool in many proofs of pointwise bounds for the finite element method (cf. [SW95]). In addition, the most relevant error notion in applications is often related to some local norm or functional instead of to the global energy error, as evidenced by the recent surge of interest in ensuring control of the error in calculating “quantities of interest” in adaptive finite element calculations instead of merely controlling the

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default global energy error (cf. [BR01]). As a final example of the applicability of local energy estimates, we mention that the estimates of [NS74] have been used to justify certain approaches to parallelization and adaptive meshing (cf. [BH00]). Thus local energy estimates are of broad and fundamental importance in finite element theory.

Here we prove local energy error estimates under the assumption that the finite element triangulation is shape regular instead of under the more restrictive assumption of quasi uniformity required in [NS74]. In other words, we essentially prove that the results of Nitsche and Schatz hold under the restrictions typically placed upon meshes in practical codes, which in particular allow for highly graded grids. Our main innovation is a novel “superapproximation” result which we state and prove in §2. In §3 we then prove a local energy bound that is valid on grids that are only assumed to be shape-regular. As in [NS74], our results are valid for operators that are only locally elliptic, so that the PDE under consideration may be degenerate or change type outside of the domain of interest. In contrast to [NS74], the results we present here are valid up to the domain boundary, allow for nonhomogeneous Neumann, Dirichlet, and mixed boundary conditions, and also require only $L_\infty$ regularity of the coefficients of the differential operator.

2. An improved superapproximation result

An essential feature of the proofs of local error estimates given in [NS74], and also of essentially all published proofs of local and maximum-norm a priori error estimates for finite element methods, is the use of superapproximation properties. In essence, superapproximation bounds establish that a function in the finite element space multiplied by any smooth function can be approximated exceptionally well by the finite element space.

In order to fix thoughts, we shall in this section assume for simplicity that $\Omega \subset \mathbb{R}^n$ is a polyhedral domain; a more general situation is considered in §3 below. Let $T_h$ be a simplicial decomposition of $\Omega$. Denote by $h_T$ the diameter of the element $T \in T_h$. We assume throughout that the elements in $T_h$ are shape-regular, that is, each simplex $T \in T_h$ contains a ball of diameter $c_1 h_T$ and is contained in a ball of radius $C h_T$, where $c_1$ and $C$ are fixed. Let also $S_h^r$ be a standard Lagrange finite element space consisting of continuous piecewise polynomials of degree $r$ – 1.

We shall use standard notation for Sobolev spaces, norms, and seminorms, e.g.,

$$\|u\|_{H^1(\Omega)} = (\int_\Omega (u^2 + |\nabla u|^2) \, dx)^{1/2}, \quad |u|_{W_h^k(\Omega)} = (\sum_{|\alpha| = k} \|D^\alpha u\|_{L_2(\Omega)}^p)^{1/p},$$

etc.

A standard superapproximation result is as follows. Let $\omega \in C^\infty(\Omega)$ with $|\omega|_{W_h^j(\Omega)} \leq C d^{-j}$, $0 \leq j \leq r$. Then for each $\chi \in S_h^r$, there exists $\eta \in S_h^r$ such that for each $T \in T_h$ satisfying $d \geq h_T$,

$$\|\omega \chi - \eta\|_{H^1(T)} \leq C \left( \frac{h_T}{d} \|\nabla \chi\|_{L_2(T)} + \frac{h_T}{d^2} \|\chi\|_{L_2(T)} \right).$$

Our modified result follows (cf. [Guz06]).

**Theorem 2.1.** Let $\omega \in C^\infty(\Omega)$ with $|\omega|_{W_h^j(\Omega)} \leq C d^{-j}$ for $0 \leq j \leq r$. Then for each $\chi \in S_h^r$, there exists $\eta \in S_h^r$ such that for each $T \in T_h$ satisfying $d \geq h_T$,

$$\|\omega^2 \chi - \eta\|_{H^1(T)} \leq C \left( \frac{h_T}{d} \|\nabla (\omega \chi)\|_{L_2(T)} + \frac{h_T}{d^2} \|\chi\|_{L_2(T)} \right).$$

**Remark 2.2.** There are two differences between (2.1) and (2.2). First, in (2.1) we consider approximation of $\omega \chi$, whereas in (2.2) we consider approximation of $\omega^2 \chi$. 

Secondly, in (2.1) the norms on the right hand side involve only $\chi$, whereas in (2.2) the $H^1$ seminorm involves $\omega \chi$. If we think of $\omega$ as a cutoff function, this distinction becomes vitally important: $\omega \chi$ has the same support as $\omega^2 \chi$, whereas the support of $\chi$ is generally larger than that of $\omega \chi$. This seemingly minor difference will allow us to establish local energy estimates on grids that are only assumed to be shape regular.

**Proof.** Let $I_h : C^0(\Omega) \rightarrow S^r_h$ be the standard Lagrange interpolant. We shall choose $\eta = I_h(\omega^2 \chi)$ in (2.2). For $T \in \mathcal{T}_h$, we may use standard approximation theory (cf. [BS02]) to calculate

$$
\|\omega^2 \chi - I_h(\omega^2 \chi)\|_{H^1(T)} \leq C h_T^{n/2} \|\omega^2 \chi - I_h(\omega^2 \chi)\|_{W^1_\infty(T)} 
\leq C h_T^{n/2+r-1} |\omega^2 \chi|_{W^1_\infty(T)}.
$$

(2.3)

Noting that $D^\alpha \chi = 0$ for all multiindices $\alpha$ with $|\alpha| = r$, recalling that $\frac{h_T}{d} \leq 1$, and employing inverse estimates, we compute

$$
C h_T^{n/2+r-1} |\omega^2 \chi|_{W^1_\infty(T)} \leq C \left( \sum_{i=2}^r h_T^{i-1} |\omega^2|_{W^i_\infty(T)} \right) \|\chi\|_{L^2(T)}
+ C h_T^{n/2+r-1} \sum_{|\alpha|=1, |\beta|=r-1} \|D^\alpha \omega D^\beta \chi\|_{L^\infty(T)}
\leq C \frac{h_T}{d} \|\chi\|_{L^2(T)} + C h_T^{n/2+r-1} \sum_{|\alpha|=1, |\beta|=r-1} \|D^\alpha \omega D^\beta \chi\|_{L^\infty(T)}.
$$

(2.4)

We next consider the terms $\|D^\alpha \omega^2 D^\beta \chi\|_{L^\infty(T)}$ above. Since $|\alpha| = 1$, we have $D^\alpha \omega^2 = 2\omega D^\alpha \omega$. Let $\hat{\omega} = \frac{1}{|T|} \int_T \omega \, dx$ so that $\|\omega - \hat{\omega}\|_{L^\infty(T)} \leq C h_T |\omega|_{W^1_\infty(T)} \leq C \frac{h_T}{d}$. Employing inverse estimates, we thus have

$$
C h_T^{n/2+r-1} \sum_{|\alpha|=1, |\beta|=r-1} \|D^\alpha \omega D^\beta \chi\|_{L^\infty(T)}
\leq C d^{-1} h_T^{n/2+r-1} \sum_{|\beta|=r-1} \|\omega D^\beta \chi\|_{L^\infty(T)}
\leq C d^{-1} h_T^{n/2+r-1} \left( \|\omega D^\beta \chi\|_{L^\infty(T)} + \|\hat{\omega} D^\beta \chi\|_{L^\infty(T)} \right)
\leq C \left( \frac{h_T}{d} \|\chi\|_{L^2(T)} + \frac{h_T}{d} |\hat{\omega}|_{H^1(T)} \right)
\leq C \left( \frac{h_T}{d} \|\chi\|_{L^2(T)} + \frac{h_T}{d} |\hat{\omega}|_{H^1(T)} + \frac{h_T}{d} |\omega^2 \chi|_{H^1(T)} \right).
$$

(2.5)

Using an inverse inequality, we find that

$$
\frac{h_T}{d} |(\hat{\omega} - \omega) \chi|_{H^1(T)} \leq \frac{h_T}{d} \left( |\omega|_{W^1_\infty(T)} \|\chi\|_{L^2(T)} + \|\hat{\omega} - \omega\|_{L^\infty(T)} |\chi|_{H^1(T)} \right)
\leq C \frac{h_T}{d} \left( \frac{1}{d} \|\chi\|_{L^2(T)} + \frac{h_T}{d} |\chi|_{H^1(T)} \right)
\leq C \frac{h_T}{d} \|\chi\|_{L^2(T)}.
$$

(2.6)

Inserting (2.6) into (2.5) and the result into (2.4) and (2.3) completes the proof of (2.2).
3. Local $H^1$ estimates

In this section we state and prove a local $H^1$ estimate that is valid on highly graded grids. We now let $\Omega$ be a domain in $\mathbb{R}^n$, and let $\Omega_0$ be a bounded subdomain of $\Omega$. We decompose $\partial \Omega \cap \partial \Omega_0$ (if it is nonempty) into a Dirichlet portion $\Gamma_D$ and a Neumann portion $\Gamma_N$. For the sake of simplicity, we assume that $\Gamma_D$ is polyhedral and that $\Gamma_N$ is either polyhedral or Lipschitz. Let $u$ satisfy

$$-\text{div}(A\nabla u) + b \cdot \nabla u + cu = f \text{ in } \Omega_0,$$

$$u = g_D \text{ on } \Gamma_D,$$

$$\frac{\partial u}{\partial n} = g_N \text{ on } \Gamma_N.$$

Here $A$ is an $n \times n$ coefficient matrix that is uniformly bounded and positive definite in $\Omega$, $b \in L^\infty(\Omega_0)^n$, $c \in L^\infty(\Omega_0)$, and $\frac{\partial}{\partial n}$ is the conormal derivative with respect to $A$. We also assume that $\Omega \subset \mathbb{R}^n$. Note that we make no assumptions about the differential equation solved by $u$ outside of $\Omega_0$.

Let $H^1_{D,0}(\Omega_0) = \{ u \in H^1(\Omega_0) : u|_{\Gamma_D} = 0 \}$, and let $H^1_D(\Omega_0) = u \in H^1(\Omega_0) : u|_{\Gamma_D} = g_D$. Also let $H^1_{N}(B) = \{ u \in H^1(\Omega_0) : u|_{\partial B} = 0 \}$ for subsets $B$ of $\Omega_0$. Thus functions in $H^1_{D,0}(\Omega_0)$ are zero on $\partial \Omega \setminus \partial \Omega_0$, but may be nonzero on portions of $\partial B$ coinciding with $\partial \Omega$, or put in other terms, functions in $H^1_D(B)$ are compactly supported in $B$ modulo $\partial \Omega$. Rewriting (3.1) in its weak form, we find that $u \in H^1_{D,0}(\Omega_0)$ satisfies

$$L(u, v) := \int_{\Omega} (A\nabla u \cdot \nabla v + b \cdot \nabla u v + cu v) \, dx$$

$$= \int_{\Omega} fv \, dx - \int_{\Gamma_N} g_N v \, d\sigma, \quad v \in H^1_{D,0}(\Omega_0) \cap H^1_N(\Omega_0).$$

Following [NS74], we do not assume that $L$ is coercive over $H^1(\Omega_0)$, but rather we make a local coercivity assumption:

**R1: Local coercivity.** There exists a constant $d_0 > 0$ such that if $B$ is the intersection of any open sphere of diameter $d \leq d_0$ with $\Omega_0$, then $L$ is coercive over $H^1_{\text{tr}}(B)$, that is, for some constant $C_1 > 0$,

$$(C_1)^{-1} ||u||^2_{H^1(B)} \leq L(u, u) \leq C_1 ||u||^2_{H^1(B)}, \quad u \in H^1_{\text{tr}}(B).$$

**Remark 3.1.** $R1$ may be satisfied in one of two ways. It may happen that $L$ is coercive over $H^1(\Omega_0)$, in which case no further argument is needed. $R1$ so long as a Poincaré inequality

$$(3.4) \quad ||u||_{L_2(B)} \leq C d ||u||_{H^1(B)}$$

holds for balls $B$ as in $R1$ having small enough diameter (cf. Remark 1.2 of [NS74]). Such Poincaré inequalities always hold for interior balls. If $B$ is the nontrivial intersection of an open ball with $\Omega_0$, then (3.4) holds for $d \leq d_1$ small enough under the restrictions we have placed on $\partial \Omega \cap \partial \Omega_0$; here $d_1$ depends on the properties of $\partial \Omega \cap \partial \Omega_0$.

Next we make assumptions concerning the finite element approximation $u_h$ of $u$. Let $T_0$ be a triangulation such that $\Omega_0 \subset \bigcup_{T \in T_0} T$ and $T \cap \Omega_0 \neq \emptyset$ for all $T \in T_0$. Let $h_T = \text{diam}(T)$ for $T \in T_0$. We denote our trial finite element space by $S_D$. We do not assume that $S_D \subset H^1_{D,0}(\Omega_0)$. In addition, we let $S_{D,0} = S_D \cap H^1_{D,0}(\Omega_0)$.
be our trial finite element space. We assume that \( u_h \) is the local finite element approximation to \( u \) on \( \Omega_0 \), that is, \( u_h \in S_D \) and
\[
L(u - u_h, v_h) = 0 \quad \text{for all } v_h \in S_{D,0} \cap H^1_0(\Omega_0).
\]
We do not explicitly fix \( u_h \) on the Dirichlet portion of the boundary, but rather implicitly assume that \( u_h|_{\Gamma_D} \) is set equal to some appropriate interpolant or projection of \( g_D \).

Next we state properties that \( S_D \) and \( S_{D,0} \) must possess in order to prove the desired local energy error estimate. Let \( \delta \leq d_0 \) be a fixed parameter, and let \( G_1 \) and \( G \) be arbitrary subsets of \( \Omega_0 \) with \( G_1 \subset G \) and \( \text{dist}(G_1, \partial G \setminus \partial \Omega) = \delta > 0 \). Then the following are assumed to hold:

**A1: Local interpolant.** There exists a local interpolant \( I \) such that for each \( u \in H^1_0(G_1), Iu \in S_D \cap H^1_0(G) \), and for each \( u \in H^1_{D,0}(\Omega_0), Iu \in S_{D,0} \).

**A2: Inverse properties.** For each \( \chi \in S_D, T \in T_h, 1 \leq p \leq q \leq \infty \), and \( 0 \leq \nu \leq s \leq r \) with \( r \) sufficiently small,
\[
\|\chi\|_{W^s_p(T)} \leq C h_T^{r-s+\frac{n}{p}-\frac{n}{q}} \|\chi\|_{W^r_q(T)}.
\]

**A3: Superapproximation.** Let \( \omega \in C^\infty(\Omega_0) \cap H^1_0(G_1) \) with \( |\omega|_{W^s_p(\Omega_0)} \leq Cd^{-j} \) for integers \( 0 \leq j \leq r \) with \( r \) sufficiently large. For each \( \chi \in S_{D,0} \) and for each \( T \in T_h \) satisfying \( \delta \leq h_T \),
\[
\|\omega^2 \chi - I(\omega^2 \chi)\|_{H^1(T)} \leq C \left( \frac{h_T}{d} \|\nabla(\omega \chi)\|_{L_2(T)} + \frac{h_T}{d^2} \|\chi\|_{L_2(T)} \right),
\]
where the interpolant \( I \) is as in A1 above.

**Remark 3.2.** A1, A2, and A3 are satisfied by standard finite element spaces defined on shape-regular triangular grids. A1 also essentially requires that the finite element mesh resolve \( G \setminus G_1 \), i.e., that \( \delta \geq K \max_{T \in G \neq \emptyset} h_T \) with \( K \) large enough.

We begin by proving a Caccioppoli-type estimate for “discrete harmonic” functions. Such a statement was also proved in [NS74] as a preliminary to local energy estimates, though the proof we give below more closely follows [SW77].

**Lemma 3.3.** Let \( G_0 \subset G \subset \Omega_0 \) be given, and let \( \text{dist}(G_0, \partial G \setminus \partial \Omega) = d \) with \( d \leq 2d_0 \) where \( d_0 \) is the parameter defined in the assumption R1. Let also A1, A2, and A3 hold with \( \delta = \frac{d}{4} \), and assume that \( u_h \in S_{D,0} \) satisfies
\[
L(u_h, v_h) = 0 \quad \text{for all } v_h \in S_{D,0} \cap H^1_0(\Omega_0).
\]
In addition let \( \max_{T \in G \neq \emptyset} \frac{h_T}{d} \leq \frac{1}{4} \). Then
\[
\|u_h\|_{H^1(G_0)} \leq C \frac{1}{d} \|u_h\|_{L_2(G)}.
\]
Here \( C \) depends only on the constants in (3.6) and (3.7) and the coefficients of \( L \).

**Proof.** We assume that \( G_0 \) is the intersection of a ball \( B_\frac{d}{2} \) of radius \( \frac{d}{2} \) with \( \Omega_0 \); the general case may be proved using a covering argument. Let then \( G_1 \) and \( G_2 \) be the intersections with \( \Omega_0 \) of balls having the same center as \( G_0 \) and having radii \( \frac{d}{2} \) and \( \frac{3d}{4} \), respectively, and without loss of generality let \( G \) be the corresponding ball of radius \( d \). Let then \( \omega \in C_0^{\infty}(G_1) \) be a cutoff function which is 1 on \( G_0 \) and which
Taking (3.15) to compute
\[ L(\omega u_h, \omega h) = L(u_h, \omega^2 u_h) \]
\[ \leq L(u_h, \omega^2 u_h) + \epsilon \frac{1}{d} \| u_h \|_{L^2(G)}^2 + \epsilon \| u_h \|_{H^1(G)}^2. \]
Next we use (3.8), (3.7), and the fact that \( \| \omega^2 u_h \|_{H^1(G)} \leq \| \omega u_h \|_{H^1(G)} + \frac{C}{d} \| u_h \|_{L^2(G)} \) to compute
\[ L(u_h, \omega^2 u_h) = L(u_h, \omega^2 u_h - I(\omega^2 u_h)) \]
\[ \leq C \sum_{T \cap G \neq \emptyset} h_T \| u_h \|_{H^1(T)} (\frac{1}{d} \omega u_h |_{H^1(T)} + \frac{1}{d^2} \| u_h \|_{L^2(T)}). \]
Using (3.6) and the fact that \( \frac{h_T}{d} \leq 1 \), we have for \( \epsilon \) as above that
\[ \epsilon h_T \| u_h \|_{H^1(T)} (\frac{1}{d} \omega u_h |_{H^1(T)} + \frac{1}{d^2} \| u_h \|_{L^2(T)}). \]
Inserting (3.13) into (3.12), noting that \( T \cap G \neq \emptyset \) implies that \( T \subset G \) (since \( \max_{T \cap G \neq \emptyset} h_T \leq \frac{d}{4} \) and carrying out further elementary manipulations then yields that for \( \epsilon > 0 \),
\[ L(u_h, \omega^2 u_h) \leq \frac{C}{d^2} \| u_h \|_{L^2(G)}^2 + \epsilon \| \omega u_h \|_{H^1(G)}^2. \]
Inserting (3.14) into (3.11) and the result into (3.10) yields
\[ \| \omega u_h \|_{H^1(G)}^2 \leq \frac{C}{d^2} \| u_h \|_{L^2(G)}^2 + 2 \epsilon \| \omega u_h \|_{H^1(G)}^2. \]
Taking \( \epsilon = \frac{1}{4} \) so that we may kick back the last term above, employing the triangle inequality, and inserting the result into (3.9) then completes the proof of (3.10). □

We now prove a local energy error estimate. In our proof below we shall follow [NS74] by using a local finite element projection in order to split the finite element error into an approximation error and a “discrete harmonic” term which may be bounded using Lemma 3.3. We note, however, that the use of a local finite element projection is not necessary, and our final local error estimate may in fact be proved with some simple modifications to the proof of Lemma 3.3 above. These two styles of proof are essentially equivalent. Local finite element projections have been used for example in [NS74], [SW77], [SW95], and [AL95] in order to prove local a priori error estimates. The methodology of Lemma 3.3 in which no local projections are used has been employed in for example [Dem04] and [Guz06] in order to prove local a priori error estimates and in [LN03] and [Dem07] in order to prove local a posteriori error estimates.
Theorem 3.4. Let $G_0 \subset G \subset \Omega_0$ be given, and let $\text{dist}(G_0, \partial G \setminus \partial \Omega) = d$ with $d \leq \min\{2d_0, d_1\}$ where $d_0$ is the parameter defined in the assumption R1 and $d_1$ is defined in Remark 3.1. Let also $A1, A2,$ and $A3$ hold with $\bar{d} = \frac{d}{16}$. In addition let $\max_{T \cap G \neq \emptyset} \frac{h_T}{d_T} \leq \frac{1}{16}$. Then

$$\|u - u_h\|_{H^1(G_0)} \leq C \min_{u_h - \chi \in S_{D,0}} \left( \|u - \chi\|_{H^1(G)} + \frac{1}{d} \|u - \chi\|_{L^2(G_0)} \right)$$

(3.16)

$$+ C \frac{1}{d} \|u - u_h\|_{L^2(G)}.$$

Here $C$ depends only on the constant $C$ in (2.2) and the coefficients of $L$.

Proof. We assume that $G_0$ is the intersection of a ball $B_{\frac{d}{2}}$ of radius $\frac{d}{2}$ with $\Omega_0$; the general case may be proved using a covering argument. Let $G_1$ be the intersection with $\Omega_0$ of a ball having the same center as $G_0$ and having radius $\frac{3d}{4}$, and without loss of generality let $G$ be the corresponding ball of radius $d$. Let then $\omega \in C_0^\infty(G)$ be a cutoff function which is 1 on $G_1$ and which satisfies $\|\omega\|_{W_0^{1,2}(G)} \leq Cd^{-j}$, $0 \leq j \leq r$. Note that we may apply Lemma 3.3 with $G_0$ on the left hand side of the estimate (3.9) and $G_1$ on the right hand side.

Next we let $P(\omega u)$ be a local finite element projection of $\omega u$. In particular, we let $P(\omega u) \in S_D \cap H^1_0(G)$ with $u_h = P(\omega u) = 0$ on $\Gamma_D \cap \partial G_1$ satisfy

$$L(\omega u - P(\omega u), v_h) = 0, \quad v_h \in S_{D,0} \cap H^1_0(G).$$

The local coercivity condition (3.3) then implies the stability estimate

$$\|P(\omega u)\|_{H^1(G)} \leq C \|\omega u\|_{H^1(G)}.$$

(3.17)

Recalling that $u_h - P(\omega u) = 0$ on $\Gamma_D \cap \partial G_1$ while employing (3.9) and using (3.4) while recalling that $\omega \equiv 1$ on $G_1$, we compute that

$$\|u - u_h\|_{H^1(G_0)} \leq \|\omega u - P(\omega u)\|_{H^1(G_0)} + \|P(\omega u) - u_h\|_{H^1(G_0)}$$

$$\leq \|\omega u - P(\omega u)\|_{H^1(G)} + \frac{C}{d} \|\omega u - u_h\|_{L^2(G_1)}$$

(3.19)

$$\leq \|\omega u - P(\omega u)\|_{H^1(G)} + \frac{C}{d} \|\|P(\omega u) - \omega u\|_{L^2(G_1)} + \|u - u_h\|_{L^2(G_1)}\|$$

$$\leq C \|\omega u - P(\omega u)\|_{H^1(G)} + \frac{C}{d} \|u - u_h\|_{L^2(G_1)}.$$

Next we employing the triangle inequality along with (3.18) while recalling that $\|\omega\|_{W_0^{1,2}(G)} \leq Cd^{-j}$ in order to find that

$$\|\omega u - P(\omega u)\|_{H^1(G)} \leq C \|\omega u\|_{H^1(G)}$$

(3.20)

$$\leq C \|u\|_{H^1(G)} + \frac{1}{d} \|u\|_{L^2(G)}.$$

In order to complete the proof of (3.16), we first insert (3.20) into (3.19) and finally write $u - u_h = (u - \chi) + (\chi - u_h)$ with $u_h - \chi \in S_{D,0}$.

References

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