A HYBRIDIZABLE AND SUPERCONVERGENT DISCONTINUOUS GALERKIN METHOD FOR BIHARMONIC PROBLEMS

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ABSTRACT. In this paper, we introduce and analyze a new discontinuous Galerkin method for solving the biharmonic problem $\Delta^2 u = f$. The method has two main, distinctive features, namely, it is amenable to an efficient implementation, and it displays new superconvergence properties. Indeed, although the method uses as separate unknowns u, ∇u, Δu and $\nabla \Delta u$, the only globally coupled degrees of freedom are those of the approximations to u and Δu on the faces of the elements. This is why we say it can be efficiently implemented. We also prove that, when polynomials of degree at most $k \geq 1$ are used on all the variables, approximations of optimal convergence rates are obtained for both u and ∇u; the approximations to Δu and $\nabla \Delta u$ converge with order $k + 1/2$ and $k - 1/2$, respectively. Moreover, both the approximation of u as well as its numerical trace superconverge in L^2-like norms, to suitably chosen projections of u with order $k+2$ for $k \geq 2$. This allows the element-by-element construction of another approximation to u converging with order $k + 2$ for $k \geq 2$. For $k = 0$, we show that the approximation to u converges with order one, up to a logarithmic factor. Numerical experiments are provided which confirm the sharpness of our theoretical estimates.

1. Introduction

In this paper, we continue our study of the LDG-H methods, introduced in [11] in the framework of second order elliptic problems, and consider the extension of the so-called *single face-hybridizable* (SFH) method, a particular LDG-H method proposed and studied in [10], to the biharmonic problem

$$\Delta^2 u = f \quad \text{in } \Omega, \tag{1.1a}$$

$$u = g \quad \text{on } \partial\Omega, \tag{1.1b}$$

$$\frac{\partial u}{\partial \boldsymbol{n}} = -\mathsf{q}_{\mathsf{N}} \quad \text{on } \partial\Omega, \tag{1.1c}$$

where $\Omega \subset \mathbb{R}^d$ is a polyhedral domain ($d \geq 2$), and $f \in L^2(\Omega)$.

Since the first equation of the above problem can be rewritten as

$$-\Delta z = f,$$
$$-\Delta u = z,$$

1991 *Mathematics Subject Classification.* 65M60,65N30,35L65.

Key words and phrases. discontinuous Galerkin methods, hybridization, superconvergence, biharmonic problems.

The first author was supported in part by the National Science Foundation (Grant DMS-0411254) and by the University of Minnesota Supercomputing Institute. The third author was supported by an NSF Mathematical Science Postdoctoral Research Fellowship (DMS-0503050).

it is natural to consider extensions of numerical methods for second-order elliptic problems to our setting. In [10], it was shown that the SFH method for second-order elliptic problems is optimally convergent in both the scalar variable and its gradient, and has superconvergence properties allowing for the construction of a new, better approximation to the scalar variable. In this paper we explore if similar results can be obtained by extending the SFH method to the biharmonic problem.

To describe our results and render clear their relevance, let us begin by presenting a short overview of the development of finite element methods for the biharmonic problem. In the past 40 years, many different finite element methods for the biharmonic problem have been devised. We bias our discussion towards methods that are not conforming and that employ piecewise polynomial approximations of arbitrary degree. A thorough discussion about conforming methods and non-conforming methods of using low polynomial degree approximations can be found in Section 6 in [8]. We begin our discussion with the mixed finite element method introduced in 1974 by Ciarlet and Raviart [9]; see also Section 7 in [8].

Ciarlet and Raviart [9] showed that, with C^0-finite elements associated with piecewise polynomials of degree $k \geq 2$, the H^1-norm of the error in u and the L^2-norm of the error in Δu converge with order $k-1$. These estimates have been subsequently improved by Scholz in [23, 24]. Indeed, in 1976, Scholz obtained an optimal error estimate of u for $k \geq 3$. Two years later, by using an L^∞-estimate, he showed that piecewise linear elements yield first order convergence in the H^1-norm of the error in u and $1/2$ order convergence in the L^2-norm of the error in Δu; he also mentioned that this approach can be extended to higher order polynomials. In 1978, Falk [15] devised a variant of Ciarlet-Raviart mixed method which gives approximations that optimally converge to u and sub-optimally converge to Δu in the L^2-norm with order $(k-1)$ for $k \geq 3$. For more details, see the review paper [18].

Another popular mixed finite element method that has been applied to the biharmonic problem is the Hellan-Herrmann-Johnson (HHJ) method. In 1980, Babuška *et.al.* [3] analyzed the HHJ method on two-dimensional convex polygonal domains. They showed that the continuous approximation to the scalar variable u converges with order $k+1$ if polynomials of degree $k \geq 1$ are used, and the approximation to the matrix of second-order partials of u converges with order k if polynomials of degree $k-1$ are used. In [1], a Lagrange multiplier was used to impose interelement continuity, and a better piecewise linear approximation to u was generated by post-processing, which automatically gave a superconvergent approximation to the gradient of u. In [13], Comodi developed similar post-processing to extend the result to the case of $k \geq 2$; she showed a superconvergence result for the distance between the approximation of u and a suitable projection of u which allowed her to use post-processing to construct an approximation converging with an optimal rate to gradient of u. In [25], Stenberg used a different post-processing to produce a new approximation to u, which converges in H^1-norm with order $k+1$ for $k \geq 1$ and converges in L^2-norm with order $k+2$ for $k \geq 3$.

Interior penalty methods have also been investigated for the biharmonic problem. In his pioneering work of 1977, Baker [5] devised an interior penalty method based on discontinuous finite elements to the biharmonic problem, and obtained an optimal error estimates for $k \geq 3$. In a sequence of papers [20, 26, 21] by Süli *et.al.*, *hp*-version of symmetric, nonsymmetric and semi-symmetric interior penalty discontinuous Galerkin finite element methods have been studied, and optimal approximations to u and $(k-1)th$ order approximation to Δu were obtained for polynomials of degree $k \geq 2$. On the other hand, C^0 interior penalty methods for plate bending problems using quadrilateral elements were analyzed in [14], and the error was shown to converge optimally in L^2-norm for smooth solutions when $k \geq 3$. Later Brenner and Sung [7] extended the analysis to polygonal domains and nonsmooth solutions, and they used post-processing to generate C^1 approximate solutions from C^0 approximate solutions.

The method we present here is also a discontinuous Galerkin method, but is formulated in terms of approximations not only to u but also to $\nabla u, \Delta u$ and $\nabla \Delta u$. As we mentioned above, this proliferation of unknowns is significantly compensated by the fact that the only globally coupled degrees of freedom are those associated to approximations to u and Δu in the element borders. The convergence properties of the method are the following. When piecewise polynomial approximations of degree $k \geq 1$ are used for all these unknowns, the approximations to both u and ∇u optimally converge with order $k+1$, and that the approximations to Δu and $\nabla \Delta u$ converge with order $k+1/2$ and $k-1/2$, respectively. We also show that suitably chosen projections of the error in u superconverge with order $k+2$ for $k \geq 2$ and with order $\frac{5}{2}$ for $k=1$. This allows us to locally construct a new approximation to u converging with order $k+2$ for $k \geq 2$ and with order $\frac{5}{2}$ for $k=1$. We also show that we can improve the above estimates for $k=1$ in the two-dimensional case, $d=2$, and for $k=0$ and $d=2,3$. Indeed, for $k=1$ and $d=2$, we show that the postprocessed approximation of u converges with order 3, up to a logarithmic factor. For $k=0$ and $d=2,3$, we show that, again up to logarithmic factors, the approximation to u, ∇u, and Δu converge with orders $1, \frac{3}{4}$ and $\frac{1}{2}$, respectively. Finally, although

Certain technical assumptions were made in order to carry out our analysis and prove the above convergence rates. In particular, we assume the domain Ω is convex and we assume H^4 regularity for the dual problem. Moreover, we assume our family of meshes are quasi-uniform. However, our numerical experiments show that the method performs very well even if we violate these assumptions. Finally, although the converge rates for z and $\boldsymbol{\sigma}$ measured in the global L^2 norm are sub-optimal with rates $k+1/2$ and $k-1/2$, respectively, our numerical experiments show that the convergence rates are optimal in any fixed interior sub-domain of Ω.

The paper is organized as follows. In Section 2, we introduce the method, discuss the characterization of its approximate solution and state our *a priori* error estimates. The proof of the characterization is presented in Appendix I and the proofs of the error estimates are given in Section 3; they use an auxiliary result on pointwise error estimates for SFH approximations to solutions of second-order problems in Appendix II. In Section 4, we discuss the extension of our results to the two-dimensional case and $k=1$, and the two- and three-dimensional case and $k=0$. In Section 5, we display numerical experiments validating the theoretical results. We end with some concluding remarks in Section 6.

2. Main results

In this section, we introduce the SFH method and show how the method under consideration can be hybridized, that is, how the only globally coupled degrees of freedom are those of the numerical traces $\widehat{u}_h$ and $\widehat{z}_h$. We show that they are the solution of a mixed formulation and obtain conditions on the local stabilization parameters τ that guarantee the existence and uniqueness of its solution. We then present a priori error estimates for all the variables as well as two superconvergence results for approximations to u. We end by showing how to use the superconvergence results to construct, in an element-by-element fashion, a new approximation $u_h^\star$ converging faster than u.

2.1. **The SFH method.** Now let us introduce the method. We begin by rewriting the problem as a first-order system as follows:

$$\boldsymbol{\sigma} + \nabla z = 0 \quad \text{in } \Omega, \tag{2.2a}$$
$$\nabla \cdot \boldsymbol{\sigma} = f \quad \text{in } \Omega, \tag{2.2b}$$
$$\boldsymbol{q} + \nabla u = 0 \quad \text{in } \Omega, \tag{2.2c}$$
$$\nabla \cdot \boldsymbol{q} = z \quad \text{in } \Omega, \tag{2.2d}$$

with the boundary conditions

$$u = g \quad \text{on } \partial\Omega, \tag{2.2e}$$
$$\boldsymbol{q} \cdot \boldsymbol{n} = \mathsf{q_N} \quad \text{on } \partial\Omega. \tag{2.2f}$$

Next, let us introduce some notation. We denote by $\Omega_h = \{K\}$ a triangulation of the domain Ω of shape-regular tetrahedra K and set $\partial\Omega_h := \{\partial K : K \in \Omega_h\}$. We associate to this triangulation the set of interior faces $\mathscr{E}_h^i$ and the set of boundary faces $\mathscr{E}_h^\partial$. We say that $e \in \mathscr{E}_h^i$ if there are two simplexes K^+ and K^- in Ω_h such that $e = \partial K^+ \cap \partial K^-$, and we say that $e \in \mathscr{E}_h^\partial$ if there is a simplex in Ω_h such that $e = \partial K \cap \partial\Omega$. We set $\mathscr{E}_h := \mathscr{E}_h^i \cup \mathscr{E}_h^\partial$.

The SFH method seeks an approximation $(\boldsymbol{\sigma}_h, z_h, \boldsymbol{q}_h, u_h, \gamma_h, \lambda_h)$ to the exact solution $(\boldsymbol{\sigma}|_\Omega, z|_\Omega, \boldsymbol{q}|_\Omega, u|_\Omega, z|_{\mathscr{E}_h}, u|_{\mathscr{E}_h\setminus\partial\Omega})$, in a finite dimensional space $\boldsymbol{V}_h \times W_h \times \boldsymbol{V}_h \times W_h \times M_h \times M_h^0$ of the form

$$\boldsymbol{V}_h := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{v}|_K \in \boldsymbol{\mathcal{P}}^k(K) \quad \forall K \in \Omega_h\}, \tag{2.3a}$$
$$W_h := \{\omega \in L^2(\Omega) : \omega|_K \in \mathcal{P}^k(K) \quad \forall K \in \Omega_h\}, \tag{2.3b}$$
$$M_h := \{\mathsf{m} \in L^2(\partial\Omega_h) : \mathsf{m}|_e \in \mathcal{P}^k(e) \quad \forall e \in \mathscr{E}_h\}, \tag{2.3c}$$
$$M_h^0 := \{\mathsf{m} \in M_h : \mathsf{m}|_{\partial\Omega} = 0\}, \tag{2.3d}$$

and determines it by requiring that

$$(\boldsymbol{\sigma}_h, \boldsymbol{\rho})_{\Omega_h} - (z_h, \nabla\cdot\boldsymbol{\rho})_{\Omega_h} + \langle \widehat{z}_h, \boldsymbol{\rho}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} = 0, \tag{2.4a}$$
$$-(\boldsymbol{\sigma}_h, \nabla\eta)_{\Omega_h} + \langle \widehat{\boldsymbol{\sigma}}_h\cdot\boldsymbol{n}, \eta\rangle_{\partial\Omega_h} = (f, \eta)_{\Omega_h}, \tag{2.4b}$$
$$(\boldsymbol{q}_h, \boldsymbol{v})_{\Omega_h} - (u_h, \nabla\cdot\boldsymbol{v})_{\Omega_h} + \langle \widehat{u}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} = 0, \tag{2.4c}$$
$$-(\boldsymbol{q}_h, \nabla\omega)_{\Omega_h} + \langle \widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, \omega\rangle_{\partial\Omega_h} = (z_h, \omega)_{\Omega_h}, \tag{2.4d}$$
$$\langle \widehat{\boldsymbol{\sigma}}_h\cdot\boldsymbol{n}, \mu\rangle_{\partial\Omega_h} = 0, \tag{2.4e}$$
$$\langle \widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}, \chi\rangle_{\partial\Omega_h} = \langle \mathsf{q_N}, \chi\rangle_{\partial\Omega}, \tag{2.4f}$$

for all $(\boldsymbol{\rho}, \eta, \boldsymbol{v}, \omega, \mu, \chi) \in \boldsymbol{V}_h \times W_h \times \boldsymbol{V}_h \times W_h \times M_h^0 \times M_h$. Here, we denote the space of polynomials of degree at most $k \geq 0$ defined on D by $\mathcal{P}^k(D)$, and set

$\boldsymbol{\mathcal{P}}^k(D) := [\mathcal{P}^k(D)]^d$. We have used the notation

$$(\boldsymbol{\rho}, \boldsymbol{v})_{\Omega_h} := \sum_{K\in\Omega_h} \int_K \boldsymbol{\rho}(x)\cdot \boldsymbol{v}(x)\,dx,$$

$$(\eta, \omega)_{\Omega_h} := \sum_{K\in\Omega_h} \int_K \eta(x)\,\omega(x)\,dx,$$

$$\langle \eta, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} := \sum_{K\in\Omega_h} \int_{\partial K} \eta(\gamma)\,\boldsymbol{v}(\gamma)\cdot\boldsymbol{n}\,d\gamma,$$

for any functions $\boldsymbol{\rho}, \boldsymbol{v}$ in $\boldsymbol{H}^1(\Omega_h) := [H^1(\Omega_h)]^d$ and η, ω in $H^1(\Omega_h)$. The outward normal unit vector to ∂K is denoted by $\boldsymbol{n}$.

The numerical traces $(\widehat{\boldsymbol{\sigma}}_h, \widehat{z}_h, \widehat{\boldsymbol{q}}_h, \widehat{u}_h)$ are defined as

$$\widehat{u}_h = \begin{cases} \mathsf{P}_\partial\, g & \text{on } \partial K \cap \partial\Omega, \\ \lambda_h & \text{otherwise,} \end{cases} \tag{2.5a}$$

$$\widehat{z}_h = \gamma_h \tag{2.5b}$$

$$\widehat{\boldsymbol{q}}_h = \boldsymbol{q}_h + \tau\,(u_h - \widehat{u}_h)\boldsymbol{n}, \tag{2.5c}$$

$$\widehat{\boldsymbol{\sigma}}_h = \boldsymbol{\sigma}_h + \tau\,(z_h - \widehat{z}_h)\boldsymbol{n}, \tag{2.5d}$$

where $\lambda_h \in M_h^0$ and $\gamma_h \in M_h$ are called Lagrange multipliers which are unknown, and P_∂ denotes an L^2-projection defined as follows. Given any function $\eta \in L^2(\mathscr{E}_h)$ and an arbitrary face $e \in \mathscr{E}_h$, the restriction of $\mathsf{P}_\partial(\eta)$ to e is defined as the element of $\mathcal{P}^k(e)$ that satisfies

$$\langle \mathsf{P}_\partial \eta - \eta, \omega\rangle_e = 0, \qquad \forall\, \omega \in \mathcal{P}^k(e). \tag{2.6}$$

The parameter τ is taken as, on each simplex $K \in \Omega_h$

$$\tau = \begin{cases} 0, & \text{on } \partial K \setminus e_K^\tau, \\ \tau_K > 0, & \text{on } e_K^\tau, \end{cases} \tag{2.7}$$

where e_K^τ is an arbitrary but fixed face of K if K does not contain any boundary face.

We also make an important assumption on the triangulation Ω_h. We assume that

(2.8a) each tetrahedra K has *at most* one boundary face,

(2.8b) if K has one boundary face, e_K^τ is the boundary face.

Experimentally, we have verified that if this assumption is violated then the approximate solution is not well defined. This assumption is thus necessary.

2.2. **Characterization of the approximate solution.** Next, we give a characterization of the approximate solution provided by the SFH method; we follow [11]. To state it, we need to introduce the *local solvers* associated with the method.

The first local solver is defined on the simplex $K \in \Omega_h$ as the mapping $\gamma \in L^2(\partial K) \to (\boldsymbol{\mathcal{S}}\gamma,\, \mathcal{Z}\gamma, \boldsymbol{\mathcal{Q}}\gamma,\, \mathcal{U}\gamma) \in \boldsymbol{\mathcal{P}}^k(K) \times \mathcal{P}^k(K) \times \boldsymbol{\mathcal{P}}^k(K) \times \mathcal{P}^k(K)$ where

$$(\boldsymbol{\mathcal{S}}\gamma, \boldsymbol{\rho})_K - (\mathcal{Z}\gamma, \nabla\cdot\boldsymbol{\rho})_K = -\langle \gamma, \boldsymbol{\rho}\cdot\boldsymbol{n}\rangle_{\partial K}, \tag{2.9a}$$

$$-(\boldsymbol{\mathcal{S}}\gamma, \nabla\eta)_K + \langle \widehat{\boldsymbol{\mathcal{S}}}\gamma\cdot\boldsymbol{n}, \eta\rangle_{\partial K} = 0, \tag{2.9b}$$

$$(\boldsymbol{\mathcal{Q}}\gamma, \boldsymbol{v})_K - (\mathcal{U}\gamma, \nabla\cdot\boldsymbol{v})_K = 0, \tag{2.9c}$$

$$-(\boldsymbol{\mathcal{Q}}\gamma, \nabla w)_K + \langle \widehat{\boldsymbol{\mathcal{Q}}}\gamma\cdot\boldsymbol{n}, w\rangle_{\partial K} = (\mathcal{Z}\gamma, w)_K, \tag{2.9d}$$

for all $(\boldsymbol{\rho}, \eta, \boldsymbol{v}, w) \in \boldsymbol{\mathcal{P}}^k(K) \times \mathcal{P}^k(K) \times \boldsymbol{\mathcal{P}}^k(K) \times \mathcal{P}^k(K)$, where

$$\widehat{\boldsymbol{\mathcal{S}}}\gamma = \boldsymbol{\mathcal{S}}\gamma + \tau(\mathcal{Z}\gamma - \mathsf{P}_\partial\gamma)\boldsymbol{n}, \tag{2.9e}$$

$$\widehat{\boldsymbol{\mathcal{Q}}}\gamma = \boldsymbol{\mathcal{Q}}\gamma + \tau\,\mathcal{U}\gamma\boldsymbol{n}. \tag{2.9f}$$

The second local solver is defined on the simplex $K \in \Omega_h$ as the mapping $\mathsf{m} \in L^2(\partial K) \to (\boldsymbol{\mathcal{S}}\mathsf{m},\, \mathcal{Z}\mathsf{m}, \boldsymbol{\mathcal{Q}}\mathsf{m},\, \mathcal{U}\mathsf{m}) \in \boldsymbol{\mathcal{P}}^k(K) \times \mathcal{P}^k(K) \times \boldsymbol{\mathcal{P}}^k(K) \times \mathcal{P}^k(K)$ where

$$(\boldsymbol{\mathcal{S}}\mathsf{m}, \boldsymbol{\rho})_K - (\mathcal{Z}\mathsf{m}, \nabla\cdot\boldsymbol{\rho})_K = 0, \tag{2.10a}$$

$$-(\boldsymbol{\mathcal{S}}\mathsf{m}, \nabla\eta)_K + \langle \widehat{\boldsymbol{\mathcal{S}}}\mathsf{m}\cdot\boldsymbol{n}, \eta\rangle_{\partial K} = 0, \tag{2.10b}$$

$$(\boldsymbol{\mathcal{Q}}\mathsf{m}, \boldsymbol{v})_K - (\mathcal{U}\mathsf{m}, \nabla\cdot\boldsymbol{v})_K = -\langle \mathsf{m}, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K}, \tag{2.10c}$$

$$-(\boldsymbol{\mathcal{Q}}\mathsf{m}, \nabla w)_K + \langle \widehat{\boldsymbol{\mathcal{Q}}}\mathsf{m}\cdot\boldsymbol{n}, w\rangle_{\partial K} = (\mathcal{Z}\mathsf{m}, w)_K, \tag{2.10d}$$

for all $(\boldsymbol{\rho}, \eta, \boldsymbol{v}, w) \in \boldsymbol{\mathcal{P}}^k(K) \times \mathcal{P}^k(K) \times \boldsymbol{\mathcal{P}}^k(K) \times \mathcal{P}^k(K)$, where

$$\widehat{\boldsymbol{\mathcal{S}}}\mathsf{m} = \boldsymbol{\mathcal{S}}\mathsf{m} + \tau\,\mathcal{Z}\mathsf{m}\,\boldsymbol{n}, \tag{2.10e}$$

$$\widehat{\boldsymbol{\mathcal{Q}}}\mathsf{m} = \boldsymbol{\mathcal{Q}}\mathsf{m} + \tau(\mathcal{U}\mathsf{m} - \mathsf{P}_\partial\mathsf{m})\boldsymbol{n}. \tag{2.10f}$$

The third local solver is defined on the simplex $K \in \Omega_h$ as the mapping $f \in L^2(K) \to (\boldsymbol{\mathcal{S}}f,\, \mathcal{Z}f, \boldsymbol{\mathcal{Q}}f,\, \mathcal{U}f) \in \boldsymbol{\mathcal{P}}^k(K) \times \mathcal{P}^k(K) \times \boldsymbol{\mathcal{P}}^k(K) \times \mathcal{P}^k(K)$ where

$$(\boldsymbol{\mathcal{S}}f, \boldsymbol{\rho})_K - (\mathcal{Z}f, \nabla\cdot\boldsymbol{\rho})_K = 0, \tag{2.11a}$$

$$-(\boldsymbol{\mathcal{S}}f, \nabla\eta)_K + \langle \widehat{\boldsymbol{\mathcal{S}}}f\cdot\boldsymbol{n}, \eta\rangle_{\partial K} = (f, \eta)_K, \tag{2.11b}$$

$$(\boldsymbol{\mathcal{Q}}f, \boldsymbol{v})_K - (\mathcal{U}f, \nabla\cdot\boldsymbol{v})_K = 0, \tag{2.11c}$$

$$-(\boldsymbol{\mathcal{Q}}f, \nabla w)_K + \langle \widehat{\boldsymbol{\mathcal{Q}}}f\cdot\boldsymbol{n}, w\rangle_{\partial K} = (\mathcal{Z}f, w)_K, \tag{2.11d}$$

for all $(\boldsymbol{\rho}, \eta, \boldsymbol{v}, w) \in \boldsymbol{\mathcal{P}}^k(K) \times \mathcal{P}^k(K) \times \boldsymbol{\mathcal{P}}^k(K) \times \mathcal{P}^k(K)$, where

$$\widehat{\boldsymbol{\mathcal{S}}}f = \boldsymbol{\mathcal{S}}f + \tau\,\mathcal{Z}f\,\boldsymbol{n}, \tag{2.11e}$$

$$\widehat{\boldsymbol{\mathcal{Q}}}f = \boldsymbol{\mathcal{Q}}f + \tau\,\mathcal{U}f\,\boldsymbol{n}. \tag{2.11f}$$

We can now state a characterization of the approximation solutions in terms of the local solvers.

Theorem 2.1. *The approximate solution* $(\boldsymbol{\sigma}_h, z_h, \boldsymbol{q}_h, u_h, \gamma_h, \lambda_h) \in \boldsymbol{V}_h \times W_h \times \boldsymbol{V}_h \times W_h \times M_h \times M_h^0$ *given by the method is well defined. Moreover, we have that*

$$\begin{aligned}(\boldsymbol{\sigma}_h, z_h, \boldsymbol{q}_h, u_h) =& (\boldsymbol{\mathcal{S}}\gamma_h,\, \mathcal{Z}\gamma_h, \boldsymbol{\mathcal{Q}}\gamma_h,\, \mathcal{U}\gamma_h)\\ &+ (\boldsymbol{\mathcal{S}}\lambda_h,\, \mathcal{Z}\lambda_h, \boldsymbol{\mathcal{Q}}\lambda_h,\, \mathcal{U}\lambda_h)\\ &+ (\boldsymbol{\mathcal{S}}g,\, \mathcal{Z}g, \boldsymbol{\mathcal{Q}}g,\, \mathcal{U}g)\\ &+ (\boldsymbol{\mathcal{S}}f,\, \mathcal{Z}f, \boldsymbol{\mathcal{Q}}f,\, \mathcal{U}f),\end{aligned}$$

where $(\gamma_h, \lambda_h) \in M_h \times M_h^0$ *satisfies*

$$\begin{aligned} a_h(\gamma_h, \chi) + b_h(\lambda_h, \chi) &= \ell_1(\chi) \qquad \forall \chi \in M_h,\\ b_h(\mu, \gamma_h) \qquad\qquad &= \ell_2(\mu) \qquad \forall \mu \in M_h^0,\end{aligned}$$

where

$$\begin{aligned} a_h(\varsigma, \chi) &:= (\mathcal{Z}\varsigma,\, \mathcal{Z}\chi)_{\Omega_h},\\ b_h(\mu, \chi) &:= \langle \mu, \boldsymbol{\mathcal{S}}\chi \cdot \boldsymbol{n}\rangle_{\partial\Omega_h} = \langle \chi, \boldsymbol{\mathcal{Q}}\mu \cdot \boldsymbol{n}\rangle_{\partial\Omega_h},\\ \ell_1(\chi) &:= -(f,\, \mathcal{U}\chi)_{\Omega_h} - \langle g, \boldsymbol{\mathcal{S}}\chi \cdot \boldsymbol{n}\rangle_{\partial\Omega} + \langle \mathsf{q}_{\mathsf{N}}, \chi\rangle_{\partial\Omega},\\ \ell_2(\mu) &:= -(f,\, \mathcal{U}\mu)_{\Omega_h},\end{aligned}$$

for all $\varsigma, \chi \in M_h$, *and* $\mu \in M_h^0$. *The solution* $(\gamma_h, \lambda_h) \in M_h \times M_h^0$ *of the above formulation exists and is unique if* $\tau_K > 0$ *for each* $K \in \Omega_h$ *and if the conditions* (2.8) *are satisfied.*

A detailed proof of this result can be found in the Appendix I. Note that the above result implies that the system of equations for the vector of degrees of freedom for γ_h and λ_h, $[\gamma_h]$ and $[\lambda_h]$, respectively, is of the form

$$\begin{bmatrix} \mathsf{A} & \mathsf{B}\\ \mathsf{B}^t & 0\end{bmatrix} \begin{bmatrix} [\gamma_h]\\ [\lambda_h]\end{bmatrix} = \begin{bmatrix} b_1\\ b_2\end{bmatrix}.$$

This is a reflection of the fact that γ_h approximates z, that λ_h approximates u, and that we can write our original problem (1.1) as

$$\begin{bmatrix} Id & \Delta\\ \Delta & 0\end{bmatrix} \begin{bmatrix} z\\ u\end{bmatrix} = \begin{bmatrix} 0\\ -f\end{bmatrix}.$$

2.3. **A priori error estimates.** Now we present a priori error estimates for the approximation $(\boldsymbol{\sigma}_h, z_h, \boldsymbol{q}_h, u_h) \in \boldsymbol{V}_h \times W_h \times \boldsymbol{V}_h \times W_h$ given by the SFH method and for the numerical trace $\widehat{u}_h$ defined by (2.5). To state them, we need to introduce new notation.

For any real-valued function η in $H^l(\Omega_h)$, we set

$$|\,\eta\,|_{H^l(\Omega_h)} := \Big(\sum_{K\in\Omega_h} |\,\eta\,|^2_{H^l(K)}\Big)^{\frac{1}{2}}.$$

For a vector-valued function $\boldsymbol{\rho} = (\sigma_1, \dots, \sigma_d) \in \boldsymbol{H}^l(\Omega_h)$ we set

$$|\,\boldsymbol{\rho}\,|_{\boldsymbol{H}^l(\Omega_h)} := \Big(\sum_{i=1}^{d} |\,\sigma_i\,|^2_{H^l(\Omega_h)}\Big)^{\frac{1}{2}}.$$

Our error estimates for the SFH approximation to (1.1) will depend on L^∞ estimates of the SFH approximation to Laplace's equation. These L^∞ are contained in the appendix and assume elliptic H^2 regularity for Laplace's equation with zero

Dirichlet boundary conditions and estimates for the first derivative of the corresponding Green's function. This assumption is satisfied if Ω is *convex*; see [16]. Therefore, for our main theorem we assume that Ω is convex. Moreover, we assume that the family of meshes $\{\Omega_h\}$ be quasi-uniform since the L^∞ estimates require this. Finally, we will also assume H^4 elliptic regularity for the bi-harmonic problem which requires more than convexity; see [6] for results on polygons. More precisely, we assume that

$$\|\boldsymbol{\zeta}\|_{\boldsymbol{H}^1(\Omega_h)} + \|\xi\|_{\boldsymbol{H}^2(\Omega_h)} + \|\boldsymbol{\psi}\|_{\boldsymbol{H}^3(\Omega_h)} + \|\varphi\|_{\boldsymbol{H}^4(\Omega_h)} \leq \mathsf{C}_{er} \|\eta\|_{L^2(\Omega)}, \tag{2.12}$$

where $(\boldsymbol{\zeta}, \xi, \boldsymbol{\psi}, \varphi)$ is the solution of the following problem:

$$\boldsymbol{\zeta} + \nabla\xi = 0 \quad \text{in } \Omega, \tag{2.13a}$$

$$\nabla\cdot\boldsymbol{\zeta} = \eta \quad \text{in } \Omega, \tag{2.13b}$$

$$\boldsymbol{\psi} + \nabla\varphi = 0 \quad \text{in } \Omega, \tag{2.13c}$$

$$\nabla\cdot\boldsymbol{\psi} = \xi \quad \text{in } \Omega, \tag{2.13d}$$

$$\varphi = 0 \quad \text{on } \partial\Omega, \tag{2.13e}$$

$$\boldsymbol{\psi}\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\Omega, \tag{2.13f}$$

Again, we would like to emphasize that the convexity of the domain and H^4 elliptic regularity are just technical assumptions for the purpose of our error analysis. From the numerical experiment in Section 5.3, we see that the method still performs well when these assumptions are violated.

The L^2-errors in the approximation of $u, \boldsymbol{q}, z$ and $\boldsymbol{\sigma}$ as well as the error in the weighted jump $\tau\,(\widehat{u}_h - u_h)$ in the norm

$$\|\widehat{u}_h - u_h\|_{L^2(\partial\Omega_h;\tau)} := \Big(\sum_{K\in\Omega_h} \tau_K \,\|(\widehat{u}_h - u_h)\|^2_{L^2(e_K^\tau)}\Big)^{1/2},$$

are given in the following theorem.

We are now ready to state our first result.

Theorem 2.2. *Suppose that the exact solution* $(u, \boldsymbol{q}, z, \boldsymbol{\sigma})$ *belongs to* $H^{k+1}(\Omega_h) \times \boldsymbol{W}^{k+1,\infty}(\Omega_h) \times H^{k+1}(\Omega_h) \times \boldsymbol{H}^{k+1}(\Omega_h)$. *If* $\tau_K > 0$ *is of order* h_K^{-1} *for all* $K \in \Omega_h$ *and* $k \geq 1$, *then for* h *sufficiently small we have*

$$\begin{aligned} \|u - u_h\|_{L^2(\Omega_h)} &\leq \mathcal{C}\, h^{k+1}, \\ \|\boldsymbol{q} - \boldsymbol{q}_h\|_{\boldsymbol{L}^2(\Omega_h)} &\leq \mathcal{C}\, h^{k+1}, \\ \|z - z_h\|_{L^2(\Omega_h)} &\leq \mathcal{C}\, h^{k+1/2}, \\ \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\boldsymbol{L}^2(\Omega_h)} &\leq \mathcal{C}\, h^{k-1/2}, \end{aligned}$$

and

$$\|\widehat{u}_h - u_h\|_{L^2(\partial\Omega_h;\tau)} \leq \mathcal{C}\, h^{k+1},$$

where

$$\mathcal{C} := C(|u|_{H^{k+1}(\Omega_h)} + |\boldsymbol{q}|_{\boldsymbol{W}^{k+1,\infty}(\Omega_h)} + |z|_{H^{k+1}(\Omega_h)} + |\boldsymbol{\sigma}|_{\boldsymbol{H}^{k+1}(\Omega_h)}),$$

for some constant C *independent of* h *and the exact solution.*

Note that this result states that, for $k \geq 1$, the convergence orders of the approximations to u and $\boldsymbol{q}$ are optimal whereas those to z and $\boldsymbol{\sigma}$ are suboptimal by $1/2$ and $3/2$. Since we actually observe these orders in our numerical results, the above result is sharp. However, our numerical experiment suggest that the converge rates for all the variables are optimal for interior sub-domains of Ω.

We also note that if we don't assume $\boldsymbol{q} \in W^{k+1,\infty}(\Omega_h)$, it is possible to prove optimal convergence rates for u and $\boldsymbol{q}$, and order k and $k-1$ for z and $\boldsymbol{\sigma}$ without using L^∞ estimates of the SFH approximation to Laplace's equation. However, our analysis required the extra regularity of the solutions in order to improve the convergence rates for z and $\boldsymbol{\sigma}$.

2.4. **Superconvergence of $\widehat{u}_h$ and u_h.** Next we present a superconvergence result. To do that, we need to introduce the following norm:

$$\| \mathsf{P}_\partial u - \widehat{u}_h \|_{L^2(\mathscr{E}_h;h)} = \Big(\sum_{K\in\Omega_h} h_K \| \mathsf{P}_\partial u - \widehat{u}_h \|^2_{L^2(\partial K)} \Big)^{1/2}.$$

We also need to introduce the projection P^ℓ. Given a function $\eta \in H^1(\Omega_h)$ and an arbitrary simplex $K \in \Omega_h$, the restriction of $\mathsf{P}^\ell \eta$ to K is defined for $\ell \geq 0$ as the element of $\mathcal{P}^\ell(K)$ that satisfies

$$(\mathsf{P}^\ell \eta - \eta, \omega)_K = 0, \qquad \forall\, \omega \in \mathcal{P}^\ell(K). \tag{2.14}$$

If $\ell < 0$, then P^ℓ is the zero operator.

Theorem 2.3. *Under the same assumption as in Theorem 2.2, we have*

$$\begin{aligned} \|\mathsf{P}^{k-1}(u-u_h)\|_{L^2(\Omega_h)} &\leq C\,\mathcal{C}\,h^{k+\min\{k+1/2,2\}}, \\ \|\mathsf{P}_\partial u - \widehat{u}_h\|_{L^2(\mathscr{E}_h;h)} &\leq C\,\mathcal{C}\,h^{k+\min\{k+1/2,2\}}. \end{aligned}$$

A similar result was proven for the scalar variable of the SFH method as applied to second-order elliptic equations [10].

2.5. **Postprocessing.** Finally, we introduce a new approximation to u, $u_h^\star$, defined as follows; see [10, 12]. On the simplex K, $u_h^\star$, is the function of $\mathcal{P}^{k+1}(K)$ given by

$$u_h^\star = \overline{u}_h + \widetilde{u}_h, \tag{2.15a}$$

where

$$\overline{u}_h = \frac{1}{|K|}\int_K u_h\,dx \tag{2.15b}$$

and $\widetilde{u}_h$ is the polynomial in $\mathcal{P}_0^{k+1}(K)$ satisfying

$$(\nabla \widetilde{u}_h, \nabla w)_K = (z_h, w)_K - \langle w, \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n}\rangle_{\partial K} \quad \forall w \in \mathcal{P}_0^{k+1}(K). \tag{2.15c}$$

Here $\mathcal{P}_0^{k+1}(K)$ is the set of polynomials in $\mathcal{P}^{k+1}(K)$ with zero mean.

We have the following result.

Theorem 2.4. *Under the same assumption as in Theorem 2.2, we have*

$$\|u - u_h^\star\|_{L^2(\Omega_h)} \leq C\,\mathcal{C}\,h^{k+\min\{k+1/2,2\}}.$$

Notice that all our results above are stated for $k \geq 1$. We show results for the case $k = 0$ in the Extensions section since the proof is more delicate.

We note that many of the results in the following section hold for $k \geq 0$. When we assume that $k \geq 1$ we will explicitly state it.

3. Proof of the error estimates

In this section, we prove all the error estimates of Section 2. Since the analysis is quite involved, let us sketch its main steps. We begin by introducing a projection which will play a key role in our analysis, $(\boldsymbol{\Pi}, \mathbb{P})$. We then use energy-like arguments to obtain preliminary estimates for $\boldsymbol{\Pi}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)$, $\boldsymbol{\Pi}(\boldsymbol{q}-\boldsymbol{q}_h)$ and $\mathbb{P}(z-z_h)$; a duality argument is then used to get a first estimate of $\mathbb{P}(u-u_h)$. Finally, we conclude by using approximation properties of the projection $(\boldsymbol{\Pi}, \mathbb{P})$. Let us point out that the estimate of $\mathbb{P}(z-z_h)$ turned out to be the most delicate. It needed the introduction of an auxiliary SFH approximation of a second-order elliptic problem as well as *pointwise* estimates error estimates presented in Appendix II. The boundary conditions we considered here are the root of this technical difficulty.

3.1. **Preliminaries: The projection $(\boldsymbol{\Pi}, \mathbb{P})$.** In this subsection, we recall the definition of the projection

$$(\boldsymbol{\Pi}, \mathbb{P}) : \boldsymbol{H}^1(\Omega_h) \times H^1(\Omega_h) \to \boldsymbol{V}_h \times W_h,$$

introduced and studied in [10].

Given a function $\boldsymbol{\rho} \in \boldsymbol{H}^1(\Omega_h)$ and an arbitrary simplex $K \in \Omega_h$, the restriction of $\boldsymbol{\Pi\rho}$ to K is defined as the element of $\boldsymbol{\mathcal{P}}^k(K)$ that satisfies

$$(\boldsymbol{\Pi\rho}-\boldsymbol{\rho}, \boldsymbol{v})_K = 0, \qquad \forall\, \boldsymbol{v} \in \boldsymbol{\mathcal{P}}^{k-1}(K), \text{ if } k \geq 1, \tag{3.16a}$$

$$\langle(\boldsymbol{\Pi\rho}-\boldsymbol{\rho})\cdot\boldsymbol{n}, \omega\rangle_e = 0, \qquad \forall\, \omega \in \mathcal{P}^k(e) \text{ and } e \neq e_K^\tau. \tag{3.16b}$$

Similarly, given a function $\eta \in H^1(\Omega_h)$ and an arbitrary simplex $K \in \Omega_h$, the restriction of $\mathbb{P}(\eta)$ to K is defined as the element of $\mathcal{P}^k(K)$ that satisfies

$$(\mathbb{P}\eta-\eta, \mathsf{w})_K = 0, \qquad \forall\, \mathsf{w} \in \mathcal{P}^{k-1}(K), \text{ if } k \geq 1, \tag{3.17a}$$

$$\langle\mathbb{P}\eta-\eta, \omega\rangle_{e_K^\tau} = 0, \qquad \forall\, \omega \in \mathcal{P}^k(e_K^\tau). \tag{3.17b}$$

A key property of these projections which will be constantly used in the analysis is contained in the following result which can be deduced from property (iii) of Proposition in [10].

Proposition 3.1. *We have*

$$\langle\mathbb{P}\eta-\mathsf{P}_\partial\eta, \boldsymbol{\Pi\rho}\cdot\boldsymbol{n}-\mathsf{P}_\partial\boldsymbol{\rho}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} = 0,$$

for all $(\boldsymbol{\rho}, \eta) \in \boldsymbol{H}^1(\Omega_h) \times H^1(\Omega_h)$.

3.2. **The error equations.** As it is customary, we begin by displaying the error equations we are going to use in the analysis. So, from the equations satisfied by the exact solution, (2.2), and those satisfied by the numerical approximation, (2.4), we obtain

$$\begin{aligned}
(\boldsymbol{e}_{\boldsymbol{\sigma}}, \boldsymbol{\rho})_{\Omega_h} - (\boldsymbol{e}_z, \nabla\cdot\boldsymbol{\rho})_{\Omega_h} + \langle z-\widehat{z}_h, \boldsymbol{\rho}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} &= 0,\\
-(\boldsymbol{e}_{\boldsymbol{\sigma}}, \nabla\eta)_{\Omega_h} + \langle(\boldsymbol{\sigma}-\widehat{\boldsymbol{\sigma}}_h)\cdot\boldsymbol{n}, \eta\rangle_{\partial\Omega_h} &= 0,\\
(\boldsymbol{e}_{\boldsymbol{q}}, \boldsymbol{v})_{\Omega_h} - (\boldsymbol{e}_u, \nabla\cdot\boldsymbol{v})_{\Omega_h} + \langle u-\widehat{u}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} &= 0,\\
-(\boldsymbol{e}_{\boldsymbol{q}}, \nabla\omega)_{\Omega_h} + \langle(\boldsymbol{q}-\widehat{\boldsymbol{q}}_h)\cdot\boldsymbol{n}, \omega\rangle_{\partial\Omega_h} &= (\boldsymbol{e}_z, \omega)_{\Omega_h},\\
\langle(\boldsymbol{\sigma}-\widehat{\boldsymbol{\sigma}}_h)\cdot\boldsymbol{n}, \mu\rangle_{\partial\Omega_h} &= 0,\\
\langle(\boldsymbol{q}-\widehat{\boldsymbol{q}}_h)\cdot\boldsymbol{n}, \chi\rangle_{\partial\Omega_h} &= 0,
\end{aligned}$$

for all $(\boldsymbol{\rho},\eta,\boldsymbol{v},\omega,\mu,\chi)\in \boldsymbol{V}_h\times W_h\times\boldsymbol{V}_h\times W_h\times M_h^0\times M_h$, where we set $e_p=p-p_h$, for $p=u,\boldsymbol{q},z$ and $\boldsymbol{\sigma}$.

Using the orthogonality property of $\mathbb{P}$, (3.17a), and that of $\boldsymbol{\Pi}$, (3.16a), we obtain, after some simple algebraic manipulations

$$\begin{aligned}
(\boldsymbol{e}_{\boldsymbol{\sigma}},\boldsymbol{\rho})_{\Omega_h}-(\boldsymbol{e}_z,\nabla\cdot\boldsymbol{\rho})_{\Omega_h}+\langle z-\widehat{z}_h,\boldsymbol{\rho}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}&=0,\\
-(\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}},\nabla\eta)_{\Omega_h}+\langle(\boldsymbol{\sigma}-\widehat{\boldsymbol{\sigma}}_h)\cdot\boldsymbol{n},\eta\rangle_{\partial\Omega_h}&=0,\\
(\boldsymbol{e}_{\boldsymbol{q}},\boldsymbol{v})_{\Omega_h}-(\mathbb{P}\boldsymbol{e}_u,\nabla\cdot\boldsymbol{v})_{\Omega_h}+\langle u-\widehat{u}_h,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}&=0,\\
-(\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{q}},\nabla\omega)_{\Omega_h}+\langle(\boldsymbol{q}-\widehat{\boldsymbol{q}}_h)\cdot\boldsymbol{n},\omega\rangle_{\partial\Omega_h}&=(\boldsymbol{e}_z,\omega)_{\Omega_h},\\
\langle(\boldsymbol{\sigma}-\widehat{\boldsymbol{\sigma}}_h)\cdot\boldsymbol{n},\mu\rangle_{\partial\Omega_h}&=0,\\
\langle(\boldsymbol{q}-\widehat{\boldsymbol{q}}_h)\cdot\boldsymbol{n},\chi\rangle_{\partial\Omega_h}&=0,
\end{aligned}$$

for all $(\boldsymbol{\rho},\eta,\boldsymbol{v},\omega,\mu,\chi)\in \boldsymbol{V}_h\times W_h\times\boldsymbol{V}_h\times W_h\times M_h^0\times M_h$.

Finally, after integrating by parts and using the definition of the projections P_∂, (2.6), we get the form of the error equations we are going to use:

$$(\boldsymbol{e}_{\boldsymbol{\sigma}},\boldsymbol{\rho})_{\Omega_h}-(\mathbb{P}\boldsymbol{e}_z,\nabla\cdot\boldsymbol{\rho})_{\Omega_h}+\langle z-\widehat{z}_h,\boldsymbol{\rho}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}=0. \tag{3.18a}$$

$$(\nabla\cdot\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}},\eta)_{\Omega_h}+\langle(\mathsf{P}_\partial\boldsymbol{\sigma}-\widehat{\boldsymbol{\sigma}}_h)\cdot\boldsymbol{n}-\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}}\cdot\boldsymbol{n},\eta\rangle_{\partial\Omega_h}=0, \tag{3.18b}$$

$$(\boldsymbol{e}_{\boldsymbol{q}},\boldsymbol{v})_{\Omega_h}-(\mathbb{P}\boldsymbol{e}_u,\nabla\cdot\boldsymbol{v})_{\Omega_h}+\langle u-\widehat{u}_h,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}=0, \tag{3.18c}$$

$$(\nabla\cdot\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{q}},\omega)_{\Omega_h}+\langle(\mathsf{P}_\partial\boldsymbol{q}-\widehat{\boldsymbol{q}}_h)\cdot\boldsymbol{n}-\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{q}}\cdot\boldsymbol{n},\omega\rangle_{\partial\Omega_h}=(\boldsymbol{e}_z,\omega)_{\Omega_h}, \tag{3.18d}$$

$$\langle(\boldsymbol{\sigma}-\widehat{\boldsymbol{\sigma}}_h)\cdot\boldsymbol{n},\mu\rangle_{\partial\Omega_h}=0, \tag{3.18e}$$

$$\langle(\boldsymbol{q}-\widehat{\boldsymbol{q}}_h)\cdot\boldsymbol{n},\chi\rangle_{\partial\Omega_h}=0, \tag{3.18f}$$

for all $(\boldsymbol{\rho},\eta,\boldsymbol{v},\omega,\mu,\chi)\in \boldsymbol{V}_h\times W_h\times\boldsymbol{V}_h\times W_h\times M_h^0\times M_h$.

3.3. **Some properties of $\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}}$.** We begin our analysis by obtaining a few simple properties of the error in $\boldsymbol{\sigma}$. They follow directly from the error equation (3.18b) and the special choice of the stabilization parameter τ, (2.7).

Lemma 3.2. *For each simplex $K\in\Omega_h$, we have that,*

$$(\widehat{\boldsymbol{\sigma}}_h-\boldsymbol{\sigma}_h)\cdot\boldsymbol{n}=\tau\,(z_h-\widehat{z}_h)=\mathsf{P}_\partial\boldsymbol{\sigma}\cdot\boldsymbol{n}-\boldsymbol{\Pi}\boldsymbol{\sigma}\cdot\boldsymbol{n}\qquad \text{on } \partial K. \tag{3.19}$$

Moreover, $\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}}\in H(div,\Omega)$ and

$$\nabla\cdot\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}}=0. \tag{3.20}$$

Proof. Setting $\mathsf{Z}:=\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}}$ and $\mathsf{w}:=(\mathsf{P}_\partial\boldsymbol{\sigma}-\widehat{\boldsymbol{\sigma}}_h)\cdot\boldsymbol{n}-\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}}\cdot\boldsymbol{n}$, we can see that the error equation (3.18b), becomes

$$(\nabla\cdot\mathsf{Z},\eta)_K+\langle\mathsf{w},\eta\rangle_{\partial K}=0\quad\forall\eta\in\mathcal{P}^k(K).$$

Note that $\mathsf{Z}\in\boldsymbol{\mathcal{P}}^k(K)$, $\mathsf{w}|_{e_K^\tau}\in\mathcal{P}^k(e_K^\tau)$, and that, on any face on $\partial K\setminus e_K^\tau$,

$$\mathsf{w}=\mathsf{P}_\partial\boldsymbol{e}_{\boldsymbol{\sigma}}\cdot\boldsymbol{n}-\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}}\cdot\boldsymbol{n}=0,$$

by definition of $\widehat{\boldsymbol{\sigma}}_h$, (2.5d), by definition of τ, (2.7), and by the orthogonality property of $\boldsymbol{\Pi}$, (3.16b). In Lemma 3.1 in [10], it was shown that this implies that $\mathsf{w}|_{\partial K}=0$ and that $\nabla\cdot\mathsf{Z}=0$, that is, that (3.19) and (3.20) hold.

To see that $\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}}\in H(div,\Omega)$, we note that, on any interior face of normal $\boldsymbol{n}$,

$$\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}}\cdot\boldsymbol{n}=(\boldsymbol{\Pi}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\cdot\boldsymbol{n}=(\mathsf{P}_\partial\boldsymbol{\sigma}-\widehat{\boldsymbol{\sigma}}_h)\cdot\boldsymbol{n},$$

by (3.19), and the result follows because the normal component of $\widehat{\boldsymbol{\sigma}}_h$ is single valued by the error equation (3.18e). This completes the proof. □

3.4. **A first estimate of $\mathbf{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}}$.** In what follows, we are going to be using the following seminorms for functions $\xi \in L^2(\partial\Omega_h)$:

$$\| \xi \|_{L^2(\partial\Omega_h;\rho)} := \Big(\sum_{e_K^\tau \subset \partial\Omega_h} \rho \, \| \xi \|^2_{L^2(e_K^\tau)} \Big)^{1/2},$$
$$\| \xi \|_{L^2(\partial\Omega;\rho)} := \Big(\sum_{e_K^\tau \subset \partial\Omega} \rho \, \| \xi \|^2_{L^2(e_K^\tau)} \Big)^{1/2},$$

where $\rho|_{e_K^\tau} \geq 0$ for all $K \in \Omega_h$. We are now ready to obtain a first estimate of the quantity $\mathbf{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}}$. It is stated in terms of the following quantity:

$$\kappa_{\partial\Omega} := \max_{K \in \Omega_h : \overline{K} \cap \partial\Omega \neq \emptyset} \big(h_K^{-1} \tau_K^{-1} \big).$$

Lemma 3.3. *We have that*

$$\begin{aligned} \|\mathbf{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}}\|_{L^2(\Omega_h)} \leq \| \mathbf{\Pi}\boldsymbol{\sigma} - \boldsymbol{\sigma} \|_{L^2(\Omega_h)} &+ C\, h^{-1} \, \| \mathbb{P} e_z \|_{L^2(\Omega_h)}, \\ &+ C\, \kappa_{\partial\Omega} \, \| (\mathsf{P}_\partial \boldsymbol{\sigma} - \mathbf{\Pi}\boldsymbol{\sigma}) \cdot \boldsymbol{n} \|_{L^2(\partial\Omega;h)}. \end{aligned}$$

Proof. Taking $\boldsymbol{\rho} := \mathbf{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}}$ in the error equation (3.18a), we obtain

$$\begin{aligned} \|\mathbf{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}}\|^2_{L^2(\Omega_h)} &= (\mathbf{\Pi}\boldsymbol{\sigma} - \boldsymbol{\sigma}, \mathbf{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}})_{\Omega_h} + (\mathbb{P} e_z, \nabla \cdot \mathbf{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}})_{\Omega_h} - \langle z - \widehat{z}_h, \mathbf{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}} \cdot \boldsymbol{n} \rangle_{\partial\Omega_h}. \\ &= (\mathbf{\Pi}\boldsymbol{\sigma} - \boldsymbol{\sigma}, \mathbf{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}})_{\Omega_h} - \langle z - \widehat{z}_h, \mathbf{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}} \cdot \boldsymbol{n} \rangle_{\partial\Omega}, \end{aligned}$$

since, by Lemma 3.2, $\mathbf{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}} \in H(div, \Omega)$ and $\nabla \cdot \mathbf{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}} = 0$. Now, by the assumption (2.8), each of the faces e lying on $\partial\Omega$ coincides with a face e_K^τ for some $K \in \Omega_h$, we can write that

$$\|\mathbf{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}}\|^2_{L^2(\Omega_h)} = (\mathbf{\Pi}\boldsymbol{\sigma} - \boldsymbol{\sigma}, \mathbf{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}})_{\Omega_h} - \sum_{e_K^\tau \subset \partial\Omega} \langle z - \widehat{z}_h, \mathbf{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}} \cdot \boldsymbol{n} \rangle_{e_K^\tau},$$

and, by the orthogonality property of $\mathbb{P}$ (3.17b), that

$$\|\mathbf{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}}\|^2_{L^2(\Omega_h)} = (\mathbf{\Pi}\boldsymbol{\sigma} - \boldsymbol{\sigma}, \mathbf{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}})_{\Omega_h} - \sum_{e_K^\tau \subset \partial\Omega} \langle \mathbb{P} z - \widehat{z}_h, \mathbf{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}} \cdot \boldsymbol{n} \rangle_{e_K^\tau}.$$

We now use the identity for $\widehat{z}_h$ given in (3.19), to get

$$\begin{aligned} \|\mathbf{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}}\|^2_{L^2(\Omega_h)} &= (\mathbf{\Pi}\boldsymbol{\sigma} - \boldsymbol{\sigma}, \mathbf{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}})_{\Omega_h} - \sum_{e_K^\tau \subset \partial\Omega} \langle \mathbb{P} e_z, \mathbf{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}} \cdot \boldsymbol{n} \rangle_{e_K^\tau}, \\ &\quad - \sum_{e_K^\tau \subset \partial\Omega} \langle \tau_K^{-1} \, (\mathsf{P}_\partial \sigma - \mathbf{\Pi}\sigma) \cdot \boldsymbol{n}, \mathbf{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}} \cdot \boldsymbol{n} \rangle_{e_K^\tau}, \end{aligned}$$

and then *weighted* Cauchy-Schwarz inequalities, to obtain

$$\begin{aligned} \|\mathbf{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}}\|^2_{L^2(\Omega_h)} \leq\ & \| \mathbf{\Pi}\boldsymbol{\sigma} - \boldsymbol{\sigma} \|_{L^2(\Omega_h)} \, \| \mathbf{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}} \|_{L^2(\Omega_h)} \\ &+ \| \mathbb{P} e_z \|_{L^2(\partial\Omega;1/h)} \, \| \mathbf{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}} \cdot \boldsymbol{n} \|_{L^2(\partial\Omega;h)}, \\ &+ \kappa_{\partial\Omega} \, \| (\mathsf{P}_\partial \sigma - \mathbf{\Pi}\sigma) \cdot \boldsymbol{n} \|_{L^2(\partial\Omega;h)} \, \| \mathbf{\Pi} \boldsymbol{e}_{\boldsymbol{\sigma}} \cdot \boldsymbol{n} \|_{L^2(\partial\Omega;h)}. \end{aligned}$$

The estimate now follows after the application of a standard inverse inequality. This completes the proof. $\square$

3.5. **A first estimate of $\boldsymbol{\Pi e_q}$.** Next, we obtain a preliminary estimate of $\boldsymbol{\Pi e_q}$. To state it, we need to introduce the following quantity:

$$\kappa_\Omega := \max_{K\in\Omega_h} (\tau_K^{-1} h_K^{-1}).$$

Lemma 3.4. *We have*

$$\begin{aligned}\|\boldsymbol{\Pi e_q}\|^2_{L^2(\Omega_h)} &+ \| \widehat{u}_h - u_h \|^2_{L^2(\partial\Omega_h;\tau)}\\ &\le C \Big(\|\boldsymbol{\Pi q} - \boldsymbol{q}\|^2_{L^2(\Omega_h)} + \|\boldsymbol{e}_z\|_{L^2(\Omega_h)} \|\mathbb{P}\boldsymbol{e}_u\|_{L^2(\Omega_h)}\\ &\qquad + \kappa_\Omega \, \| (\boldsymbol{\Pi q} - \mathsf{P}_\partial \boldsymbol{q}) \cdot \boldsymbol{n}\|^2_{L^2(\partial\Omega_h;h)} \Big).\end{aligned}$$

Proof. Taking $\boldsymbol{v} := \boldsymbol{\Pi e_q}$ in the error equation (3.18c), we obtain that

$$\|\boldsymbol{\Pi e_q}\|^2_{L^2(\Omega_h)} = (\boldsymbol{\Pi q} - \boldsymbol{q}, \boldsymbol{\Pi e_q})_{\Omega_h} - \langle u - \widehat{u}_h, \boldsymbol{\Pi e_q} \cdot \boldsymbol{n}\rangle_{\partial\Omega_h} + (\mathbb{P}\boldsymbol{e}_u, \nabla\cdot \boldsymbol{\Pi e_q})_{\Omega_h},$$

and taking $\omega := \mathbb{P}\boldsymbol{e}_u$ in the error equation (3.18d), that

$$\begin{aligned}\|\boldsymbol{\Pi e_q}\|^2_{L^2(\Omega_h)} = &(\boldsymbol{\Pi q} - \boldsymbol{q}, \boldsymbol{\Pi e_q})_{\Omega_h} + (\boldsymbol{e}_z, \mathbb{P}\boldsymbol{e}_u)_{\Omega_h} - \langle u - \widehat{u}_h, \boldsymbol{\Pi e_q} \cdot \boldsymbol{n}\rangle_{\partial\Omega_h}\\ &- \langle (\mathsf{P}_\partial \boldsymbol{q} - \widehat{\boldsymbol{q}}_h)\cdot\boldsymbol{n} - \boldsymbol{\Pi e_q}\cdot\boldsymbol{n}, \mathbb{P}\boldsymbol{e}_u\rangle_{\partial\Omega_h}.\end{aligned}$$

Let us work on the last two terms of the above right-hand side. We have

$$\begin{aligned}T :=& - \langle u - \widehat{u}_h, \boldsymbol{\Pi e_q} \cdot \boldsymbol{n}\rangle_{\partial\Omega_h} - \langle (\mathsf{P}_\partial \boldsymbol{q} - \widehat{\boldsymbol{q}}_h)\cdot\boldsymbol{n} - \boldsymbol{\Pi e_q}\cdot\boldsymbol{n}, \mathbb{P}\boldsymbol{e}_u\rangle_{\partial\Omega_h}\\ =& \langle u - \widehat{u}_h - \mathbb{P}\boldsymbol{e}_u, (\mathsf{P}_\partial \boldsymbol{q} - \widehat{\boldsymbol{q}}_h)\cdot\boldsymbol{n} - \boldsymbol{\Pi e_q}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} - \langle \mathsf{P}_\partial u - \widehat{u}_h, (\boldsymbol{q} - \widehat{\boldsymbol{q}}_h)\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}\\ =& \langle u - \widehat{u}_h - \mathbb{P}\boldsymbol{e}_u, (\mathsf{P}_\partial \boldsymbol{q} - \widehat{\boldsymbol{q}}_h)\cdot\boldsymbol{n} - \boldsymbol{\Pi e_q}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}\end{aligned}$$

by the error equation (3.18f). Hence

$$\begin{aligned}T =& \langle (\mathsf{P}_\partial u - \mathbb{P}u) + (u_h - \widehat{u}_h), (\mathsf{P}_\partial \boldsymbol{q} - \boldsymbol{\Pi q})\cdot\boldsymbol{n} + (\boldsymbol{q}_h - \widehat{\boldsymbol{q}}_h)\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}\\ =& \langle \mathsf{P}_\partial u - \mathbb{P}u, (\mathsf{P}_\partial \boldsymbol{q} - \boldsymbol{\Pi q})\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} + \langle \mathsf{P}_\partial u - \mathbb{P}u, (\boldsymbol{q}_h - \widehat{\boldsymbol{q}}_h)\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}\\ &+ \langle u_h - \widehat{u}_h, (\mathsf{P}_\partial \boldsymbol{q} - \boldsymbol{\Pi q})\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} + \langle u_h - \widehat{u}_h, (\boldsymbol{q}_h - \widehat{\boldsymbol{q}}_h)\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}.\end{aligned}$$

The first term of the above right-hand side is equal to zero by Proposition 3.1. The second term is also equal to zero by the definition of the numerical flux $\widehat{\boldsymbol{q}}_h$, (2.5c), the definition of τ, (2.7), and the orthogonality property (3.17b) of the projection $\mathbb{P}$. Hence, we get

$$\begin{aligned}T =& \langle u_h - \widehat{u}_h, (\mathsf{P}_\partial \boldsymbol{q} - \boldsymbol{\Pi q})\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} + \langle u_h - \widehat{u}_h, (\boldsymbol{q}_h - \widehat{\boldsymbol{q}}_h)\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}\\ =& \langle u_h - \widehat{u}_h, (\mathsf{P}_\partial \boldsymbol{q} - \boldsymbol{\Pi q})\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} - \| \widehat{u}_h - u_h \|^2_{L^2(\partial\Omega_h;\tau)}.\\ =& \sum_{K\in\Omega_h} \langle u_h - \widehat{u}_h, (\mathsf{P}_\partial \boldsymbol{q} - \boldsymbol{\Pi q})\cdot\boldsymbol{n}\rangle_{e_K^\tau} - \| \widehat{u}_h - u_h \|^2_{L^2(\partial\Omega_h;\tau)},\end{aligned}$$

by the orthogonality property of the projection $\boldsymbol{\Pi}$, (3.16b). The result now easily follows after straightforward applications of Cauchy-Schwarz and Young's inequalities. This completes the proof. □

3.6. **A first estimate of $\mathbb{P}\boldsymbol{e}_z$.** Estimating $\mathbb{P}\boldsymbol{e}_z$ turns out to be considerably more involved than obtaining the previous estimates. We are going thus to proceed in three steps.

Step 1: An identity for $\|\mathbb{P}\boldsymbol{e}_z\|_{L^2(\Omega_h)}$. We begin by obtaining an expression for the L^2-norm of $\mathbb{P}\boldsymbol{e}_z$.

Lemma 3.5.

$$\begin{aligned}\|\mathbb{P}\boldsymbol{e}_z\|^2_{L^2(\Omega_h)} = &-(z-\mathbb{P}z,\mathbb{P}\boldsymbol{e}_z)_{\Omega_h} - (\boldsymbol{\Pi\sigma}-\boldsymbol{\sigma},\boldsymbol{\Pi e_q})_{\Omega_h} + (\boldsymbol{\Pi q}-\boldsymbol{q},\boldsymbol{\Pi e_\sigma})_{\Omega_h}\\ &+\sum_{K\in\Omega_h}\tau_K^{-1}\,\langle(\mathsf{P}_\partial\boldsymbol{q}-\boldsymbol{\Pi q})\cdot\boldsymbol{n},(\boldsymbol{\Pi\sigma}-\mathsf{P}_\partial\boldsymbol{\sigma})\cdot\boldsymbol{n}\rangle_{e_K^\tau}\\ &-\sum_{K\in\Omega_h}\langle(u_h-\widehat{u}_h)\cdot\boldsymbol{n},(\boldsymbol{\Pi\sigma}-\mathsf{P}_\partial\boldsymbol{\sigma})\cdot\boldsymbol{n}\rangle_{e_K^\tau}.\end{aligned}$$

Proof. Taking $\omega := \mathbb{P}\boldsymbol{e}_z$ in the error equation (3.18d), we obtain that

$$\begin{aligned}\|\mathbb{P}\boldsymbol{e}_z\|^2_{L^2(\Omega_h)} = &-(z-\mathbb{P}z,\mathbb{P}\boldsymbol{e}_z)_{\Omega_h} + (\nabla\cdot\boldsymbol{\Pi e_q},\mathbb{P}\boldsymbol{e}_z)_{\Omega_h}\\ &+\langle(\mathsf{P}_\partial\boldsymbol{q}-\widehat{\boldsymbol{q}}_h)\cdot\boldsymbol{n}-\boldsymbol{\Pi e_q}\cdot\boldsymbol{n},\mathbb{P}\boldsymbol{e}_z\rangle_{\partial\Omega_h},\end{aligned}$$

and taking $\boldsymbol{\rho}=\boldsymbol{\Pi e_q}$ in the error equation (3.18a), that

$$\begin{aligned}\|\mathbb{P}\boldsymbol{e}_z\|^2_{L^2(\Omega_h)} = &-(z-\mathbb{P}z,\mathbb{P}\boldsymbol{e}_z)_{\Omega_h} - (\boldsymbol{\Pi\sigma}-\boldsymbol{\sigma},\boldsymbol{\Pi e_q})_{\Omega_h} + (\boldsymbol{\Pi e_\sigma},\boldsymbol{\Pi e_q})_{\Omega_h}\\ &+\langle z-\widehat{z}_h,\boldsymbol{\Pi e_q}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}+\langle(\mathsf{P}_\partial\boldsymbol{q}-\widehat{\boldsymbol{q}}_h)\cdot\boldsymbol{n}-\boldsymbol{\Pi e_q}\cdot\boldsymbol{n},\mathbb{P}\boldsymbol{e}_z\rangle_{\partial\Omega_h}.\end{aligned}$$

Finally, taking $\boldsymbol{v}=\boldsymbol{\Pi e_\sigma}$ in the error equation (3.18c), and using the fact that, by Lemma 3.2, $\boldsymbol{\Pi e_\sigma}$ is in $H(div,\Omega)$ and that $\nabla\cdot\boldsymbol{\Pi e_\sigma}=0$, we get

$$\begin{aligned}\|\mathbb{P}\boldsymbol{e}_z\|^2_{L^2(\Omega_h)} = &-(z-\mathbb{P}z,\mathbb{P}\boldsymbol{e}_z)_{\Omega_h} - (\boldsymbol{\Pi\sigma}-\boldsymbol{\sigma},\boldsymbol{\Pi e_q})_{\Omega_h} + (\boldsymbol{\Pi q}-\boldsymbol{q},\boldsymbol{\Pi e_\sigma})_{\Omega_h}\\ &+\langle z-\widehat{z}_h,\boldsymbol{\Pi e_q}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}+\langle(\mathsf{P}_\partial\boldsymbol{q}-\widehat{\boldsymbol{q}}_h)\cdot\boldsymbol{n}-\boldsymbol{\Pi e_q}\cdot\boldsymbol{n},\mathbb{P}\boldsymbol{e}_z\rangle_{\partial\Omega_h}\\ = &-(z-\mathbb{P}z,\mathbb{P}\boldsymbol{e}_z)_{\Omega_h} - (\boldsymbol{\Pi\sigma}-\boldsymbol{\sigma},\boldsymbol{\Pi e_q})_{\Omega_h} + (\boldsymbol{\Pi q}-\boldsymbol{q},\boldsymbol{\Pi e_\sigma})_{\Omega_h}\\ &+\langle(\mathsf{P}_\partial\boldsymbol{q}-\widehat{\boldsymbol{q}}_h)\cdot\boldsymbol{n}-\boldsymbol{\Pi e_q}\cdot\boldsymbol{n},\mathbb{P}\boldsymbol{e}_z-(z-\widehat{z}_h)\rangle_{\partial\Omega_h}\\ &+\langle\mathsf{P}_\partial z-\widehat{z}_h,(\boldsymbol{q}-\widehat{\boldsymbol{q}}_h)\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}\\ = &-(z-\mathbb{P}z,\mathbb{P}\boldsymbol{e}_z)_{\Omega_h} - (\boldsymbol{\Pi\sigma}-\boldsymbol{\sigma},\boldsymbol{\Pi e_q})_{\Omega_h} + (\boldsymbol{\Pi q}-\boldsymbol{q},\boldsymbol{\Pi e_\sigma})_{\Omega_h}\\ &+\langle(\mathsf{P}_\partial\boldsymbol{q}-\widehat{\boldsymbol{q}}_h)\cdot\boldsymbol{n}-\boldsymbol{\Pi e_q}\cdot\boldsymbol{n},\mathbb{P}\boldsymbol{e}_z-(z-\widehat{z}_h\rangle_{\partial\Omega_h},\end{aligned}$$

by the error equation (3.18f). Rewriting the last term of the above right-hand side, we obtain

$$\begin{aligned}\|\mathbb{P}\boldsymbol{e}_z\|^2_{L^2(\Omega_h)} = &-(z-\mathbb{P}z,\mathbb{P}\boldsymbol{e}_z)_{\Omega_h} - (\boldsymbol{\Pi\sigma}-\boldsymbol{\sigma},\boldsymbol{\Pi e_q})_{\Omega_h} + (\boldsymbol{\Pi q}-\boldsymbol{q},\boldsymbol{\Pi e_\sigma})_{\Omega_h}\\ &+\langle(\mathsf{P}_\partial\boldsymbol{q}-\boldsymbol{\Pi q})\cdot\boldsymbol{n},\widehat{z}_h-z_h\rangle_{\partial\Omega_h}+\langle(\boldsymbol{q}_h-\widehat{\boldsymbol{q}}_h)\cdot\boldsymbol{n},\widehat{z}_h-z_h\rangle_{\partial\Omega_h}\\ &+\langle(\mathsf{P}_\partial\boldsymbol{q}-\boldsymbol{\Pi q})\cdot\boldsymbol{n},\mathbb{P}z-\mathsf{P}_\partial z\rangle_{\partial\Omega_h}+\langle(\boldsymbol{q}_h-\widehat{\boldsymbol{q}}_h)\cdot\boldsymbol{n},\mathbb{P}z-\mathsf{P}_\partial z\rangle_{\partial\Omega_h}.\end{aligned}$$

The before-the-last term of the above right-hand side is equal to zero by Proposition 3.1 and the last by the definition of the numerical trace $\widehat{\boldsymbol{q}}_h$, (2.5c), the definition of τ, (2.7), and the orthogonality property of the projection $\mathbb{P}$, (3.17b). As a consequence, we have that

$$\begin{aligned}\|\mathbb{P}\boldsymbol{e}_z\|^2_{L^2(\Omega_h)} =& -(z-\mathbb{P}z,\mathbb{P}\boldsymbol{e}_z)_{\Omega_h} - (\boldsymbol{\Pi\sigma}-\boldsymbol{\sigma},\boldsymbol{\Pi e_q})_{\Omega_h} + (\boldsymbol{\Pi q}-\boldsymbol{q},\boldsymbol{\Pi e_\sigma})_{\Omega_h}\\ &+\langle(\mathsf{P}_\partial\boldsymbol{q}-\boldsymbol{\Pi q})\cdot\boldsymbol{n},\widehat{z}_h-z_h\rangle_{\partial\Omega_h}+\langle(\boldsymbol{q}_h-\widehat{\boldsymbol{q}}_h)\cdot\boldsymbol{n},\widehat{z}_h-z_h\rangle_{\partial\Omega_h}\\ =& -(z-\mathbb{P}z,\mathbb{P}\boldsymbol{e}_z)_{\Omega_h} - (\boldsymbol{\Pi\sigma}-\boldsymbol{\sigma},\boldsymbol{\Pi e_q})_{\Omega_h} + (\boldsymbol{\Pi q}-\boldsymbol{q},\boldsymbol{\Pi e_\sigma})_{\Omega_h}\\ &+\sum_{K\in\Omega_h}\langle(\mathsf{P}_\partial\boldsymbol{q}-\boldsymbol{\Pi q})\cdot\boldsymbol{n},\widehat{z}_h-z_h\rangle_{e_K^\tau}\\ &-\sum_{K\in\Omega_h}\tau_K\,\langle\widehat{u}_h-u_h,\widehat{z}_h-z_h\rangle_{e_K^\tau},\end{aligned}$$

by the orthogonality property projection $\boldsymbol{\Pi}$ (3.16) and by the definition of the numerical trace $\widehat{\boldsymbol{q}}_h$, (2.5c). The result now follows by using the identity for $\widehat{z}_h$ (3.19). This completes the proof. □

Step 2: An identity for the term $(\boldsymbol{\Pi q}-\boldsymbol{q},\boldsymbol{\Pi e_\sigma})_{\Omega_h}$. If we use a simple Cauchy-Schwarz inequality to estimate the term $(\boldsymbol{\Pi q}-\boldsymbol{q},\boldsymbol{\Pi e_\sigma})_{\Omega_h}$, we would lose a factor $h^{1/2}$. To prevent this, we make use of the auxiliary approximation of $\boldsymbol{q}$, $\tilde{\boldsymbol{q}}_h$, we define next.

The function $(\tilde{\boldsymbol{q}}_h,\tilde{u}_h,\widehat{\tilde{u}}_h)$ is the element of $V_h\times W_h\times M_h$ satisfying the equations

$$(\tilde{\boldsymbol{q}}_h,\boldsymbol{v})_{\Omega_h}-(\tilde{u}_h,\nabla\cdot\boldsymbol{v})_{\Omega_h}+\langle\widehat{\tilde{u}}_h,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}=0, \tag{3.21a}$$

$$-(\tilde{\boldsymbol{q}}_h,\nabla\omega)_{\Omega_h}+\langle\widehat{\tilde{\boldsymbol{q}}}_h\cdot\boldsymbol{n},\omega\rangle_{\partial\Omega_h}=(z,\omega)_{\Omega_h}, \tag{3.21b}$$

$$\langle\widehat{\tilde{\boldsymbol{q}}}_h\cdot\boldsymbol{n},\chi\rangle_{\partial\Omega_h}=0, \tag{3.21c}$$

for every $(\boldsymbol{v},\omega,\mu)\in V_h\times W_h\times M_h^0$. Here

$$\widehat{\tilde{\boldsymbol{q}}}_h=\tilde{\boldsymbol{q}}_h+\tau(\tilde{u}_h-\widehat{\tilde{u}}_h)\boldsymbol{n},\quad\text{and}\quad\widehat{\tilde{u}}_h=\mathsf{P}_\partial g\quad\text{on }\partial\Omega. \tag{3.21d}$$

We are going to the properties of this function gathered in the following result; see [10] for the proof.

Proposition 3.6. *We have that* $\boldsymbol{\Pi q}-\tilde{\boldsymbol{q}}_h\in H(div,\Omega)$ *and* $\nabla\cdot(\boldsymbol{\Pi q}-\tilde{\boldsymbol{q}}_h)=0$.

Lemma 3.7. *We have*

$$\begin{aligned}(\boldsymbol{\Pi q}-\boldsymbol{q},\boldsymbol{\Pi e_\sigma})_{\Omega_h}=&(\boldsymbol{\Pi q}-\tilde{\boldsymbol{q}}_h,\boldsymbol{\Pi\sigma}-\boldsymbol{\sigma})_{\Omega_h}+\langle\mathbb{P}e_z,(\boldsymbol{\Pi q}-\tilde{\boldsymbol{q}}_h)\cdot\boldsymbol{n}\rangle_{\partial\Omega}\\ &+\sum_{e_K^\tau\subset\partial\Omega}\tau_K^{-1}\,\langle(\mathsf{P}_\partial\boldsymbol{\sigma}-\boldsymbol{\Pi\sigma})\cdot\boldsymbol{n},(\boldsymbol{\Pi q}-\tilde{\boldsymbol{q}}_h)\cdot\boldsymbol{n}\rangle_{e_K^\tau}.\end{aligned}$$

Proof. We have

$$\begin{aligned}T:=&(\boldsymbol{\Pi q}-\boldsymbol{q},\boldsymbol{\Pi e_\sigma})_{\Omega_h}\\ =&(\boldsymbol{\Pi q}-\tilde{\boldsymbol{q}}_h,\boldsymbol{\Pi e_\sigma})_{\Omega_h}+(\tilde{\boldsymbol{q}}_h-\boldsymbol{q},\boldsymbol{\Pi e_\sigma})_{\Omega_h}\\ =&(\boldsymbol{\Pi q}-\tilde{\boldsymbol{q}}_h,\boldsymbol{\Pi e_\sigma})_{\Omega_h}+(\tilde{u}_h-u,\nabla\cdot\boldsymbol{\Pi e_\sigma})_{\Omega_h}-\langle\widehat{\tilde{u}}_h-u,\boldsymbol{\Pi e_\sigma}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h},\end{aligned}$$

by the first equation defining $\tilde{\boldsymbol{q}}_h$ with $\boldsymbol{v} := \boldsymbol{\Pi e_\sigma}$, (3.21a), and since $\boldsymbol{q} = -\nabla u$ by the equation (2.2d). Now, since by Lemma 3.2, $\nabla \cdot \boldsymbol{\Pi e_\sigma} = 0$ and $\boldsymbol{\Pi e_\sigma} \in H(div, \Omega)$, we obtain that

$$\begin{aligned} T &= (\boldsymbol{\Pi q} - \tilde{\boldsymbol{q}}_h, \boldsymbol{\Pi e_\sigma})_{\Omega_h} - \langle \widehat{u}_h - u, \boldsymbol{\Pi e_\sigma} \cdot \boldsymbol{n} \rangle_{\partial\Omega} \\ &= (\boldsymbol{\Pi q} - \tilde{\boldsymbol{q}}_h, \boldsymbol{\Pi e_\sigma})_{\Omega_h}, \end{aligned}$$

by the definition of the numerical trace $\widehat{u}_h$ on $\partial\Omega_h$. Therefore, by the error equation (3.18a) with $\boldsymbol{\rho} := \boldsymbol{\Pi q}_h - \tilde{\boldsymbol{q}}_h$, we get

$$T = (\boldsymbol{\Pi q} - \tilde{\boldsymbol{q}}_h, \boldsymbol{\Pi \sigma} - \boldsymbol{\sigma})_{\Omega_h} + \langle z - \widehat{z}_h, (\boldsymbol{\Pi q} - \tilde{\boldsymbol{q}}_h) \cdot \boldsymbol{n} \rangle_{\partial\Omega},$$

since, by Proposition 3.6, $\nabla \cdot (\boldsymbol{\Pi q} - \tilde{\boldsymbol{q}}_h) = 0$ and $\boldsymbol{\Pi q} - \tilde{\boldsymbol{q}}_h \in H(div, \Omega)$. Now, by property (3.17b) of the projection $\mathbb{P}$,

$$\begin{aligned} T &= (\boldsymbol{\Pi q} - \tilde{\boldsymbol{q}}_h, \boldsymbol{\Pi \sigma} - \boldsymbol{\sigma})_{\Omega_h} + \langle \mathbb{P} z - \widehat{z}_h, (\boldsymbol{\Pi q} - \tilde{\boldsymbol{q}}_h) \cdot \boldsymbol{n} \rangle_{\partial\Omega} \\ &= (\boldsymbol{\Pi q} - \tilde{\boldsymbol{q}}_h, \boldsymbol{\Pi \sigma} - \boldsymbol{\sigma})_{\Omega_h} + \langle \mathbb{P} z - z_h, (\boldsymbol{\Pi q} - \tilde{\boldsymbol{q}}_h) \cdot \boldsymbol{n} \rangle_{\partial\Omega} \\ &\quad + \langle z_h - \widehat{z}_h, (\boldsymbol{\Pi q} - \tilde{\boldsymbol{q}}_h) \cdot \boldsymbol{n} \rangle_{\partial\Omega} \\ &= (\boldsymbol{\Pi q} - \tilde{\boldsymbol{q}}_h, \boldsymbol{\Pi \sigma} - \boldsymbol{\sigma})_{\Omega_h} + \langle \mathbb{P} e_z, (\boldsymbol{\Pi q} - \tilde{\boldsymbol{q}}_h) \cdot \boldsymbol{n} \rangle_{\partial\Omega} \\ &\quad + \sum_{e_K^\tau \subset \partial\Omega} \tau_K^{-1} \, \langle (\mathsf{P}_\partial \boldsymbol{\sigma} - \boldsymbol{\Pi \sigma}) \cdot \boldsymbol{n}, (\boldsymbol{\Pi q} - \tilde{\boldsymbol{q}}_h) \cdot \boldsymbol{n} \rangle_{e_K^\tau}, \end{aligned}$$

by the identity (3.19). This completes the proof. □

Step 3: Estimating $\|\mathbb{P}\boldsymbol{e}_z\|_{L^2(\Omega_h)}$.
The preliminary estimate of $\|\mathbb{P}\boldsymbol{e}_z\|_{L^2(\Omega_h)}$ is contained in the following result.

Corollary 3.8. *Let $l(k) = \log(\frac{1}{h})$ if $k = 0$ and $l(k)=1$ otherwise. Then,*

$$\begin{aligned} \|\mathbb{P}\boldsymbol{e}_z\|^2_{L^2(\Omega_h)} \le\ & \| z - \mathbb{P}z \|_{L^2(\Omega_h)} \, \| \mathbb{P}\boldsymbol{e}_z \|_{L^2(\Omega_h)} + \| \boldsymbol{\Pi\sigma} - \boldsymbol{\sigma} \|_{L^2(\Omega_h)} \, \| \boldsymbol{\Pi e_q} \|_{L^2(\Omega_h)} \\ & + \| \boldsymbol{\Pi q} - \boldsymbol{q} \|_{L^2(\Omega_h)} \, \| \boldsymbol{\Pi\sigma} - \boldsymbol{\sigma} \|_{L^2(\Omega_h)} \\ & + C \, \| \mathbb{P} e_z \|_{L^2(\Omega_h)} \times \\ & \quad \Big(h^{-1/2} \, \|\boldsymbol{q} - \boldsymbol{\Pi q}\|_{L^\infty(\Omega_h)} + l(k) h^{1/2} \, \|\nabla \cdot (\boldsymbol{q} - \boldsymbol{\Pi q})\|_{L^\infty(\Omega_h)} \Big) \\ & + C \, \kappa_{\partial\Omega} \, \| (\mathsf{P}_\partial \boldsymbol{\sigma} - \boldsymbol{\Pi\sigma}) \cdot \boldsymbol{n} \|_{L^2(\partial\Omega;h)} \, \| \boldsymbol{\Pi q} - \boldsymbol{q} \|_{L^2(\Omega_h)} \\ & + \kappa_\Omega \, \| (\mathsf{P}_\partial \boldsymbol{q} - \boldsymbol{\Pi q}) \cdot \boldsymbol{n} \|_{L^2(\partial\Omega_h;h)} \, \| (\boldsymbol{\Pi\sigma} - \mathsf{P}_\partial \boldsymbol{\sigma}) \cdot \boldsymbol{n} \|_{L^2(\partial\Omega_h;h)} \\ & + \kappa_\Omega^{1/2} \, \| u_h - \widehat{u}_h \|_{L^2(\partial\Omega_h;\tau)} \, \| (\boldsymbol{\Pi\sigma} - \mathsf{P}_\partial \boldsymbol{\sigma}) \cdot \boldsymbol{n} \|_{L^2(\partial\Omega_h;h)}. \end{aligned}$$

To prove this corollary, we need two key estimates contained in the following result.

Proposition 3.9. *We have*

$$\begin{aligned} &\|\boldsymbol{\Pi q} - \tilde{\boldsymbol{q}}_h\|_{L^2(\Omega_h)} \le \|\boldsymbol{q} - \boldsymbol{\Pi q}\|_{L^2(\Omega_h)}, \\ &\|\boldsymbol{\Pi q} - \tilde{\boldsymbol{q}}_h\|_{L^\infty(\Omega_h)} \le C \, \left(\|\boldsymbol{q} - \boldsymbol{\Pi q}\|_{L^\infty(\Omega_h)} + l(k) h \, \|\nabla \cdot (\boldsymbol{q} - \boldsymbol{\Pi q})\|_{L^\infty(\Omega_h)} \right). \end{aligned}$$

The first estimates was proven in [10]. The second is proven in the Appendix II; see Theorem 6.3. We are now ready to prove Corollary 3.8.

Proof. Applying weighted Cauchy-Schwarz inequalities to the identity of Lemma 3.5, we obtain

$$\begin{aligned}\|\mathbb{P}\boldsymbol{e}_z\|^2_{L^2(\Omega_h)} \le\ & \|\, z - \mathbb{P}z\,\|_{L^2(\Omega_h)}\, \|\,\mathbb{P}\boldsymbol{e}_z\,\|_{L^2(\Omega_h)} + \|\,\boldsymbol{\Pi\sigma} - \boldsymbol{\sigma}\,\|_{L^2(\Omega_h)}\, \|\,\boldsymbol{\Pi e_q}\,\|_{L^2(\Omega_h)}\\ &+ |\,(\boldsymbol{\Pi q} - \boldsymbol{q}, \boldsymbol{\Pi e_\sigma})_{\Omega_h}\,|\\ &+ \kappa_\Omega\, \|\,(\mathsf{P}_\partial \boldsymbol{q} - \boldsymbol{\Pi q})\cdot\boldsymbol{n}\,\|_{L^2(\partial\Omega_h;h)}\, \|\,(\boldsymbol{\Pi\sigma} - \mathsf{P}_\partial\boldsymbol{\sigma})\cdot\boldsymbol{n}\,\|_{L^2(\partial\Omega_h;h)}\\ &+ \kappa_\Omega^{1/2}\, \|\,u_h - \widehat{u}_h\,\|_{L^2(\partial\Omega_h;\tau)}\, \|\,(\boldsymbol{\Pi\sigma} - \mathsf{P}_\partial\boldsymbol{\sigma})\cdot\boldsymbol{n}\,\|_{L^2(\partial\Omega_h;h)}.\end{aligned}$$

We now use the expression of the third term of the above right-hand given by the identity of Lemma 3.7 to obtain

$$\begin{aligned}|\,(\boldsymbol{\Pi q} - \boldsymbol{q}, \boldsymbol{\Pi e_\sigma})_{\Omega_h}\,| \le\ & \|\,\boldsymbol{\Pi q} - \tilde{\boldsymbol{q}}_h\,\|_{L^2(\Omega_h)}\, \|\,\boldsymbol{\Pi\sigma} - \boldsymbol{\sigma}\,\|_{L^2(\Omega_h)}\\ &+ C\, \|\,\mathbb{P}e_z\,\|_{L^2(\partial\Omega)}\, \|\,(\boldsymbol{\Pi q} - \tilde{\boldsymbol{q}}_h)\cdot\boldsymbol{n}\,\|_{L^\infty(\partial\Omega)}\\ &+ \kappa_{\partial\Omega}\, \|\,(\mathsf{P}_\partial\boldsymbol{\sigma} - \boldsymbol{\Pi\sigma})\cdot\boldsymbol{n}\,\|_{L^2(\partial\Omega;h)}\, \|\,(\boldsymbol{\Pi q} - \tilde{\boldsymbol{q}}_h)\cdot\boldsymbol{n}\,\|_{L^2(\partial\Omega;h)}\\ \le\ & \|\,\boldsymbol{\Pi q} - \tilde{\boldsymbol{q}}_h\,\|_{L^2(\Omega_h)}\, \|\,\boldsymbol{\Pi\sigma} - \boldsymbol{\sigma}\,\|_{L^2(\Omega_h)}\\ &+ C\, \|\,\mathbb{P}e_z\,\|_{L^2(\partial\Omega)}\, \|\,\boldsymbol{\Pi q} - \tilde{\boldsymbol{q}}_h\,\|_{L^\infty(\Omega_h)}\\ &+ C\,\kappa_{\partial\Omega}\, \|\,(\mathsf{P}_\partial\boldsymbol{\sigma} - \boldsymbol{\Pi\sigma})\cdot\boldsymbol{n}\,\|_{L^2(\partial\Omega;h)}\, \|\,\boldsymbol{\Pi q} - \tilde{\boldsymbol{q}}_h\,\|_{L^2(\Omega_h)},\end{aligned}$$

by a standard inverse inequality. Finally, using the estimates of Proposition 3.9, we obtain that

$$\begin{aligned}|\,(\boldsymbol{\Pi q} - \boldsymbol{q}, \boldsymbol{\Pi e_\sigma})_{\Omega_h}\,| \le\ & \|\,\boldsymbol{\Pi q} - \boldsymbol{q}\,\|_{L^2(\Omega_h)}\, \|\,\boldsymbol{\Pi\sigma} - \boldsymbol{\sigma}\,\|_{L^2(\Omega_h)}\\ &+ C\,\|\,\mathbb{P}e_z\,\|_{L^2(\partial\Omega)}\times\\ &\quad \big(\|\boldsymbol{q} - \boldsymbol{\Pi q}\|_{L^\infty(\Omega_h)} + l(k)\, h\, \|\nabla\cdot(\boldsymbol{q} - \boldsymbol{\Pi q})\|_{L^\infty(\Omega_h)}\big)\\ &+ C\,\kappa_{\partial\Omega}\, \|\,(\mathsf{P}_\partial\boldsymbol{\sigma} - \boldsymbol{\Pi\sigma})\cdot\boldsymbol{n}\,\|_{L^2(\partial\Omega;h)}\, \|\,\boldsymbol{\Pi q} - \boldsymbol{q}\,\|_{L^2(\Omega_h)},\end{aligned}$$

and the result follows by using an inverse inequality. This completes the proof. □

3.7. **A first estimate of $\mathbb{P}e_u$.** To obtain the estimate of $\mathbb{P}e_u$, we begin by obtaining the following result.

Lemma 3.10. *We have*

$$\begin{aligned}\|\mathbb{P}\boldsymbol{e}_u\|^2_{L^2(\Omega_h)} =\ & (\boldsymbol{e_q}, \boldsymbol{\Pi\zeta} - \boldsymbol{\zeta})_{\Omega_h} + \sum_{K\in\Omega_h} \langle \widehat{u}_h - u_h, (\mathsf{P}_\partial\boldsymbol{\zeta} - \boldsymbol{\Pi\zeta})\cdot\boldsymbol{n}\rangle_{e_K^\tau}\\ &- (\boldsymbol{q} - \boldsymbol{\Pi q}, \nabla(\mathbb{P}\xi - \xi))_{\Omega_h} - (\boldsymbol{e}_z, \mathbb{P}\xi - \xi)_{\Omega_h}\\ &+ (\nabla(z - \mathbb{P}z), \boldsymbol{\psi} - \boldsymbol{\Pi\psi})_{\Omega_h}\\ &- \sum_{K\in\Omega_h} \tau_K^{-1}\, \langle(\mathsf{P}_\partial\boldsymbol{\sigma} - \boldsymbol{\Pi\sigma})\cdot\boldsymbol{n}, (\mathsf{P}_\partial\boldsymbol{\psi} - \boldsymbol{\Pi\psi})\cdot\boldsymbol{n}\rangle_{e_K^\tau}\\ &- (\boldsymbol{\sigma} - \boldsymbol{\Pi\sigma}, \boldsymbol{\Pi\psi} - \boldsymbol{\psi})_{\Omega_h} + (\boldsymbol{\sigma} - \boldsymbol{\Pi\sigma}, \nabla(\varphi - \mathbb{P}\varphi))_{\Omega_h}\\ &- (\boldsymbol{\Pi e_\sigma}, \boldsymbol{\Pi\psi} - \boldsymbol{\psi})_{\Omega_h}.\end{aligned}$$

Proof. By the adjoint equation (2.13b) with $\eta := \mathbb{P}e_u$, we have that

$$\begin{aligned}\|\mathbb{P}\boldsymbol{e}_u\|^2_{L^2(\Omega_h)} &= (\mathbb{P}\boldsymbol{e}_u, \nabla\cdot\boldsymbol{\zeta})_{\Omega_h}\\ &= (\mathbb{P}\boldsymbol{e}_u, \nabla\cdot\boldsymbol{\Pi\zeta})_{\Omega_h} + (\mathbb{P}\boldsymbol{e}_u, \nabla\cdot(\boldsymbol{\zeta} - \boldsymbol{\Pi\zeta}))_{\Omega_h}\\ &= (\mathbb{P}\boldsymbol{e}_u, \nabla\cdot\boldsymbol{\Pi\zeta})_{\Omega_h} + \langle\mathbb{P}\boldsymbol{e}_u, (\boldsymbol{\zeta} - \boldsymbol{\Pi\zeta})\cdot\boldsymbol{n}\rangle_{\partial\Omega_h},\end{aligned}$$

after integrating by parts and using the orthogonality property (3.16a) of the projection $\mathbf{\Pi}$. Now, taking $\boldsymbol{v} := \mathbf{\Pi}\boldsymbol{\zeta}$ in the error equation (3.18c), we obtain that

$$\begin{aligned}\|\mathbb{P}\boldsymbol{e}_u\|^2_{L^2(\Omega_h)} &= (\boldsymbol{e}_{\boldsymbol{q}}, \mathbf{\Pi}\boldsymbol{\zeta})_{\Omega_h} + \langle u - \widehat{u}_h, \mathbf{\Pi}\boldsymbol{\zeta}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} + \langle \mathbb{P}\boldsymbol{e}_u, (\boldsymbol{\zeta} - \mathbf{\Pi}\boldsymbol{\zeta})\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}\\ &= (\boldsymbol{e}_{\boldsymbol{q}}, \mathbf{\Pi}\boldsymbol{\zeta} - \boldsymbol{\zeta})_{\Omega_h} - (\boldsymbol{e}_{\boldsymbol{q}}, \nabla\xi)_{\Omega_h} + \langle u - \widehat{u}_h, \boldsymbol{\zeta}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}\\ &\quad + \langle \mathbb{P}\boldsymbol{e}_u - u + \widehat{u}_h, (\boldsymbol{\zeta} - \mathbf{\Pi}\boldsymbol{\zeta})\cdot\boldsymbol{n}\rangle_{\partial\Omega_h},\end{aligned}$$

by the adjoint equation (2.13a). Since $\mathsf{P}_\partial u - \widehat{u}_h$ and $\boldsymbol{\zeta}$ are single-valued, and since $\mathsf{P}_\partial u - \widehat{u}_h = 0$ on $\partial\Omega$, we have

$$\begin{aligned}\|\mathbb{P}\boldsymbol{e}_u\|^2_{L^2(\Omega_h)} &= (\boldsymbol{e}_{\boldsymbol{q}}, \mathbf{\Pi}\boldsymbol{\zeta} - \boldsymbol{\zeta})_{\Omega_h} - (\boldsymbol{e}_{\boldsymbol{q}}, \nabla\xi)_{\Omega_h} + \langle \mathbb{P}\boldsymbol{e}_u - u + \widehat{u}_h, (\boldsymbol{\zeta} - \mathbf{\Pi}\boldsymbol{\zeta})\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}\\ &= (\boldsymbol{e}_{\boldsymbol{q}}, \mathbf{\Pi}\boldsymbol{\zeta} - \boldsymbol{\zeta})_{\Omega_h} - (\boldsymbol{e}_{\boldsymbol{q}}, \nabla\xi)_{\Omega_h} + \langle \mathbb{P}u - u + \widehat{u}_h - u_h, (\boldsymbol{\zeta} - \mathbf{\Pi}\boldsymbol{\zeta})\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}\\ &= (\boldsymbol{e}_{\boldsymbol{q}}, \mathbf{\Pi}\boldsymbol{\zeta} - \boldsymbol{\zeta})_{\Omega_h} - (\boldsymbol{e}_{\boldsymbol{q}}, \nabla\xi)_{\Omega_h} + \langle \widehat{u}_h - u_h, (\boldsymbol{\zeta} - \mathbf{\Pi}\boldsymbol{\zeta})\cdot\boldsymbol{n}\rangle_{\partial\Omega_h},\end{aligned}$$

by Proposition 3.1. Hence

$$\|\mathbb{P}\boldsymbol{e}_u\|^2_{L^2(\Omega_h)} = (\boldsymbol{e}_{\boldsymbol{q}}, \mathbf{\Pi}\boldsymbol{\zeta} - \boldsymbol{\zeta})_{\Omega_h} + \sum_{K\in\Omega_h} \langle \widehat{u}_h - u_h, (\mathsf{P}_\partial\boldsymbol{\zeta} - \mathbf{\Pi}\boldsymbol{\zeta})\cdot\boldsymbol{n}\rangle_{e^\tau_K} - (\boldsymbol{e}_{\boldsymbol{q}}, \nabla\xi)_{\Omega_h},$$

by the orthogonality property of the projection $\mathbf{\Pi}$ (3.16b).

Let us now work on the last term of the above right-hand side. We have

$$\begin{aligned}T &:= (\boldsymbol{e}_{\boldsymbol{q}}, \nabla\xi)_{\Omega_h}\\ &= (\boldsymbol{e}_{\boldsymbol{q}}, \nabla(\mathbb{P}\xi - \xi))_{\Omega_h} - (\boldsymbol{e}_{\boldsymbol{q}}, \nabla\mathbb{P}\xi)_{\Omega_h}\\ &= (\boldsymbol{e}_{\boldsymbol{q}}, \nabla(\mathbb{P}\xi - \xi))_{\Omega_h} - (\mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{q}}, \nabla\mathbb{P}\xi)_{\Omega_h},\end{aligned}$$

by the orthogonality property (3.16a) of $\mathbf{\Pi}$. Integrating by parts, we get

$$T = (\boldsymbol{e}_{\boldsymbol{q}}, \nabla(\mathbb{P}\xi - \xi))_{\Omega_h} + (\nabla\cdot\mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{q}}, \mathbb{P}\xi)_{\Omega_h} - \langle \mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{q}}\cdot\boldsymbol{n}, \mathbb{P}\xi\rangle_{\partial\Omega_h},$$

and by the error equation (3.18d) with $\omega := \mathbb{P}\xi$,

$$\begin{aligned}T &= (\boldsymbol{e}_{\boldsymbol{q}}, \nabla(\mathbb{P}\xi - \xi))_{\Omega_h} + (\boldsymbol{e}_z, \mathbb{P}\xi)_{\Omega_h} - \langle (\mathsf{P}_\partial\boldsymbol{q} - \widehat{\boldsymbol{q}}_h)\cdot\boldsymbol{n}, \mathbb{P}\xi\rangle_{\partial\Omega_h}\\ &= (\boldsymbol{q} - \mathbf{\Pi}\boldsymbol{q}, \nabla(\mathbb{P}\xi - \xi))_{\Omega_h} + (\mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{q}}, \nabla(\mathbb{P}\xi - \xi))_{\Omega_h}\\ &\quad + (\boldsymbol{e}_z, \mathbb{P}\xi)_{\Omega_h} - \langle (\mathsf{P}_\partial\boldsymbol{q} - \widehat{\boldsymbol{q}}_h)\cdot\boldsymbol{n}, \mathbb{P}\xi\rangle_{\partial\Omega_h}.\end{aligned}$$

Integrating by parts and using the orthogonality property of the projection $\mathbb{P}$, (3.17a) , we obtain

$$\begin{aligned}T &= (\boldsymbol{q} - \mathbf{\Pi}\boldsymbol{q}, \nabla(\mathbb{P}\xi - \xi))_{\Omega_h} + \langle \mathbf{\Pi}\boldsymbol{e}_{\boldsymbol{q}}\cdot\boldsymbol{n}, \mathbb{P}\xi - \xi\rangle_{\partial\Omega_h}\\ &\quad + (\boldsymbol{e}_z, \mathbb{P}\xi)_{\Omega_h} - \langle (\mathsf{P}_\partial\boldsymbol{q} - \widehat{\boldsymbol{q}}_h)\cdot\boldsymbol{n}, \mathbb{P}\xi\rangle_{\partial\Omega_h}\\ &= (\boldsymbol{q} - \mathbf{\Pi}\boldsymbol{q}, \nabla(\mathbb{P}\xi - \xi))_{\Omega_h} + (\boldsymbol{e}_z, \mathbb{P}\xi)_{\Omega_h}\\ &\quad + \langle (\mathbf{\Pi}\boldsymbol{q} - \mathsf{P}_\partial\boldsymbol{q})\cdot\boldsymbol{n}, \mathbb{P}\xi - \xi\rangle_{\partial\Omega_h} - \langle (\boldsymbol{q}_h - \widehat{\boldsymbol{q}}_h)\cdot\boldsymbol{n}, \mathbb{P}\xi - \xi\rangle_{\partial\Omega_h}.\end{aligned}$$

The third term of the above right-hand side is equal to zero by Proposition 3.1 and the fourth by the definition of the numerical trace $\widehat{\boldsymbol{q}}_h$, (2.5c), the definition of τ, (2.7), and the orthogonality property of the projection $\mathbb{P}$, (3.17b). We thus obtain that

$$T = (\boldsymbol{q} - \mathbf{\Pi}\boldsymbol{q}, \nabla(\mathbb{P}\xi - \xi))_{\Omega_h} + (\boldsymbol{e}_z, \mathbb{P}\xi)_{\Omega_h}.$$

Let us now work on the last term of the above identity. We have

$$\begin{aligned} U &:= (\boldsymbol{e}_z, \mathbb{P}\xi)_{\Omega_h} \\ &= (\boldsymbol{e}_z, \mathbb{P}\xi - \xi)_{\Omega_h} + (\boldsymbol{e}_z, \xi)_{\Omega_h} \\ &= (\boldsymbol{e}_z, \mathbb{P}\xi - \xi)_{\Omega_h} + (\boldsymbol{e}_z, \nabla\cdot(\boldsymbol{\psi} - \boldsymbol{\Pi}\boldsymbol{\psi}))_{\Omega_h} + (\boldsymbol{e}_z, \nabla\cdot\boldsymbol{\Pi}\boldsymbol{\psi})_{\Omega_h}, \end{aligned}$$

by the adjoint equation (2.13d). Integrating by parts, we get

$$\begin{aligned} U = &(\boldsymbol{e}_z, \mathbb{P}\xi - \xi)_{\Omega_h} - (\nabla\boldsymbol{e}_z, \boldsymbol{\psi} - \boldsymbol{\Pi}\boldsymbol{\psi})_{\Omega_h} \\ &+ \langle \boldsymbol{e}_z, (\boldsymbol{\psi} - \boldsymbol{\Pi}\boldsymbol{\psi})\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} + (\boldsymbol{e}_z, \nabla\cdot\boldsymbol{\Pi}\boldsymbol{\psi})_{\Omega_h} \\ = &(\boldsymbol{e}_z, \mathbb{P}\xi - \xi)_{\Omega_h} - (\nabla(z - \mathbb{P}z), \boldsymbol{\psi} - \boldsymbol{\Pi}\boldsymbol{\psi})_{\Omega_h} \\ &+ \langle \boldsymbol{e}_z, (\boldsymbol{\psi} - \boldsymbol{\Pi}\boldsymbol{\psi})\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} + (\boldsymbol{e}_z, \nabla\cdot\boldsymbol{\Pi}\boldsymbol{\psi})_{\Omega_h}, \end{aligned}$$

by the orthogonality property (3.16a) of $\boldsymbol{\Pi}$. Now, by the error equation (3.18a) with $\boldsymbol{\rho} := \boldsymbol{\Pi}\boldsymbol{\psi}$, we have

$$\begin{aligned} U = &(\boldsymbol{e}_z, \mathbb{P}\xi - \xi)_{\Omega_h} - (\nabla(z - \mathbb{P}z), \boldsymbol{\psi} - \boldsymbol{\Pi}\boldsymbol{\psi})_{\Omega_h} \\ &+ \langle \boldsymbol{e}_z, (\boldsymbol{\psi} - \boldsymbol{\Pi}\boldsymbol{\psi})\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} + (\boldsymbol{e}_{\boldsymbol{\sigma}}, \boldsymbol{\Pi}\boldsymbol{\psi})_{\Omega_h} + \langle z - \widehat{z}_h, \boldsymbol{\Pi}\boldsymbol{\psi}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} \\ = &(\boldsymbol{e}_z, \mathbb{P}\xi - \xi)_{\Omega_h} - (\nabla(z - \mathbb{P}z), \boldsymbol{\psi} - \boldsymbol{\Pi}\boldsymbol{\psi})_{\Omega_h} \\ &+ \langle z_h - \widehat{z}_h, (\boldsymbol{\psi} - \boldsymbol{\Pi}\boldsymbol{\psi})\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} + (\boldsymbol{e}_{\boldsymbol{\sigma}}, \boldsymbol{\Pi}\boldsymbol{\psi})_{\Omega_h} \\ = &(\boldsymbol{e}_z, \mathbb{P}\xi - \xi)_{\Omega_h} - (\nabla(z - \mathbb{P}z), \boldsymbol{\psi} - \boldsymbol{\Pi}\boldsymbol{\psi})_{\Omega_h} \\ &+ \sum_{K\in\Omega_h} \tau_K^{-1} \langle (\mathsf{P}_\partial\boldsymbol{\sigma} - \boldsymbol{\Pi}\boldsymbol{\sigma})\cdot\boldsymbol{n}, (\mathsf{P}_\partial\boldsymbol{\psi} - \boldsymbol{\Pi}\boldsymbol{\psi})\cdot\boldsymbol{n}\rangle_{e_K^\tau} + (\boldsymbol{e}_{\boldsymbol{\sigma}}, \boldsymbol{\Pi}\boldsymbol{\psi})_{\Omega_h}, \end{aligned}$$

by the identity (3.19). Let us now work on the last term of the above right-hand side. We have

$$\begin{aligned} V &:= (\boldsymbol{e}_{\boldsymbol{\sigma}}, \boldsymbol{\Pi}\boldsymbol{\psi})_{\Omega_h} \\ &= (\boldsymbol{\sigma} - \boldsymbol{\Pi}\boldsymbol{\sigma}, \boldsymbol{\Pi}\boldsymbol{\psi} - \boldsymbol{\psi})_{\Omega_h} - (\boldsymbol{\sigma} - \boldsymbol{\Pi}\boldsymbol{\sigma}, \nabla\varphi)_{\Omega_h} \\ &\quad + (\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}}, \boldsymbol{\Pi}\boldsymbol{\psi} - \boldsymbol{\psi})_{\Omega_h} + (\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}}, \nabla\varphi)_{\Omega_h}, \end{aligned}$$

by the adjoint equation (2.13c). By the orthogonality property (3.16a) of $\boldsymbol{\Pi}$,

$$\begin{aligned} V = &(\boldsymbol{\sigma} - \boldsymbol{\Pi}\boldsymbol{\sigma}, \boldsymbol{\Pi}\boldsymbol{\psi} - \boldsymbol{\psi})_{\Omega_h} - (\boldsymbol{\sigma} - \boldsymbol{\Pi}\boldsymbol{\sigma}, \nabla(\varphi - \mathbb{P}\varphi))_{\Omega_h} \\ &+ (\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}}, \boldsymbol{\Pi}\boldsymbol{\psi} - \boldsymbol{\psi})_{\Omega_h} + (\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}}, \nabla\varphi)_{\Omega_h} \\ = &(\boldsymbol{\sigma} - \boldsymbol{\Pi}\boldsymbol{\sigma}, \boldsymbol{\Pi}\boldsymbol{\psi} - \boldsymbol{\psi})_{\Omega_h} - (\boldsymbol{\sigma} - \boldsymbol{\Pi}\boldsymbol{\sigma}, \nabla(\varphi - \mathbb{P}\varphi))_{\Omega_h} \\ &+ (\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}}, \boldsymbol{\Pi}\boldsymbol{\psi} - \boldsymbol{\psi})_{\Omega_h} \end{aligned}$$

since

$$(\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}}, \nabla\varphi)_{\Omega_h} = -(\nabla\cdot\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}}, \varphi)_{\Omega_h} + \langle \boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}}\cdot\boldsymbol{n}, \varphi\rangle_{\partial\Omega_h} = 0,$$

by Lemma 3.2 and the boundary condition for φ of the adjoint problem (2.13e). This completes the proof. □

Now, a straightforward application of weighted Cauchy-Schwarz inequalities to the expression for $\|\mathbb{P}\boldsymbol{e}_u\|^2_{L^2(\Omega_h)}$ given by Lemma 3.10 gives us the estimate we sought.

Corollary 3.11.

$$
\begin{aligned}
\|\mathbb{P}\boldsymbol{e}_u\|^2_{L^2(\Omega_h)} \le{}& \|\,\boldsymbol{e}_{\boldsymbol{q}}\,\|_{L^2(\Omega_h)}\,\|\,\boldsymbol{\Pi\zeta}-\boldsymbol{\zeta}\,\|_{L^2(\Omega_h)}\\
&+\kappa_\Omega^{1/2}\|\,\widehat{u}_h-u_h\,\|_{L^2(\partial\Omega_h;\tau)}\,\|\,(\mathsf{P}_\partial\boldsymbol{\zeta}-\boldsymbol{\Pi\zeta})\cdot\boldsymbol{n}\,\|_{L^2(\partial\Omega_h;h)}\\
&+\|\,\boldsymbol{q}-\boldsymbol{\Pi q}\|_{L^2(\Omega_h)}\,\|\,\nabla(\mathbb{P}\xi-\xi)\,\|_{L^2(\Omega_h)}\\
&+\|\,\boldsymbol{e}_z\,\|_{L^2(\Omega_h)}\,\|\,\mathbb{P}\xi-\xi\,\|_{L^2(\Omega_h)}\\
&+\|\,\nabla(z-\mathbb{P}z)\,\|_{L^2(\Omega_h)}\,\|\,\boldsymbol{\psi}-\boldsymbol{\Pi\psi}\,\|_{L^2(\Omega_h)}\\
&+\kappa_\Omega\,\|\,(\mathsf{P}_\partial\boldsymbol{\sigma}-\boldsymbol{\Pi\sigma})\cdot\boldsymbol{n}\,\|_{L^2(\partial\Omega_h;h)}\,\|\,(\mathsf{P}_\partial\boldsymbol{\psi}-\boldsymbol{\Pi\psi})\cdot\boldsymbol{n}\,\|_{L^2(\partial\Omega_h;h)}\\
&+\|\,\boldsymbol{\sigma}-\boldsymbol{\Pi\sigma}\,\|_{L^2(\Omega_h)}\,\|\,\boldsymbol{\psi}-\boldsymbol{\Pi\psi}\,\|_{L^2(\Omega_h)}\\
&+\|\,\boldsymbol{\sigma}-\boldsymbol{\Pi\sigma}\,\|_{L^2(\Omega_h)}\,\|\,\nabla(\varphi-\mathbb{P}\varphi)\,\|_{L^2(\Omega_h)}\\
&+\|\,\boldsymbol{\Pi e_\sigma}\,\|_{L^2(\Omega_h)}\,\|\,\boldsymbol{\Pi\psi}-\boldsymbol{\psi}\,\|_{L^2(\Omega_h)}.
\end{aligned}
$$

3.8. **Final estimates.** In this section we combine the intermediate error estimates to obtain the final estimates for all the variables. We are going to use the following approximation result.

Proposition 3.12. *For any* $(\boldsymbol{\rho},\eta)\in\boldsymbol{H}^{k+1}(\Omega_h)\times H^{k+1}(\Omega_h)$ *we have*

$$\|\,\boldsymbol{\Pi\rho}\cdot\boldsymbol{n}-\mathsf{P}_\partial\boldsymbol{\rho}\cdot\boldsymbol{n}\,\|_{L^2(\partial\Omega_h;h)}\le C\,h^{k+1}\,|\,\nabla\cdot\boldsymbol{\rho}\,|_{H^{k+1}(\Omega_h)}, \tag{3.22a}$$

$$\|\,\boldsymbol{\Pi\rho}-\boldsymbol{\rho}\,\|_{L^2(\Omega_h)}\le C\,h^{k+1}\,|\,\nabla\cdot\boldsymbol{\rho}\,|_{H^{k+1}(\Omega_h)}, \tag{3.22b}$$

$$\|\,\boldsymbol{\Pi\rho}-\boldsymbol{\rho}\,\|_{L^\infty(\Omega_h)}\le C\,h^{k+1}\,|\,\nabla\cdot\boldsymbol{\rho}\,|_{W^{k+1,\infty}(\Omega_h)}, \tag{3.22c}$$

$$\|\,\nabla\cdot(\boldsymbol{q}-\boldsymbol{\Pi q})\,\|_{L^\infty(\Omega_h)}\le C\,h^{k+1}\,|\,\nabla\cdot\boldsymbol{\rho}\,|_{W^{k+1,\infty}(\Omega_h)}, \tag{3.22d}$$

$$\|\,\mathbb{P}\eta-\eta\,\|_{L^2(\Omega_h)}\le C\,h^{k+1}\,|\,\nabla\eta\,|_{\boldsymbol{H}^{k+1}(\Omega_h)} \tag{3.22e}$$

where C depends only on k and the shape-regularity parameters of the simplex K.

These estimates can be proven as in [10].

So, from the estimates for $\|\,\boldsymbol{\Pi e_\sigma}\,\|_{L^2(\Omega_h)}$, $\|\,\boldsymbol{\Pi e_q}\,\|_{L^2(\Omega_h)}$ and $\|\,\mathbb{P}e_z\,\|_{L^2(\Omega_h)}$, in Lemmas 3.3, 3.4 and 3.8, respectively, we get

$$
\begin{aligned}
\|\boldsymbol{\Pi e_\sigma}\|_{L^2(\Omega_h)} &\le C\,\mathcal{C}\,h^{k+1}+C\,h^{-1}\,\|\,\mathbb{P}e_z\,\|_{L^2(\Omega_h)},\\
\|\boldsymbol{\Pi e_q}\|_{L^2(\Omega_h)} &\le C\,(1+\kappa_\Omega^{1/2})\mathcal{C}\,h^{k+1}+\|\mathbb{P}\boldsymbol{e}_z\|^{1/2}_{L^2(\Omega_h)}\,\|\mathbb{P}\boldsymbol{e}_u\|^{1/2}_{L^2(\Omega_h)}\\
\|\,\widehat{u}_h-u_h\,\|_{L^2(\partial\Omega_h;\tau)} &\le C\,(1+\kappa_\Omega^{1/2})\mathcal{C}\,h^{k+1}+\|\mathbb{P}\boldsymbol{e}_z\|^{1/2}_{L^2(\Omega_h)}\,\|\mathbb{P}\boldsymbol{e}_u\|^{1/2}_{L^2(\Omega_h)}\\
\|\mathbb{P}e_z\|_{L^2(\Omega_h)} &\le C\,(l(k)+h^{1/2}\kappa_\Omega^{1/2})\mathcal{C}\,h^{k+1/2}+C\,\|\,\boldsymbol{\Pi e_q}\,\|_{L^2(\Omega_h)}\\
&\quad+C\kappa_\Omega^{1/2}\|\,u_h-\widehat{u}_h\,\|_{L^2(\partial\Omega_h;\tau)}.
\end{aligned}
$$

The remaining estimate requires a more careful handling. Indeed, from the estimate of $\| \mathbb{P}e_u \|_{L^2(\Omega_h)}$ in Lemma 3.11, we get

$$
\begin{aligned}
\|\mathbb{P}\boldsymbol{e}_u\|^2_{L^2(\Omega_h)} \leq {} & C\, h \,\| \boldsymbol{e}_{\boldsymbol{q}} \|_{L^2(\Omega_h)} \, |\, \boldsymbol{\zeta} \,|_{\boldsymbol{H}^1(\Omega_h)} \\
& + C\, \kappa_\Omega^{1/2} h \,\| \widehat{u}_h - u_h \|_{L^2(\partial\Omega_h;\tau)} \, |\, \boldsymbol{\zeta} \,|_{\boldsymbol{H}^1(\Omega_h)} \\
& + C\, h^{\min\{k,1\}} \,\| \boldsymbol{q} - \boldsymbol{\Pi q} \|_{L^2(\Omega_h)} \, |\, \xi \,|_{H^2(\Omega_h)} \\
& + C\, h^{\min\{k,1\}+1} \,\| \boldsymbol{e}_z \|_{L^2(\Omega_h)} \, |\, \xi \,|_{H^2(\Omega_h)} \\
& + C\, h^{\min\{k,1\}+1} \,\| \nabla(z - \mathbb{P}z) \|_{L^2(\Omega_h)} \, |\, \boldsymbol{\psi} \,|_{H^3(\Omega_h)} \\
& + C\, h^{\min\{k,2\}+1} \,\kappa_\Omega \,\| (\mathsf{P}_\partial \boldsymbol{\sigma} - \boldsymbol{\Pi\sigma}) \cdot \boldsymbol{n} \|_{L^2(\partial\Omega_h;h)} \, |\, \boldsymbol{\psi} \,|_{H^3(\partial\Omega_h;h)} \\
& + C\, h^{\min\{k,1\}+1} \| \boldsymbol{\sigma} - \boldsymbol{\Pi\sigma} \|_{L^2(\Omega_h)} \, |\, \boldsymbol{\psi} \,|_{H^3(\Omega_h)} \\
& + C\, h^{\min\{k,3\}} \| \boldsymbol{\sigma} - \boldsymbol{\Pi\sigma} \|_{L^2(\Omega_h)} \, |\, \varphi \,|_{H^4(\Omega_h)} \\
& + C\, h^{\min\{k,2\}+1} \| \boldsymbol{\Pi e_\sigma} \|_{L^2(\Omega_h)} \, |\, \boldsymbol{\psi} \,|_{H^3(\Omega_h)}.
\end{aligned}
$$

and by the elliptic regularity estimate (2.12), we obtain

$$
\begin{aligned}
\|\mathbb{P}\boldsymbol{e}_u\|_{L^2(\Omega_h)} \leq {} & C\, \mathcal{C}\, (1 + \kappa_{\partial\Omega}\, h)\, h^{\min\{k,1\}+k+1} \\
& + C\, h \,\| \boldsymbol{\Pi e_q} \|_{L^2(\Omega_h)} + C\, \kappa_\Omega^{1/2} h \,\| \widehat{u}_h - u_h \|_{L^2(\partial\Omega_h;\tau)} \\
& + C\, h^{\min\{k,1\}+1} \| \mathbb{P}\boldsymbol{e}_z \|_{L^2(\Omega_h)} + C\, h^{\min\{k,2\}+1} \| \boldsymbol{\Pi e_\sigma} \|_{L^2(\Omega_h)}.
\end{aligned}
$$

We can immediately see that the optimal choice for κ_Ω is to be of order one. So, if we take $\tau|_K$ to be of order h_K^{-1} the above estimates become

(3.23)
$$\|\boldsymbol{\Pi e_\sigma}\|_{L^2(\Omega_h)} \leq C\, \mathcal{C}\, h^{k+1} + C\, h^{-1} \,\| \mathbb{P}e_z \|_{L^2(\Omega_h)},$$

(3.24)
$$\|\boldsymbol{\Pi e_q}\|_{L^2(\Omega_h)} \leq C\, \mathcal{C}\, h^{k+1} + \|\mathbb{P}\boldsymbol{e}_z\|^{1/2}_{L^2(\Omega_h)} \, \|\mathbb{P}\boldsymbol{e}_u\|^{1/2}_{L^2(\Omega_h)}$$

(3.25)
$$\| \widehat{u}_h - u_h \|_{L^2(\partial\Omega_h;\tau)} \leq C\, \mathcal{C}\, h^{k+1} + \|\mathbb{P}\boldsymbol{e}_z\|^{1/2}_{L^2(\Omega_h)} \, \|\mathbb{P}\boldsymbol{e}_u\|^{1/2}_{L^2(\Omega_h)}$$

(3.26)
$$
\begin{aligned}
\|\mathbb{P}e_z\|_{L^2(\Omega_h)} &\leq C\, \mathcal{C}\, l(k)\, h^{k+1/2} + C \,\| \boldsymbol{\Pi e_q} \|_{L^2(\Omega_h)} + C \| u_h - \widehat{u}_h \|_{L^2(\partial\Omega_h;\tau)}, \\
\|\mathbb{P}\boldsymbol{e}_u\|_{L^2(\Omega_h)} &\leq C\, \mathcal{C}\, h^{\min\{k,1\}+k+1} + C\, h \,\| \boldsymbol{\Pi e_q} \|_{L^2(\Omega_h)} + C\, h \,\| \widehat{u}_h - u_h \|_{L^2(\partial\Omega_h;\tau)} \\
&\quad + C\, h^{\min\{k,1\}+1} \,\| \mathbb{P}\boldsymbol{e}_z \|_{L^2(\Omega_h)} + C\, h^{\min\{k,2\}+1} \| \boldsymbol{\Pi e_\sigma} \|_{L^2(\Omega_h)}.
\end{aligned}
$$

If we assume $k \geq 1$ and apply some simple algebraic manipulations, and if we assume that h is small enough, we get

$$
\begin{aligned}
\|\boldsymbol{\Pi e_\sigma}\|_{L^2(\Omega_h)} &\leq C\, \mathcal{C}\, h^{k-1/2} \\
\|\boldsymbol{\Pi e_q}\|_{L^2(\Omega_h)} &\leq C\, \mathcal{C}\, h^{k+1} \\
\| \widehat{u}_h - u_h \|_{L^2(\partial\Omega_h;\tau)} &\leq C\, \mathcal{C}\, h^{k+1} \\
\|\mathbb{P}e_z\|_{L^2(\Omega_h)} &\leq C\, \mathcal{C}\, h^{k+1/2} \\
\|\mathbb{P}\boldsymbol{e}_u\|_{L^2(\Omega_h)} &\leq C\, \mathcal{C}\, h^{k+1+\min\{k-1/2,1\}}.
\end{aligned}
$$

The proof is now complete if we apply the triangle inequality. We would like to point out that since $h^{\min\{k,2\}+1}\| \boldsymbol{\Pi e_\sigma} \|_{L^2(\Omega_h)}$ appears in the estimate of $\|\mathbb{P}\boldsymbol{e}_u\|_{L^2(\Omega_h)}$ it was important that we assumed that $k \geq 1$. In fact, because of this term we were not able to get an estimate for $k = 0$. In the Extension section we prove a different estimate for $\|\mathbb{P}\boldsymbol{e}_u\|_{L^2(\Omega_h)}$ which will allow us to prove error estimates in the case $k = 0$.

3.9. **Proof of Theorems 2.2 and 2.3.** If we now use the approximation properties of $\mathbb{P}$ and $\boldsymbol{\Pi}$ of Proposition 3.12, the estimates of Theorem 2.2 follow immediately. Note that the first estimate of Theorem 2.3 follows from the fact that $\| \mathsf{P}^{k-1} e_u \|_{L^2(\Omega_h)} \leq \|\mathbb{P}\boldsymbol{e}_u\|_{L^2(\Omega_h)}$. It remains to prove the second estimate of Theorem 2.3

The proof is similar to that of Theorem 2.8 in [10]. For each simplex K, we have that on the face e^τ_K, by definition of the projection $\mathbb{P}$, (3.17),

$$\begin{aligned}\| \mathsf{P}_\partial u - \widehat{u}_h \|_{L^2(e^\tau_K)} &= \| \mathbb{P}u - \widehat{u}_h \|_{L^2(e^\tau_K)} \\ &\leq \| \mathbb{P}u - u_h \|_{L^2(e^\tau_K)} + \| u_h - \widehat{u}_h \|_{L^2(e^\tau_K)}.\end{aligned}$$

By using a classical inverse inequality, we can conclude that

$$h_K^{1/2} \| \mathsf{P}_\partial u - \widehat{u}_h \|_{L^2(e^\tau_K)} \leq C \left(\| \mathbb{P}u - u_h \|_{L^2(K)} + h_K^{1/2} \| u_h - \widehat{u}_h \|_{L^2(e^\tau_K)} \right).$$

Now we consider the error in the faces e of K which are different from the face e^τ_K. By the error equation (3.18c), we have that, for all $\boldsymbol{v} \in \boldsymbol{\mathcal{P}}^k(K)$,

$$\begin{aligned}\langle \widehat{u}_h - \mathsf{P}_\partial u, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial K\setminus e^\tau_K} &= (\boldsymbol{q} - \boldsymbol{q}_h, \boldsymbol{v})_K - (\mathbb{P}u - u_h, \nabla\cdot\boldsymbol{v})_K \\ &\quad - \langle \widehat{u}_h - \mathsf{P}_\partial u, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{e^\tau_K}.\end{aligned}$$

Taking $\boldsymbol{v} := \boldsymbol{Z}$ given by Lemma 3.2 in [10] with $z = \widehat{u}_h - \mathsf{P}_\partial u$, we obtain that

$$\begin{aligned}\| \widehat{u}_h - \mathsf{P}_\partial u \|_{L^2(\partial K\setminus e^\tau_K)} \leq C\, (& h_K^{1/2} \| \boldsymbol{q} - \boldsymbol{q}_h \|_{\boldsymbol{L}^2(K)} + h_K^{-1/2} \| \mathbb{P}u - u_h \|_{L^2(K)} \\ & + \| \widehat{u}_h - \mathsf{P}_\partial u \|_{L^2(e^\tau_K)}),\end{aligned}$$

and using the estimate for the error in e^τ_K,

$$\begin{aligned}h_K^{1/2} \| \widehat{u}_h - \mathsf{P}_\partial u \|_{L^2(\partial K\setminus e^\tau_K)} \leq C\, (& \| \mathbb{P}u - u_h \|_{L^2(K)} + h_K \| \boldsymbol{q} - \boldsymbol{q}_h \|_{\boldsymbol{L}^2(K)} \\ & + h_K^{1/2} \| u_h - \widehat{u}_h \|_{L^2(e^\tau_K)}).\end{aligned}$$

As a consequence

$$\begin{aligned}\| \mathsf{P}_\partial u - \widehat{u}_h \|_{L^2(\mathscr{E}_h;h)} \leq\ C\ (& \| \mathbb{P}u - u_h \|_{L^2(\Omega_h)} + h \| \boldsymbol{q} - \boldsymbol{q}_h \|_{\boldsymbol{L}^2(\Omega_h)} \\ & + h\,\kappa_\Omega \, \|(\widehat{u}_h - u_h)\|_{L^2(\mathscr{E}_h;\tau)}).\end{aligned}$$

The result now follows from Theorems 2.2. This completes the proof of Theorem 2.3.

3.10. **Proof of Theorem 2.4.** The proof of Theorem 2.4 is almost identical to that of Theorem 2.9 in [10]. We define $\overline{u}|_K := \frac{1}{|K|}(u,1)_K$ for every $K \in \Omega_h$, and let $\tilde{u} := u - \overline{u}$. Note that, by the definition of $u_h^\star$, (2.15a), we have

$$\| u - u_h^\star \|_{L^2(K)} \leq \| \overline{u} - \overline{u}_h \|_{L^2(K)} + \| \tilde{u} - \tilde{u}_h \|_{L^2(K)},$$

We estimate each of the two terms of the right-hand side separately.

We begin by estimating the error $\overline{u} - \overline{u}_h$. Since $\overline{u} - \overline{u}_h = \mathsf{P}^0(u - u_h)$, we get

$$\| \overline{u} - \overline{u}_h \|_{L^2(K)} \leq \| \mathsf{P}^{k-1}(u - u_h) \|_{L^2(K)},$$

for $k \geq 1$.

Hence,

$$\| \overline{u} - \overline{u}_h \|_{L^2(\Omega_h)} \leq \| \mathsf{P}^{k-1}(u - u_h) \|_{L^2(\Omega_h)} \leq C\, \mathcal{C}\, h^{k+1+\min\{k-1/2,1\}},$$

which follows from Theorem 2.3.

Now we estimate the error $\tilde{u} - \tilde{u}_h$. Note that by Poincaré's inequality, we have

$$\| \tilde{u} - \tilde{u}_h \|_{L^2(K)} \leq C\, h_K \| \nabla(\tilde{u} - \tilde{u}_h) \|_{L^2(K)},$$

so it is enough to estimate the error in the gradient. By the definition of $\tilde{u}_h$, (2.15c),

$$(\nabla(\tilde{u} - \tilde{u}_h), \nabla w)_K = (z - z_h, w)_K - \langle w, (\boldsymbol{q} - \widehat{\boldsymbol{q}}_h) \cdot \boldsymbol{n} \rangle_{\partial K} \qquad \forall w \in \mathcal{P}_0^{k+1}(K).$$

Then we have

$$(\nabla(\mathsf{P}^{k+1}\tilde{u} - \tilde{u}_h), \nabla w)_K = \sum_{i=1}^{3} T_i,$$

where

$$\begin{aligned} T_1 =& (\nabla(\mathsf{P}^{k+1}\tilde{u} - \tilde{u}), \nabla w)_K, \\ T_2 =& (z - z_h, w)_K, \\ T_3 =& - \langle w, (\boldsymbol{q} - \widehat{\boldsymbol{q}}_h) \cdot \boldsymbol{n} \rangle_{\partial K}. \end{aligned}$$

By using Cauchy-Schwarz inequality, we get that

$$T_1 \leq \| \nabla(\mathsf{P}^{k+1}\tilde{u} - \tilde{u}) \|_{L^2(K)} \| \nabla w \|_{L^2(K)},$$

and

$$\begin{aligned} T_2 \leq& \| z - z_h \|_{L^2(K)} \| w \|_{L^2(K)} \\ \leq& C h_K \| z - z_h \|_{L^2(K)} \| \nabla w \|_{L^2(K)}, \end{aligned}$$

by Poincaré's inequality. Similar to the proof of Theorem 2.9 [10], we get that

$$\begin{aligned} T_3 \leq\ C \| \nabla w \|_{L^2(K)} \Big(& \| \boldsymbol{q} - \boldsymbol{q}_h \|_{L^2(K)} + h_K \| z - \mathsf{P}^k z \|_{L^2(K)} \\ & + h_K^{1/2} \tau_K^{1/2} \| \tau_K (u_h - \widehat{u}_h) \|_{L^2(e_K^\tau)} \Big). \end{aligned}$$

Hence, we have

$$(\nabla(\mathsf{P}^{k+1}\tilde{u} - \tilde{u}_h), \nabla w)_K = \sum_{i=1}^{3} T_i \leq\ C \| \nabla w \|_{L^2(K)} \Theta_K,$$

where

$$\begin{aligned} \Theta_K =& \| \nabla(\mathsf{P}^{k+1}\tilde{u} - \tilde{u}) \|_{L^2(K)} + h_K \| z - z_h \|_{L^2(K)} + \| \boldsymbol{q} - \boldsymbol{q}_h \|_{L^2(K)} \\ & + h_K \| z - \mathsf{P}^k z \|_{L^2(K)} + h_K^{1/2} \tau_K^{1/2} \| \tau_K (u_h - \widehat{u}_h) \|_{L^2(e_K^\tau)}. \end{aligned}$$

Taking $w = \mathsf{P}^{k+1}\tilde{u} - \tilde{u}_h$, we get that

$$\| \nabla(\mathsf{P}^{k+1}\tilde{u} - \tilde{u}_h) \|_{L^2(K)} \leq\ C\, \Theta_K.$$

This implies, after using Poincaré's inequality, that

$$\| \mathsf{P}^{k+1}\tilde{u} - \tilde{u}_h \|_{L^2(\Omega_h)} \leq Ch(\sum_{K\in\Omega_h} \Theta_K^2)^{1/2} \leq C\,\mathcal{C}\,h^{k+2}.$$

by Theorem 2.2 and the well-known approximation properties of P^{k+1} and P^k. This completes the proof of Theorem 2.4

4. Extensions

In this section we will show how to improve the error estimates in Theorems 2.3 and 2.4 in the case of linear approximations, $k = 1$, and dimension $d = 2$. Moreover, we will be able to prove error estimates for u, $\boldsymbol{q}$ and z in the case $k = 0$ and $d = 2, 3$.

We start by stating the improved result for $k = 1$ and $d = 2$.

Theorem 4.1. *We assume the same hypotheses of Theorem 2.2 and we further assume that $k = 1$ and $d = 2$. We have,*

$$\begin{aligned}
\|\mathsf{P}^{k-1}(u - u_h)\|_{L^2(\Omega_h)} &\leq C\,\mathcal{C}\,\log(\frac{1}{h})\,h^3,\\
\|\mathsf{P}_\partial u - \widehat{u}_h\|_{L^2(\mathscr{E}_h;h)} &\leq C\,\mathcal{C}\,\log(\frac{1}{h})\,h^3,\\
\|u - u_h^\star\|_{L^2(\Omega_h)} &\leq C\,\mathcal{C}\,\log(\frac{1}{h})\,h^3,
\end{aligned}$$

where C is independent of h, and the exact solution.

Next we state a result for $k = 0$ and $d = 2, 3$.

Theorem 4.2. *We assume the same hypotheses of Theorem 2.2 and further assume that $k = 0$ and $d = 2, 3$. Then,*

$$\begin{aligned}
\|u - u_h\|_{L^2(\Omega_h)} &\leq C\,\mathcal{C}\,\log(\frac{1}{h})^2\,h,\\
\|\boldsymbol{q} - \boldsymbol{q}_h\|_{\boldsymbol{L}^2(\Omega_h)} &\leq C\,\mathcal{C}\,\log(\frac{1}{h})^{\frac{3}{2}}\,h^{\frac{3}{4}},\\
\|z - z_h\|_{L^2(\Omega_h)} &\leq C\,\mathcal{C}\,\log(\frac{1}{h})h^{\frac{1}{2}},
\end{aligned}$$

and

$$\| \widehat{u}_h - u_h \|_{L^2(\partial\Omega_h;\tau)} \leq C\,\mathcal{C}\,\log(\frac{1}{h})^{\frac{3}{2}} h^{\frac{3}{4}}.$$

Notice that this results gives quasi-optimal error estimates for u. Moreover, we get sub-optimal error estimates for $\boldsymbol{q}$ and z. Note that we do not state error estimates for $\boldsymbol{\sigma}$. This is because $\boldsymbol{\sigma}_h$ does not converge to $\boldsymbol{\sigma}$ for $k = 0$ as our numerical experiments demonstrate.

4.1. A different estimate for $\mathbb{P}\boldsymbol{e}_u$.

In order to prove these results we will need to improve the estimates for $\mathbb{P}\boldsymbol{e}_u$. We start by writing an identity that is different than the one giving in Lemma 3.10. To do that we need to define an auxiliary variable, $\tilde{\boldsymbol{\psi}}_h \in \boldsymbol{V}_h$. This function along with $\tilde{\phi}_h \in W_h$ and $\widehat{\tilde{\phi}}_h \in M_h^0$, solve

$$(\tilde{\boldsymbol{\psi}}_h, \boldsymbol{v})_{\Omega_h} - (\tilde{\phi}_h, \nabla\cdot\boldsymbol{v})_{\Omega_h} + \langle \widehat{\tilde{\phi}}_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} = 0, \tag{4.27a}$$

$$-(\tilde{\boldsymbol{\psi}}_h, \nabla\omega)_{\Omega_h} + \langle \widehat{\tilde{\boldsymbol{\psi}}}_h\cdot\boldsymbol{n}, \omega\rangle_{\partial\Omega_h} = (\xi, \omega)_{\Omega_h}, \tag{4.27b}$$

$$\langle \widehat{\tilde{\boldsymbol{\psi}}}_h\cdot\boldsymbol{n}, \chi\rangle_{\partial\Omega_h} = 0, \tag{4.27c}$$

for every $(\boldsymbol{v}, \omega, \mu) \in \boldsymbol{V}_h \times W_h \times M_h^0$. Here

$$\widehat{\tilde{\boldsymbol{\psi}}}_h = \tilde{\boldsymbol{\psi}} + \tau(\tilde{\phi}_h - \widehat{\tilde{\phi}}_h)\boldsymbol{n}, \quad \text{and} \quad \widehat{\tilde{\phi}}_h = 0 \quad \text{on}\,\partial\Omega. \tag{4.27d}$$

In other words, $(\tilde{\boldsymbol{\psi}}_h, \tilde{\phi}_h, \widehat{\tilde{\phi}}_h)$ is the SFH approximation to the second-order problem

$$\begin{aligned}\boldsymbol{\psi} + \nabla\phi =& 0 \quad \Omega,\\ \nabla\cdot\boldsymbol{\psi} =& \xi \quad \Omega,\\ \phi =& 0 \quad\quad \partial\Omega.\end{aligned}$$

We are now ready to state the result.

Lemma 4.3. *We have*

$$\begin{aligned}\|\mathbb{P}\boldsymbol{e}_u\|^2_{L^2(\Omega_h)} = & (\boldsymbol{e}_{\boldsymbol{q}}, \boldsymbol{\Pi}\boldsymbol{\zeta} - \boldsymbol{\zeta})_{\Omega_h} + \sum_{K\in\Omega_h} \langle \widehat{u}_h - u_h, (\mathsf{P}_\partial\boldsymbol{\zeta} - \boldsymbol{\Pi}\boldsymbol{\zeta})\cdot\boldsymbol{n}\rangle_{e_K^\tau}\\ & - (\boldsymbol{q} - \boldsymbol{\Pi}\boldsymbol{q}, \nabla(\mathbb{P}\xi - \xi))_{\Omega_h} - (\boldsymbol{e}_z, \mathbb{P}\xi - \xi)_{\Omega_h}\\ & + (\nabla(z - \mathbb{P}z), \boldsymbol{\psi} - \boldsymbol{\Pi}\boldsymbol{\psi})_{\Omega_h}\\ & - \sum_{K\in\Omega_h} \tau_K^{-1} \langle (\mathsf{P}_\partial\boldsymbol{\sigma} - \boldsymbol{\Pi}\boldsymbol{\sigma})\cdot\boldsymbol{n}, (\mathsf{P}_\partial\boldsymbol{\psi} - \boldsymbol{\Pi}\boldsymbol{\psi})\cdot\boldsymbol{n}\rangle_{e_K^\tau}\\ & + (\boldsymbol{\sigma} - \boldsymbol{\Pi}\boldsymbol{\sigma}, \tilde{\boldsymbol{\psi}}_h - \mathsf{P}^{k-1}\boldsymbol{\psi})_{\Omega_h} + \langle \mathbb{P}\boldsymbol{e}_z, (\boldsymbol{\Pi}\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h)\cdot\boldsymbol{n}\rangle_{\partial\Omega}\\ & + \langle \frac{1}{\tau}(\mathsf{P}_\partial\boldsymbol{\sigma} - \boldsymbol{\Pi}\boldsymbol{\sigma})\cdot\boldsymbol{n}, (\boldsymbol{\Pi}\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h)\cdot\boldsymbol{n}\rangle_{\partial\Omega}.\end{aligned}$$

Proof. The proof is exactly the same as the proof of Lemma 3.10 the only difference being how we treat V. To this end, we apply some simple algebraic manipulations to obtain

$$V = (\boldsymbol{e}_{\boldsymbol{\sigma}}, \boldsymbol{\Pi}\boldsymbol{\psi})_{\Omega_h} = T_1 + T_2 + T_3 + T_4,$$

where

$$\begin{aligned}T_1 :=& (\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\psi}}_h)_{\Omega_h},\\ T_2 :=& (\boldsymbol{\sigma} - \boldsymbol{\Pi}\boldsymbol{\sigma}, \boldsymbol{\psi})_{\Omega_h},\\ T_3 :=& (\boldsymbol{\sigma} - \boldsymbol{\Pi}\boldsymbol{\sigma}, \tilde{\boldsymbol{\psi}}_h - \boldsymbol{\psi})_{\Omega_h},\\ T_4 :=& (\boldsymbol{e}_{\boldsymbol{\sigma}}, \boldsymbol{\Pi}\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h)_{\Omega_h}.\end{aligned}$$

We now simplify $T_1, \cdots, T_4$. By using (4.27a), (3.20), and the fact that $\widehat{\tilde{\phi}}_h = 0$ on $\partial\Omega$ we get $T_1 = 0$. If we apply (3.16a) we get

$$T_2 = (\boldsymbol{\sigma} - \boldsymbol{\Pi}\boldsymbol{\sigma}, \boldsymbol{\psi} - \mathsf{P}^{k-1}\boldsymbol{\psi})_{\Omega_h}.$$

We leave T_3 the same and we now simplify T_4. By applying (7.40) we get

$$T_4 = \langle z - \widehat{z}_h, (\boldsymbol{\Pi}\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h)\cdot\boldsymbol{n}\rangle_{\partial\Omega},$$

where we applied Proposition 3.6 with $\boldsymbol{q}$ and $\tilde{\boldsymbol{q}}_h$ replaced with $\boldsymbol{\psi}$ and $\tilde{\boldsymbol{\psi}}_h$, respectively. If we now apply (2.7) and (2.8b) we get

$$\begin{aligned}T_4 =&\langle \mathbb{P}z - \widehat{z}_h, (\boldsymbol{\Pi\psi} - \tilde{\boldsymbol{\psi}}_h)\cdot \boldsymbol{n}\rangle_{\partial\Omega}\\ =&\langle \mathbb{P}\boldsymbol{e}_z, (\boldsymbol{\Pi\psi} - \tilde{\boldsymbol{\psi}}_h)\cdot \boldsymbol{n}\rangle_{\partial\Omega} + \langle z_h - \widehat{z}_h, (\boldsymbol{\Pi\psi} - \tilde{\boldsymbol{\psi}}_h)\cdot \boldsymbol{n}\rangle_{\partial\Omega}.\end{aligned}$$

Finally, if we apply (3.19) we get

$$T_4 =\langle \mathbb{P}\boldsymbol{e}_z, (\boldsymbol{\Pi\psi} - \tilde{\boldsymbol{\psi}}_h)\cdot \boldsymbol{n}\rangle_{\partial\Omega} + \langle \frac{1}{\tau}(\mathsf{P}_\partial\boldsymbol{\sigma} - \boldsymbol{\Pi\sigma})\cdot\boldsymbol{n}, (\boldsymbol{\Pi\psi} - \tilde{\boldsymbol{\psi}}_h)\cdot \boldsymbol{n}\rangle_{\partial\Omega}.$$

Hence,

$$\begin{aligned}V =&(\boldsymbol{\sigma} - \boldsymbol{\Pi\sigma}, \boldsymbol{\psi} - \mathsf{P}^{k-1}\boldsymbol{\psi})_{\Omega_h} + (\boldsymbol{\sigma} - \boldsymbol{\Pi\sigma}, \tilde{\boldsymbol{\psi}}_h - \boldsymbol{\psi})_{\Omega_h}\\ &+ \langle \mathbb{P}\boldsymbol{e}_z, (\boldsymbol{\Pi\psi} - \tilde{\boldsymbol{\psi}}_h)\cdot \boldsymbol{n}\rangle_{\partial\Omega} + \langle \frac{1}{\tau}(\mathsf{P}_\partial\boldsymbol{\sigma} - \boldsymbol{\Pi\sigma})\cdot\boldsymbol{n}, (\boldsymbol{\Pi\psi} - \tilde{\boldsymbol{\psi}}_h)\cdot \boldsymbol{n}\rangle_{\partial\Omega}\\ =&(\boldsymbol{\sigma} - \boldsymbol{\Pi\sigma}, \tilde{\boldsymbol{\psi}}_h - \mathsf{P}^{k-1}\boldsymbol{\psi})_{\Omega_h} + \langle \mathbb{P}\boldsymbol{e}_z, (\boldsymbol{\Pi\psi} - \tilde{\boldsymbol{\psi}}_h)\cdot \boldsymbol{n}\rangle_{\partial\Omega}\\ &+ \langle \frac{1}{\tau}(\mathsf{P}_\partial\boldsymbol{\sigma} - \boldsymbol{\Pi\sigma})\cdot\boldsymbol{n}, (\boldsymbol{\Pi\psi} - \tilde{\boldsymbol{\psi}}_h)\cdot \boldsymbol{n}\rangle_{\partial\Omega}.\end{aligned}$$

The proof of the lemma is complete once we use this result for V in the proof of Lemma 3.10. □

Now we can state a result analogues to Corollary 3.11.

Corollary 4.4. *Let $l(k) = 1$ if $k \geq 1$ and $l(k) = \log(\frac{1}{h})$ if $k = 0$. Then,*

$$\begin{aligned}\|\mathbb{P}\boldsymbol{e}_u\|^2_{L^2(\Omega_h)} \leq &\|\,\boldsymbol{e}_{\boldsymbol{q}}\,\|_{L^2(\Omega_h)}\,\|\,\boldsymbol{\Pi\zeta} - \boldsymbol{\zeta}\,\|_{L^2(\Omega_h)}\\ &+ \kappa_\Omega^{1/2}\|\,\widehat{u}_h - u_h\,\|_{L^2(\partial\Omega_h;\tau)}\,\|\,(\mathsf{P}_\partial\boldsymbol{\zeta} - \boldsymbol{\Pi\zeta})\cdot\boldsymbol{n}\,\|_{L^2(\partial\Omega_h;h)}\\ &+ \|\,\boldsymbol{q} - \boldsymbol{\Pi q}\|_{L^2(\Omega_h)}\,\|\,\nabla(\mathbb{P}\xi - \xi)\,\|_{L^2(\Omega_h)}\\ &+ \|\,\boldsymbol{e}_z\,\|_{L^2(\Omega_h)}\,\|\,\mathbb{P}\xi - \xi\,\|_{L^2(\Omega_h)}\\ &+ \|\,\nabla(z - \mathbb{P}z)\,\|_{L^2(\Omega_h)}\,\|\,\boldsymbol{\psi} - \boldsymbol{\Pi\psi}\,\|_{L^2(\Omega_h)}\\ &+ \kappa_\Omega\,\|\,(\mathsf{P}_\partial\boldsymbol{\sigma} - \boldsymbol{\Pi\sigma})\cdot\boldsymbol{n}\,\|_{L^2(\partial\Omega_h;h)}\,\|\,(\mathsf{P}_\partial\boldsymbol{\psi} - \boldsymbol{\Pi\psi})\cdot\boldsymbol{n}\,\|_{L^2(\partial\Omega_h;h)}\\ &+ \|\,\boldsymbol{\sigma} - \boldsymbol{\Pi\sigma}\,\|_{L^2(\Omega_h)}\,\|\tilde{\boldsymbol{\psi}}_h - \mathsf{P}^{k-1}\boldsymbol{\psi}\|_{L^2(\Omega_h)}\\ &+ C\,h^{-1/2}\|\mathbb{P}\boldsymbol{e}_z\|_{L^2(\Omega_h)}\|\boldsymbol{\Pi\psi} - \tilde{\boldsymbol{\psi}}_h\|_{L^2(\partial\Omega)}\\ &+ \kappa_{\partial\Omega}\,\|\,(\mathsf{P}_\partial\boldsymbol{\sigma} - \boldsymbol{\Pi\sigma})\cdot\boldsymbol{n}\,\|_{L^2(\partial\Omega;h)}\,\|\,(\mathsf{P}_\partial\boldsymbol{\psi} - \boldsymbol{\Pi\psi})\cdot\boldsymbol{n}\,\|_{L^2(\partial\Omega;h)}.\end{aligned}$$

If we use approximation properties of $\mathbb{P}$ and $\boldsymbol{\Pi}$ we are able to prove the following corollary.

Corollary 4.5. *If τ_K is of order $\frac{1}{h_K}$ for all $K \in \Omega_h$, then*

$$\begin{aligned}\|\mathbb{P}\boldsymbol{e}_u\|_{L^2(\Omega_h)} \leq &C\,\mathcal{C}\,h^{\min\{k,1\}+k+1} + C\,h\,\|\,\boldsymbol{\Pi e}_{\boldsymbol{q}}\,\|_{L^2(\Omega_h)} + C\,h\,\|\,\widehat{u}_h - u_h\,\|_{L^2(\partial\Omega_h;\tau)}\\ &+ C\,h^{\min\{k,1\}+1}\,\|\,\mathbb{P}\boldsymbol{e}_z\,\|_{L^2(\Omega_h)}\\ &+ C\,h^{-1/4}\|\mathbb{P}\boldsymbol{e}_z\|^{1/2}_{L^2(\Omega_h)}\times\\ &\quad(\|\boldsymbol{\psi} - \boldsymbol{\Pi\psi}\|_{L^\infty(\Omega)} + l(k)\,h\|\nabla\cdot(\boldsymbol{\psi} - \boldsymbol{\Pi\psi})\|_{L^\infty(\Omega_h)})^{1/2}\end{aligned}$$

Proof. This result follows from Corollary 4.4 once we use

$$\|\tilde{\boldsymbol{\psi}}_h - \mathsf{P}^{k-1}\boldsymbol{\psi}\|_{L^2(\Omega_h)} \leq C\;h^{\min\{k,3\}}|\boldsymbol{\psi}|_{H^k(\Omega)},$$

and

$$\|\boldsymbol{\Pi}\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h\|_{L^2(\partial\Omega)} \le C\,(\|\boldsymbol{\psi} - \boldsymbol{\Pi}\boldsymbol{\psi}\|_{L^\infty(\Omega)} + l(k)\,h\|\nabla\cdot(\boldsymbol{\psi} - \boldsymbol{\Pi}\boldsymbol{\psi})\|_{L^\infty(\Omega_h)}).$$

The first inequality follows from estimates contained in [10]. The second inequality follows from the fact

$$\|\boldsymbol{\Pi}\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h\|_{L^2(\partial\Omega)} \le C\,\|\boldsymbol{\Pi}\boldsymbol{\psi} - \tilde{\boldsymbol{\psi}}_h\|_{L^\infty(\Omega)},$$

and Theorem 6.3 contained in the appendix. □

4.2. **Proof of Theorem 4.2.** We will estimate the last term of the right-hand side of the inequality in Corollary 4.5.

After using approximation properties of $\boldsymbol{\Pi}$ we get

$$\|\boldsymbol{\psi} - \boldsymbol{\Pi}\boldsymbol{\psi}\|_{L^\infty(\Omega)} + \log(\frac{1}{h})\,h\|\nabla\cdot(\boldsymbol{\psi} - \boldsymbol{\Pi}\boldsymbol{\psi})\|_{L^\infty(\Omega_h)} \le C\,\log(\frac{1}{h})\,h\|\boldsymbol{\psi}\|_{W^{1,\infty}(\Omega)}.$$

We next apply the following Sobolev inequality which holds in dimension $d = 2, 3$; see [8].

$$\|\boldsymbol{\psi}\|_{W^{1,\infty}(\Omega)} \le C\,\|\boldsymbol{\psi}\|_{H^3(\Omega)} \le C\,\|\mathbb{P}\boldsymbol{e}_u\|_{L^2(\Omega_h)}.$$

In the last inequality we used elliptic regularity (2.12).

Hence,

$$\begin{aligned}\|\mathbb{P}\boldsymbol{e}_u\|_{L^2(\Omega_h)} \le{}& C\,\mathcal{C}\,h + C\,h\,\|\,\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{q}}\,\|_{L^2(\Omega_h)} + C\,h\,\|\,\widehat{u}_h - u_h\,\|_{L^2(\partial\Omega_h;\tau)}\\ &+ C\,h\,\|\,\mathbb{P}\boldsymbol{e}_z\,\|_{L^2(\Omega_h)}\\ &+ C\,\log(\frac{1}{h})h^{1/2}\|\mathbb{P}\boldsymbol{e}_z\|_{L^2(\Omega_h)}.\end{aligned}$$

If we combine this inequality with (3.23), (3.24), (3.25) and (3.26) and apply algebraic manipulations we arrive at our result.

4.3. **Proof of Theorem 4.1.** We start by using Corollary 4.5 to estimate $\|\mathbb{P}\boldsymbol{e}_u\|_{L^2(\Omega_h)}$. If we use approximation properties of $\boldsymbol{\Pi}$

$$\|\boldsymbol{\psi} - \boldsymbol{\Pi}\boldsymbol{\psi}\|_{L^\infty(\Omega)} +\, h\|\nabla\cdot(\boldsymbol{\psi} - \boldsymbol{\Pi}\boldsymbol{\psi})\|_{L^\infty(\Omega_h)} \le C\,h^{2-2/p}\|\boldsymbol{\psi}\|_{W^{2,p}(\Omega)}$$

for $p > 2$. Here we used that $d = 2$.

Finally, since we are assuming $d = 2$ the following Sobolev inequality [8] holds

$$\|\boldsymbol{\psi}\|_{W^{2,p}(\Omega)} \le C\,p\,\|\boldsymbol{\psi}\|_{H^3(\Omega)} \le Cp\|\mathbb{P}\boldsymbol{e}_u\|_{L^2(\Omega_h)}.$$

for any $p < \infty$.

Therefore, we arrive at

$$\|\boldsymbol{\psi} - \boldsymbol{\Pi}\boldsymbol{\psi}\|_{L^\infty(\Omega)} +\, h\|\nabla\cdot(\boldsymbol{\psi} - \boldsymbol{\Pi}\boldsymbol{\psi})\|_{L^\infty(\Omega_h)} \le C\,p\,h^{2-2/p}\|\mathbb{P}\boldsymbol{e}_u\|_{L^2(\Omega_h)}.$$

Hence, applying Corollary 4.5 we get

$$\begin{aligned}\|\mathbb{P}\boldsymbol{e}_u\|_{L^2(\Omega_h)} \le{}& C\,\mathcal{C}\,h^3 + C\,h\,\|\,\boldsymbol{\Pi}\boldsymbol{e}_{\boldsymbol{q}}\,\|_{L^2(\Omega_h)} + C\,h\,\|\,\widehat{u}_h - u_h\,\|_{L^2(\partial\Omega_h;\tau)}\\ &+ C\,h^2\,\|\,\mathbb{P}\boldsymbol{e}_z\,\|_{L^2(\Omega_h)}\\ &+ C\,p\,h^{3/2-2/p}\|\mathbb{P}\boldsymbol{e}_z\|_{L^2(\Omega_h)}.\end{aligned}$$

If we use the above inequality and Theorem 2.2, we get

$$\|\mathbb{P}\boldsymbol{e}_u\|_{L^2(\Omega_h)} \le C\,\mathcal{C}\,p\,h^{3-\frac{2}{p}}.$$

If we let $p = 2\,\log(\frac{1}{h})$, then $h = e^{\frac{-p}{2}}$. So, we see that

$$h^{\frac{-2}{p}} = (e^{\frac{-p}{2}})^{\frac{-2}{p}} = e.$$

Hence,

$$\|\mathbb{P}\boldsymbol{e}_u\|_{L^2(\Omega_h)} \leq C\,\mathcal{C}\log(\frac{1}{h})\,h^3.$$

Theorem 4.1 now follows if we use the above inequality in the proof of Theorems 2.3 and 2.4.

5. Numerical results

In this section, we provide numerical experiments that support our theoretical convergence results of the SFH method. We also investigate the convergence properties of the SFH method in interior sub-domains. Finally, we explore the convergence properties of the SFH method for non-smooth solutions on non-convex domains using graded meshes.

5.1. **Smooth solution.** In this subsection, we carry out numerical experiments to validate the theoretical convergence properties of the SFH method for the biharmonic problem. In particular, the solution is smooth and the domain is such that the H^4 regularity assumption is satisfied.

We use uniform meshes obtained by discretizing $\Omega = (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2})$ with squares of side 2^{-l} which are then divided into two triangles as indicated in Fig. 1; the resulting mesh is denoted by "mesh=l".

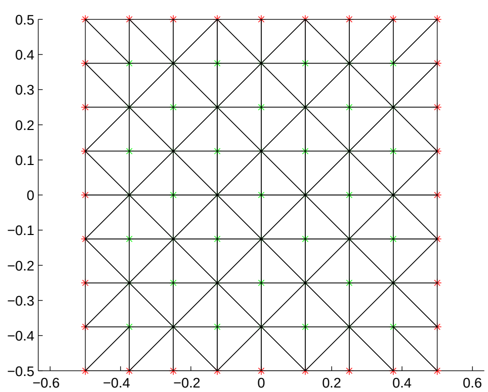

FIGURE 1. Example of a mesh with $h = 1/2^3$.

The test problem is obtained by choosing g and f so that the exact solution is $u(x, y) = x^4\,y^3$ on the domain Ω. The history of convergence of the SFH method with

$$\tau_K = 1/h = 2^l,$$

on the "mesh=l", is displayed in Table 1 for polynomials of degree $k = 0, 1, 2, 3$.

In Table 1, we observe that for k=1,2, 3, the quantities $\|u - u_h\|_{L^2(\Omega)}$ and $\|\boldsymbol{q}-\boldsymbol{q}_h\|_{\boldsymbol{L}^2(\Omega)}$ have optimal convergence rates, and $\|z-z_h\|_{L^2(\Omega)}$ and $\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\boldsymbol{L}^2(\Omega)}$ converge with order $k+1/2$ and $k-1/2$, respectively, as predicted by Theorem 2.2. We also see that $\|P_\partial u - \widehat{u}_h\|_{L^2(\mathcal{E}_h;h)}$ superconverges and $\|u - u_h^\star\|_{L^2(\Omega)}$ converges with rate $O(h^{k+2})$ for $k = 1, 2, 3$, which agrees with the conclusion of Theorems 2.3 and 4.1.

TABLE 1. History of convergence of the SFH method.

	mesh	$\|u-u_h\|_{L^2(\Omega)}$		$\|u-u_h^\star\|_{L^2(\Omega)}$		$\|\boldsymbol{q}-\boldsymbol{q}_h\|_{L^2(\Omega)}$		$\|z-z_h\|_{L^2(\Omega)}$		$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{L^2(\Omega)}$	
k	ℓ	error	order	error	order	error	order	error	order	error	order
0	1	.66e-2	-	.15e-2	-	.11e-1	-	.83e-1	-	.16e+1	-
	2	.33e-2	1.00	.97e-3	0.66	.95e-2	0.16	.62e-1	0.42	.10e+1	0.58
	3	.13e-2	1.40	.41e-3	1.23	.59e-2	0.69	.37e-1	0.77	.54e-0	0.95
	4	.41e-3	1.61	.11e-3	1.87	.29e-2	1.02	.19e-1	0.91	.33e-0	0.71
	5	.14e-3	1.52	.25e-4	2.15	.14e-2	1.10	.11e-1	0.88	.34e-0	-0.06
	6	.61e-4	1.24	.61e-5	2.04	.64e-3	1.08	.60e-2	0.81	.45e-0	-0.37
	7	.29e-4	1.06	.27e-5	1.20	.31e-3	1.05	.36e-2	0.74	.60e-0	-0.43
	8	.15e-4	1.01	.14e-5	0.89	.15e-3	1.02	.23e-2	0.66	.83e-0	-0.46
1	1	.10e-1	-	.29e-2	-	.21e-1	-	.15e-0	-	.15e+1	-
	2	.13e-2	2.94	.26e-3	3.47	.40e-2	2.40	.38e-1	2.03	.57e-0	1.38
	3	.19e-3	2.83	.20e-4	3.72	.88e-3	2.18	.94e-2	2.00	.31e-0	0.89
	4	.30e-4	2.68	.20e-5	3.33	.20e-3	2.15	.34e-2	1.45	.21e-0	0.55
	5	.56e-5	2.41	.22e-6	3.13	.47e-4	2.10	.13e-2	1.38	.15e-0	0.47
	6	.12e-5	2.16	.27e-7	3.06	.11e-4	2.04	.49e-3	1.43	.11e-0	0.47
	7	.30e-6	2.05	.33e-8	3.03	.28e-5	2.01	.18e-3	1.46	.78e-1	0.48
2	1	.35e-2	-	.31e-3	-	.56e-2	-	.56e-1	-	.62e-0	-
	2	.23e-3	3.91	.16e-4	4.31	.62e-3	3.18	.58e-2	3.27	.18e-0	1.78
	3	.17e-4	3.75	.93e-6	4.07	.74e-4	3.08	.11e-2	2.46	.65e-1	1.46
	4	.14e-5	3.61	.52e-7	4.15	.84e-5	3.14	.21e-3	2.32	.24e-1	1.42
	5	.14e-6	3.34	.31e-8	4.09	.99e-6	3.07	.40e-4	2.41	.90e-2	1.45
	6	.16e-7	3.12	.19e-9	4.04	.12e-6	3.02	.72e-5	2.46	.32e-2	1.47
3	1	.75e-3	-	.43e-4	-	.11e-2	-	.90e-2	-	.11e-0	-
	2	.28e-4	4.76	.12e-5	5.12	.72e-4	3.94	.53e-3	4.07	.21e-1	2.37
	3	.11e-5	4.73	.34e-7	5.18	.41e-5	4.15	.54e-4	3.30	.46e-2	2.21
	4	.45e-7	4.54	.94e-9	5.17	.23e-6	4.13	.55e-5	3.31	.92e-3	2.32
	5	.24e-8	4.26	.28e-10	5.09	.14e-7	4.05	.52e-6	3.40	.17e-3	2.41
	6	.14e-9	4.08	.84e-12	5.03	.87e-9	4.01	.47e-7	3.45	.32e-4	2.45

For the case of $k=0$, numerical results show that the approximations to u have optimal convergence order, in agreement with Theorem 4.2, up to a logarithmic factor. Numerically, it appears, that the approximation to $\boldsymbol{q}$ converges in an optimal way, which suggest that our error estimate for $\boldsymbol{q}$ is not sharp; see Theorem 4.2. Theoretically we have no conclusion about the convergence of the approximation of $\boldsymbol{\sigma}$ for $k=0$.

5.2. **Interior sub-domains.** In Theorem 2.2, we have that the convergence rates of z and $\boldsymbol{\sigma}$ in the domain are $k+1/2$ and $k-1/2$, respectively, if polynomials of degree up to k are used. The previous numerical experiment shows that these convergence rates are actually sharp. However, if we consider a fixed interior sub-domain, we observe optimal convergence rates of all the variables.

Here, we use the same test problem as in the first numerical experiment. The domain $\Omega=(-0.5,0.5)\times(-0.5,0.5)$ is discretized by uniform meshes; see Fig. 1. The exact solution is $u(x,y)=x^4\,y^3$. In Table 2, we observe that for $k=0,1,2$ the convergence rates of $u,\boldsymbol{q},z$ and $\boldsymbol{\sigma}$ are all optimal in the subdomain $\Omega_0 := (-0.4375,0.4375)\times(-0.4375,0.4375)$.

5.3. **Non-convex domain and graded meshes.** In our error analysis, we assume that the domain is convex and, in fact, we assume H^4 regularity for the dual problem. Moreover, we also assumed that our family of meshes are quasi-uniform. Here, we test the SFH method on a non-convex domain where we use highly graded

Table 2. History of convergence of the SFH method in the subdomain $\Omega_0 = (-0.4375, 0.4375) \times (-0.4375, 0.4375)$.

	mesh	$\|u-u_h\|_{L^2(\Omega_0)}$		$\|u-u_h^\star\|_{L^2(\Omega_0)}$		$\|\boldsymbol{q}-\boldsymbol{q}_h\|_{L^2(\Omega_0)}$		$\|z-z_h\|_{L^2(\Omega_0)}$		$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{L^2(\Omega_0)}$	
k	ℓ	error	order	error	order	error	order	error	order	error	order
0	4	.14e-3	-	.81e-4	-	.11e-2	-	.86e-2	-	.97e-1	-
	5	.51e-4	1.49	.16e-4	2.36	.53e-3	0.99	.43e-2	1.00	.41e-1	1.25
	6	.23e-4	1.15	.43e-5	1.89	.27e-3	0.99	.22e-2	0.99	.20e-1	1.05
	7	.12e-4	0.99	.23e-5	0.87	.13e-3	0.99	.11e-2	1.01	.99e-1	1.00
1	4	.80e-5	-	.88e-6	-	.88e-4	-	.55e-3	-	.44e-2	-
	5	.21e-5	1.94	.11e-6	2.95	.23e-4	1.95	.14e-3	1.94	.11e-2	2.06
	6	.52e-6	2.00	.14e-7	3.00	.57e-5	2.00	.36e-4	1.99	.26e-3	2.01
	7	.13e-6	1.99	.17e-8	3.02	.14e-5	2.00	.90e-5	1.99	.65e-3	2.00
2	4	.45e-6	-	.27e-7	-	.44e-5	-	.22e-4	-	.16e-3	-
	5	.62e-7	2.88	.17e-8	3.98	.57e-6	2.96	.29e-5	2.94	.17e-4	3.25
	6	.78e-8	2.98	.11e-9	3.99	.72e-7	2.99	.36e-6	2.98	.21e-5	3.01

meshes near the re-entrant corner. We observe that the method still converges well although our technical assumptions are violated.

We consider the non-convex domain Ω, which has vertices $(0,0), (0.5,0), (0.5,0.5)$, $(-0.5,0.5), (-0.5,-0.25)$ and $(-0.25,-0.25)$; see Fig. 2. Following Grisvard [19], we choose the exact solution to be the function

$$u(r,\theta) = r^{1+a}U(\theta)$$

where r is the distance to the origin and θ measures the angle from the positive x-axis. Here, a is the solution of the equation

$$\sin(a\theta) = a\sin(\theta),$$

and

$$U(\theta) = A(5\pi/4)B(\theta) - A(\theta)B(5\pi/4),$$

where

$$A(\theta) = \sin((a-1)\theta)/(a-1) - \sin((a+1)\theta)/(a+1),$$
$$B(\theta) = \cos((a-1)\theta) - \cos((a+1)\theta).$$

Notice that $a \approx 1.3$, so $u \in H^3(\Omega)$, however, $u \notin H^3(\Omega)$. In particular, $\boldsymbol{\sigma} \notin H^1(\Omega)$.

We use graded meshes ([4, 2, 22]) to capture the singularity at the origin; see Fig. 2. As usual, h denotes the largest mesh size of the triangulation. The size of an arbitrary triangle K, h_K, are chosen such that

$$h_K \leq C\, h^{\frac{1}{1-\beta}}$$

for triangles which have a vertex at the origin, and

$$h_K = C\, h\, r_K^\beta$$

for triangles which do not have a vertex at the origin. Here r_K is the distance of K to the origin. In general, to get accurate results the parameter $\beta < 1$ is chosen large enough depending on the polynomial degree k and the regularity of the exact solution. However, larger β gives more refined meshes. For our problem, $\beta = 1/2$ is a good choice for $k = 0, 1$.

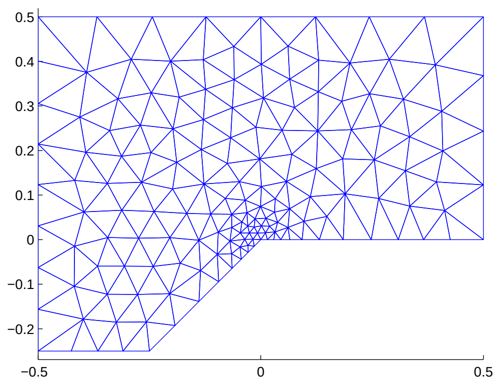

FIGURE 2. Example of a graded mesh.

For the graded meshes, we compute the orders of convergence in terms of the total number of degrees of freedom "N". We assume that a generic error "e" is of the form

$$e(N) = CN^{-order/2},$$

since in the uniform case we can take $h = N^{-1/2}$. The orders of convergence are given by

$$order(i) = \frac{\log(e(N_{i-1})/e(N_i))}{\log((N_{i-1}/N_i)^{-1/2})}.$$

Notice that the number of triangles in the mesh is proportional to the total number of degrees of freedom "N". We obtain the errors and orders of convergence for the approximate solutions using polynomials of degree $k = 0, 1$, which are displayed in Table 3. We can see that the convergence rates are still consistent with those predicted by the theory for smooth solutions. We expect similar results for higher polynomial degrees if properly graded meshes are used.

TABLE 3. History of convergence of the SFH method for a re-entrant corner problem.

	mesh	$\|u-u_h\|_{L^2(\Omega)}$		$\|u-u_h^\star\|_{L^2(\Omega)}$		$\|\boldsymbol{q}-\boldsymbol{q}_h\|_{L^2(\Omega)}$		$\|z-z_h\|_{L^2(\Omega)}$		$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{L^2(\Omega)}$	
k	$\#triangles$	error	order	error	order	error	order	error	order	error	order
0	94	.22e-0	-	.20e-1	-	.38 e-0	-	.11e+1	-	.12e+2	-
	280	.87e-1	1.72	.83e-2	1.66	.23e-0	0.97	.95e-0	0.31	.20e+2	-0.83
	1100	.36e-1	1.30	.23e-2	1.85	.11e-0	1.00	.61e-0	0.65	.21e+2	-0.09
	4366	.17e-2	1.10	.67e-3	1.81	.58e-1	0.98	.33e-0	0.89	.23e+2	-0.15
	18056	.79e-2	1.06	.15e-3	2.07	.28e-1	1.04	.23e-0	0.51	.31e+2	-0.43
	74062	.38e-2	1.03	.32e-4	2.26	.13e-1	1.03	.15e-0	0.57	.44e+2	-0.48
1	94	.11e-1	-	.270e-3	-	.11 e-1	-	.13e-0	-	.41e+1	-
	280	.26e-2	2.59	.47e-4	3.19	.36e-2	2.02	.62e-1	1.40	.31e+1	0.51
	1100	.67e-3	1.96	.71e-5	2.76	.98e-3	1.89	.23e-1	1.47	.21e+1	0.56
	4366	.16e-3	2.07	.95e-6	2.91	.26e-3	1.95	.84e-2	1.45	.15e+1	0.50
	18056	.38e-4	2.06	.11e-6	3.10	.61e-4	2.03	.29e-1	1.48	.11e+1	0.49
	74062	.89e-5	2.04	.12e-7	3.05	.15e-4	2.02	.10e-2	1.50	.75e-0	0.49

6. Concluding remarks

We can easily extend our results to more general boundary conditions. Extensions to other fourth-order systems of equations arising in computational mechanics constitutes the subject of ongoing research.

Let us end by pointing out that the loss of optimality we have observed in the approximation of z and $\boldsymbol{\sigma}$ does not take place in the one-dimensional case. How to prevent it and obtain methods optimally convergent remains an interesting open problem.

Ackknowledgements We would like to thank Manil Suri for valuable discussions about graded meshes. We also thank Monique Dauge and Serge Nicaise for providing precise references about how to construct singular solutions of the biharmonic problem.

Appendix I: Proof of the characterization result

In this section, we prove the characterization of the approximation, Theorem 2.1. We proceed in several steps.

6.1. **Step 1: Rewriting the conservativity conditions.** As we are going to see next, the weak formulation defining the Lagrange multipliers λ_h and γ_h in Theorem 2.1 is nothing but a rewriting of the conservativity conditions (2.4e) and (2.4f). To show this, we need the following auxiliary result whose proof will be presented in detail later.

Lemma 6.1. *for any* $\gamma \in M_h$ *and* $\mathsf{m} \in M_h^0$, *we have*

$$\begin{aligned}
&(i) && \langle \chi, \widehat{\mathbf{Q}}\gamma \cdot \boldsymbol{n}\rangle_{\partial\Omega_h} &&= (\mathcal{Z}\chi, \mathcal{Z}\gamma)_{\Omega_h},\\
&(ii) && \langle \chi, \widehat{\mathbf{Q}}\mathsf{m} \cdot \boldsymbol{n}\rangle_{\partial\Omega_h} &&= \langle \mathsf{m}, \mathbf{S}\chi \cdot \boldsymbol{n}\rangle_{\partial\Omega_h},\\
&(iii) && \langle \chi, \widehat{\mathbf{Q}} f \cdot \boldsymbol{n}\rangle_{\partial\Omega_h} &&= (f, \mathcal{U}\chi)_{\Omega_h},\\
&(iv) && \langle \mu, \widehat{\mathbf{S}}\gamma \cdot \boldsymbol{n}\rangle_{\partial\Omega_h} &&= \langle \gamma, \mathbf{Q}\mu \cdot \boldsymbol{n}\rangle_{\partial\Omega_h},\\
&(v) && \langle \mu, \widehat{\mathbf{S}}\mathsf{m} \cdot \boldsymbol{n}\rangle_{\partial\Omega_h} &&= 0,\\
&(vi) && \langle \mu, \widehat{\mathbf{S}} f\rangle_{\partial\Omega_h} &&= (f, \mathcal{U}\mu)_{\Omega_h}.
\end{aligned}$$

From the conservativity conditions (2.4e) and (2.4f), we get

$$\langle \chi, \widehat{\mathbf{Q}}\gamma + \widehat{\mathbf{Q}}\lambda + \widehat{\mathbf{Q}}g + \widehat{\mathbf{Q}}f) \cdot \boldsymbol{n}\rangle_{\partial\Omega_h} = \langle \chi, \mathsf{q}_{\mathsf{N}}\rangle_{\partial\Omega}, \tag{6.28a}$$

$$\langle \mu, \widehat{\mathbf{S}}\gamma + \widehat{\mathbf{S}}\lambda + \widehat{\mathbf{S}}g + \widehat{\mathbf{S}}f) \cdot \boldsymbol{n}\rangle_{\partial\Omega_h} = 0. \tag{6.28b}$$

Then we only need to apply Lemma 6.1 to substitute for the terms on the left hand side of above equations to see that the weak formulation for $(\gamma_h, \lambda_h) \in M_h \times M_h^0$ is nothing but a rewriting of the conservativity conditions (2.4e) and (2.4f).

6.2. **Step 2: Existence and uniqueness.** Here, we show that $(\gamma_h, \lambda_h) \in M_h \times M_h^0$ is uniquely defined provided $\tau_K > 0$ for all $K \in \Omega_h$ and the assumptions (2.8) are satisfied. To do that, we are going to use a key auxiliary result which easily follows from the definitions of the local solvers by arguments similar to those used in [10].

Lemma 6.2. *Assume that* $\tau_k > 0$ *for all* $K \in \Omega_h$. *Then for any* $\gamma \in M_h$ *and* $\mathsf{m} \in M_h^0$, *we have*

$$\nabla \cdot \boldsymbol{\mathcal{S}}\gamma = 0, \quad (\widehat{\boldsymbol{\mathcal{S}}}\gamma - \boldsymbol{\mathcal{S}}\gamma) \cdot \boldsymbol{n}|_{\mathscr{E}_h} = \tau(\mathcal{Z}\gamma - \gamma)|_{\mathscr{E}_h} = 0, \tag{6.29}$$

$$\nabla \cdot \boldsymbol{\mathfrak{Q}}\mathsf{m} = 0, \quad (\widehat{\boldsymbol{\mathfrak{Q}}}\mathsf{m} - \boldsymbol{\mathfrak{Q}}\mathsf{m}) \cdot \boldsymbol{n}|_{\mathscr{E}_h} = \tau(\mathcal{U}\mathsf{m} - \mathsf{m})|_{\mathscr{E}_h} = 0, \tag{6.30}$$

$$\boldsymbol{\mathcal{S}}\mathsf{m} = \mathcal{Z}\mathsf{m} = 0, \quad \widehat{\boldsymbol{\mathcal{S}}}\mathsf{m} = 0. \tag{6.31}$$

We only have to show that the only solution $(\gamma_h, \lambda_h) \in M_h \times M_h^0$ of the formulation

$$\begin{aligned} a_h(\gamma_h, \chi) + b_h(\lambda_h, \chi) &= 0) \qquad \forall \chi \in M_h,\\ b_h(\mu, \gamma_h) \qquad\qquad\quad &= 0 \qquad \forall \mu \in M_h^0, \end{aligned}$$

is the trivial one.

To do that, we begin by noting that if we take $\chi := \gamma_h$ and $\mu := \lambda_h$, we obtain that $a_h(\gamma_h, \gamma_h) = 0$ which implies that $\mathcal{Z}\gamma_h = 0$ on Ω_h. This implies that $\tau\,\gamma_h = 0$ on $\partial\Omega_h$, by property (6.29) of Lemma 6.2. We can also see that the second equation implies that $\boldsymbol{\mathcal{S}}\gamma_h \cdot \boldsymbol{n} = 0$ on $\partial\Omega_h \setminus \partial\Omega$. Hence, the first equation defining the first local solver, (2.9a), gives, for any $K \in \Omega_h$,

$$(\boldsymbol{\mathcal{S}}\gamma_h, \boldsymbol{\rho})_K = \langle \gamma_h, \boldsymbol{\rho} \cdot \boldsymbol{n} \rangle_{\partial K \setminus e_K^\tau} \qquad \text{for all } \boldsymbol{\rho} \in \boldsymbol{\mathcal{P}}^k(K).$$

Now, taking $\boldsymbol{\rho} := \boldsymbol{\mathcal{S}}\gamma_h$, we get that

$$(\boldsymbol{\mathcal{S}}\gamma_h, \boldsymbol{\mathcal{S}}\gamma_h)_K = \langle \gamma_h, \boldsymbol{\mathcal{S}}\gamma_h \cdot \boldsymbol{n} \rangle_{\{\partial K \setminus e_K^\tau\} \cap \partial\Omega} = 0,$$

provided $\{\partial K \setminus e_K^\tau\} \cap \partial\Omega = \emptyset$, that is, provide the condition (2.8) in satisfied. This implies that $\boldsymbol{\mathcal{S}}\gamma_h = 0$ and hence that

$$\langle \gamma_h, \boldsymbol{\rho} \cdot \boldsymbol{n} \rangle_{\partial K \setminus e_K^\tau} = 0 \qquad \text{for all } \boldsymbol{\rho} \in \boldsymbol{\mathcal{P}}^k(K).$$

This implies that $\gamma_h = 0$ on $\partial K \setminus \eta$ and hence on ∂K.

Let us now show that λ_h is also equal to zero. Since the first equation of the weak formulation is

$$b_h(\lambda_h, \chi) = 0 \qquad \forall \chi \in M_h,$$

we see that $\boldsymbol{\mathfrak{Q}}\lambda_h \cdot \boldsymbol{n} = 0$ on $\partial\Omega_h$. If we take $\boldsymbol{v} := \boldsymbol{\mathfrak{Q}}\lambda_h$ in the third equation defining the second local solver, (2.10c), we obtain

$$(\boldsymbol{\mathfrak{Q}}\lambda_h, \boldsymbol{\mathfrak{Q}}\lambda_h)_K = -(\mathcal{U}\lambda_h, \nabla \cdot \boldsymbol{\mathfrak{Q}}\lambda_h)_K = 0,$$

by property (6.30) of Lemma 6.2. This implies that $\boldsymbol{\mathfrak{Q}}\lambda_h = 0$ on K and so, the equation (2.10c) becomes, after a simple integration by parts,

$$-(\nabla\,\mathcal{U}\lambda_h, \boldsymbol{v})_K = \langle \lambda_h - \mathcal{U}\lambda_h, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial K} = \langle \lambda_h - \mathcal{U}\lambda_h, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial K \setminus e_K^\tau},$$

by property (6.30) of Lemma 6.2. This immediately implies, see Lemma 3.2 in [10] that $\nabla\,\mathcal{U}\lambda_h = 0$ on K and that $\mathcal{U}\lambda_h = \lambda_h$ on $\partial K \setminus e_K^\tau$ and hence on ∂K. As a consequence λ_h is a constant on $\mathscr{E}_h$ and since $\lambda_h \in M_h^0$, λ_h is identically equal to zero on $\mathscr{E}_h$.

This completes the proof of Theorem 2.1.

6.3. **Step 3: Proof of the auxiliary Lemma 6.1.** Now let us prove Lemma 6.1.

(i) By taking $\gamma = \chi$, and $\boldsymbol{\rho} = \mathbf{\mathcal{Q}}\gamma$ in the definition of the local solvers (2.9a), we have that

$$\begin{aligned}\langle \chi, \widehat{\mathbf{\mathcal{Q}}}\gamma \cdot \boldsymbol{n}\rangle_{\partial\Omega_h} &= \langle \chi, (\widehat{\mathbf{\mathcal{Q}}}\gamma - \mathbf{\mathcal{Q}}\gamma)\cdot \boldsymbol{n}\rangle_{\partial\Omega_h} + \langle \chi, \mathbf{\mathcal{Q}}\gamma \cdot \boldsymbol{n}\rangle_{\partial\Omega_h}\\ &= \langle \chi, (\widehat{\mathbf{\mathcal{Q}}}\gamma - \mathbf{\mathcal{Q}}\gamma)\cdot \boldsymbol{n}\rangle_{\partial\Omega_h} + (\mathbf{\mathcal{S}}\chi, \mathbf{\mathcal{Q}}\gamma)_{\Omega_h} + (\mathcal{Z}\chi, \nabla\cdot \mathbf{\mathcal{Q}}\gamma)_{\Omega_h}.\end{aligned}$$

Then we rewrite the last two terms on the right hand side of the last equality. By taking $\boldsymbol{v} = \mathbf{\mathcal{S}}\chi$ in the definition (2.9c) and integrating by parts, we obtain that

$$\begin{aligned}(\mathbf{\mathcal{S}}\chi, \mathbf{\mathcal{Q}}\gamma)_{\Omega_h} &= -(\mathcal{U}\gamma, \nabla\cdot \mathbf{\mathcal{S}}\chi)_{\Omega_h}\\ &= -\langle \mathcal{U}\gamma, \mathbf{\mathcal{S}}\chi\cdot \boldsymbol{n}\rangle_{\partial\Omega_h} + (\mathbf{\mathcal{S}}\chi, \nabla\mathcal{U}\gamma)_{\Omega_h}\\ &= \langle (\widehat{\mathbf{\mathcal{S}}}\chi - \mathbf{\mathcal{S}}\chi)\cdot \boldsymbol{n}, \mathcal{U}\gamma\rangle_{\partial\Omega_h}\\ &= 0\end{aligned}$$

by Lemma 6.2 (6.29). Using integration by parts and taking $\omega = \mathcal{Z}\chi$ in the definition of the local solvers (2.9d) we get

$$\begin{aligned}(\mathcal{Z}\chi, \nabla\cdot \mathbf{\mathcal{Q}}\gamma)_{\Omega_h} &= \langle \mathcal{Z}\chi, \mathbf{\mathcal{Q}}\gamma\cdot \boldsymbol{n}\rangle_{\partial\Omega_h} - (\nabla\mathcal{Z}\chi, \mathbf{\mathcal{Q}}\gamma)_{\Omega_h}\\ &= -\langle \mathcal{Z}\chi, (\widehat{\mathbf{\mathcal{Q}}}\gamma - \mathbf{\mathcal{Q}}\gamma)\cdot \boldsymbol{n}\rangle_{\partial\Omega_h} + (\mathcal{Z}\chi, \mathcal{Z}\gamma)_{\Omega_h}.\end{aligned}$$

Hence

$$\langle \chi, \widehat{\mathbf{\mathcal{Q}}}\gamma\cdot \boldsymbol{n}\rangle_{\partial\Omega_h} = (\mathcal{Z}\chi, \mathcal{Z}\gamma)_{\Omega_h} + \langle \chi - \mathcal{Z}\chi, (\widehat{\mathbf{\mathcal{Q}}}\gamma - \mathbf{\mathcal{Q}}\gamma)\cdot \boldsymbol{n}\rangle_{\partial\Omega_h}.$$

By Lemma 6.2 (6.29) we have

$$\langle \chi - \mathcal{Z}\chi, (\widehat{\mathbf{\mathcal{Q}}}\gamma - \mathbf{\mathcal{Q}}\gamma)\cdot \boldsymbol{n}\rangle_{\partial\Omega_h} = \langle \tau(\chi - \mathcal{Z}\chi), \mathcal{U}\gamma\rangle_{\partial\Omega_h} = 0,$$

so we get

$$\langle \chi, \widehat{\mathbf{\mathcal{Q}}}\gamma\cdot \boldsymbol{n}\rangle_{\partial\Omega_h} = (\mathcal{Z}\chi, \mathcal{Z}\gamma)_{\Omega_h}.$$

(ii) Using Lemma 6.2 (6.30), and taking $\gamma = \chi$ and $\boldsymbol{\rho} = \mathbf{\mathcal{Q}}\lambda$ in the definition of the local solvers (2.9a), we have that

$$\begin{aligned}\langle \chi, \widehat{\mathbf{\mathcal{Q}}}\mathsf{m}\cdot \boldsymbol{n}\rangle_{\partial\Omega_h} &= \langle \chi, \mathbf{\mathcal{Q}}\mathsf{m}\cdot \boldsymbol{n}\rangle_{\partial\Omega_h}\\ &= (\mathbf{\mathcal{S}}\chi, \mathbf{\mathcal{Q}}\mathsf{m})_{\Omega_h} + (\mathcal{Z}\chi, \nabla\cdot \mathbf{\mathcal{Q}}\mathsf{m})_O h\\ &= (\mathbf{\mathcal{S}}\chi, \mathbf{\mathcal{Q}}\mathsf{m})_{\Omega_h}.\end{aligned}$$

Then in the definition of the local solvers (2.10c) taking $\boldsymbol{v} = \mathbf{\mathcal{S}}\chi$, we get

$$\langle \chi, \widehat{\mathbf{\mathcal{Q}}}\mathsf{m}\cdot \boldsymbol{n}\rangle_{\partial\Omega_h} = \langle \mathsf{m}, \mathbf{\mathcal{S}}\chi\cdot \boldsymbol{n}\rangle_{\partial\Omega_h} - (\mathcal{U}\mathsf{m}, \nabla\cdot \mathbf{\mathcal{S}}\chi)_{\Omega_h}.$$

From Lemma 6.2 (6.30) we know that $\nabla\cdot \mathbf{\mathcal{S}}\chi = 0$. Hence

$$\langle \chi, \widehat{\mathbf{\mathcal{Q}}}\mathsf{m}\cdot \boldsymbol{n}\rangle_{\partial\Omega_h} = \langle \mathsf{m}, \mathbf{\mathcal{S}}\chi\cdot \boldsymbol{n}\rangle_{\partial\Omega_h}.$$

(iii) Taking $\gamma = \chi$ and $\boldsymbol{\rho} = \boldsymbol{\mathcal{Q}} f$ in the definition of the local solvers (2.9a), we get

$$\begin{aligned}\langle \chi, \widehat{\boldsymbol{\mathcal{Q}}} f \cdot \boldsymbol{n}\rangle_{\partial\Omega_h} &= \langle \chi, (\widehat{\boldsymbol{\mathcal{Q}}} f - \boldsymbol{\mathcal{Q}} f)\cdot \boldsymbol{n}\rangle_{\partial\Omega_h} + \langle \chi, \boldsymbol{\mathcal{Q}} f \cdot \boldsymbol{n}\rangle_{\partial\Omega_h} \\ &= \langle \chi, (\widehat{\boldsymbol{\mathcal{Q}}} f - \boldsymbol{\mathcal{Q}} f)\cdot \boldsymbol{n}\rangle_{\partial\Omega_h} + (\boldsymbol{\mathcal{S}}\chi, \boldsymbol{\mathcal{Q}} f)_{\Omega_h} + (\mathcal{Z}\chi, \nabla\cdot \boldsymbol{\mathcal{Q}} f)_{\Omega_h}.\end{aligned}$$

Taking $\boldsymbol{v} = \boldsymbol{\mathcal{S}}\chi$ in the definition of the local solvers (2.11c), we get

$$(\boldsymbol{\mathcal{S}}\chi, \boldsymbol{\mathcal{Q}} f)_{\Omega_h} = -(\mathcal{U}\chi, \nabla\cdot\boldsymbol{\mathcal{S}}\chi)_{\Omega_h} = 0$$

by Lemma 6.2 (6.29). So we have

$$\begin{aligned}\langle \chi, \widehat{\boldsymbol{\mathcal{Q}}} f \cdot \boldsymbol{n}\rangle_{\partial\Omega_h} &= \langle \chi, (\widehat{\boldsymbol{\mathcal{Q}}} f - \boldsymbol{\mathcal{Q}} f)\cdot \boldsymbol{n}\rangle_{\partial\Omega_h} + (\mathcal{Z}\chi, \nabla\cdot \boldsymbol{\mathcal{Q}} f)_{\Omega_h} \\ &= \langle \chi, (\widehat{\boldsymbol{\mathcal{Q}}} f - \boldsymbol{\mathcal{Q}} f)\cdot \boldsymbol{n}\rangle_{\partial\Omega_h} + \langle \mathcal{Z}\chi, \boldsymbol{\mathcal{Q}} f\cdot \boldsymbol{n}\rangle_{\partial\Omega_h} - (\nabla\mathcal{Z}\chi, \boldsymbol{\mathcal{Q}} f)_{\Omega_h}\end{aligned}$$

by using integration by parts. Then using the definition of the local solvers (2.11d) by taking $\omega = \mathcal{Z}\chi$

$$\begin{aligned}\langle \chi, \widehat{\boldsymbol{\mathcal{Q}}} f \cdot \boldsymbol{n}\rangle_{\partial\Omega_h} &= \langle \chi, (\widehat{\boldsymbol{\mathcal{Q}}} f - \boldsymbol{\mathcal{Q}} f)\cdot \boldsymbol{n}\rangle_{\partial\Omega_h} + \langle \mathcal{Z}\chi, \boldsymbol{\mathcal{Q}} f\cdot \boldsymbol{n}\rangle_{\partial\Omega_h} \\ &\quad - \langle \mathcal{Z}\chi, \widehat{\boldsymbol{\mathcal{Q}}} f\cdot \boldsymbol{n}\rangle_{\partial\Omega_h} + (\mathcal{Z} f, \mathcal{Z}\chi)_{\Omega_h} \\ &= \langle \chi - \mathcal{Z}\chi, (\widehat{\boldsymbol{\mathcal{Q}}} f - \boldsymbol{\mathcal{Q}} f)\cdot \boldsymbol{n}\rangle_{\partial\Omega_h} + (\mathcal{Z} f, \mathcal{Z}\chi)_{\Omega_h}.\end{aligned}$$

From the definition (2.11f) and Lemma 6.2 (6.29) we get that

$$\langle \chi - \mathcal{Z}\chi, (\widehat{\boldsymbol{\mathcal{Q}}} f - \boldsymbol{\mathcal{Q}} f)\cdot \boldsymbol{n}\rangle_{\partial\Omega_h} = \langle \tau(\chi - \mathcal{Z}\chi), \mathcal{U} f)\rangle_{\partial\Omega_h} + (\mathcal{Z} f, \mathcal{Z}\chi)_{\Omega_h} = 0.$$

Hence

$$\langle \chi, \widehat{\boldsymbol{\mathcal{Q}}} f \cdot \boldsymbol{n}\rangle_{\partial\Omega_h} = (\mathcal{Z} f, \mathcal{Z}\chi)_{\Omega_h}.$$

Then we use the definition of the local solvers (2.9d) by taking $\gamma = \chi$ and $\omega = \mathcal{Z} f$, and we obtain that

$$\begin{aligned}\langle \chi, \widehat{\boldsymbol{\mathcal{Q}}} f \cdot \boldsymbol{n}\rangle_{\partial\Omega_h} &= \langle \widehat{\boldsymbol{\mathcal{Q}}}\chi\cdot\boldsymbol{n}, \mathcal{Z} f\rangle_{\partial\Omega_h} - (\boldsymbol{\mathcal{Q}}\chi, \nabla\mathcal{Z} f)_{\Omega_h} \\ &= \langle (\widehat{\boldsymbol{\mathcal{Q}}}\chi - \boldsymbol{\mathcal{Q}}\chi)\cdot\boldsymbol{n}, \mathcal{Z} f\rangle_{\partial\Omega_h} + (\nabla\cdot\boldsymbol{\mathcal{Q}}\chi, \mathcal{Z} f)_{\Omega_h}\end{aligned}$$

by integration by parts. Taking $\boldsymbol{\rho} = \boldsymbol{\mathcal{Q}}\chi$ in the definition of the local solvers (2.11a) and taking $(\gamma, \boldsymbol{v}) = (\chi, \boldsymbol{\mathcal{S}} f)$ in the definition (2.9c), we get that

$$\begin{aligned}(\nabla\cdot\boldsymbol{\mathcal{Q}}\chi, \mathcal{Z} f)_{\Omega_h} &= -(\boldsymbol{\mathcal{S}} f, \boldsymbol{\mathcal{Q}}\chi)_{\Omega_h} \\ &= (\mathcal{U}\chi, \nabla\cdot\boldsymbol{\mathcal{S}} f \\ &= \langle \mathcal{U}\chi, \boldsymbol{\mathcal{S}} f\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} - (\nabla\mathcal{U}\chi, \boldsymbol{\mathcal{S}} f)_{\Omega_h} \\ &= \langle \mathcal{U}\chi, (\boldsymbol{\mathcal{S}} f - \widehat{\boldsymbol{\mathcal{S}}} f)\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} + (f, \mathcal{U}\chi)_{\Omega_h}\end{aligned}$$

by integrating by parts and then taking $\eta = \mathcal{U}\chi$ in the definition of the local solvers (2.11b). Therefore

$$\langle \chi, \widehat{\boldsymbol{\mathcal{Q}}} f \cdot \boldsymbol{n}\rangle_{\partial\Omega_h} = \langle (\widehat{\boldsymbol{\mathcal{Q}}}\chi - \boldsymbol{\mathcal{Q}}\chi)\cdot\boldsymbol{n}, \mathcal{Z} f\rangle_{\partial\Omega_h} + \langle \mathcal{U}\chi, (\boldsymbol{\mathcal{S}} f - \widehat{\boldsymbol{\mathcal{S}}} f)\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} + (f, \mathcal{U}\chi)_{\Omega_h}.$$

From the definitions (2.9f) and (2.11e) we get

$$\langle (\widehat{\boldsymbol{\mathcal{Q}}}\chi - \boldsymbol{\mathcal{Q}}\chi)\cdot\boldsymbol{n}, \mathcal{Z} f\rangle_{\partial\Omega_h} + \langle \mathcal{U}\chi, (\boldsymbol{\mathcal{S}} f - \widehat{\boldsymbol{\mathcal{S}}} f)\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} = \langle \tau\,\mathcal{U}\chi, \mathcal{Z} f\rangle_{\partial\Omega_h} - \langle \mathcal{U}\chi, \tau\,\mathcal{U} f\rangle_{\partial\Omega_h} = 0.$$

So we get

$$\langle \chi, \widehat{\boldsymbol{\mathcal{Q}}} f \cdot \boldsymbol{n}\rangle_{\partial\Omega_h} = (f, \mathcal{U}\chi)_{\Omega_h}.$$

(iv) From Lemma 6.2 (6.30) we have that

$$\langle \mu, \widehat{\mathcal{S}}\gamma \cdot \boldsymbol{n}\rangle_{\partial\Omega_h} = \langle \mu, \boldsymbol{\mathcal{S}}\gamma \cdot \boldsymbol{n}\rangle_{\partial\Omega_h}.$$

Then taking $\mathsf{m} = \mu$ and $\boldsymbol{v} = \boldsymbol{\mathcal{S}}\gamma$ in the definition of the local solvers (2.10c), we get

$$\begin{aligned}\langle \mu, \widehat{\mathcal{S}}\gamma \cdot \boldsymbol{n}\rangle_{\partial\Omega_h} &= (\boldsymbol{\mathcal{Q}}\mu, \boldsymbol{\mathcal{S}}\gamma)_{\Omega_h} + (\mathcal{U}\mu, \nabla\cdot\boldsymbol{\mathcal{S}}\gamma)_{\Omega_h}\\ &= (\boldsymbol{\mathcal{Q}}\mu, \boldsymbol{\mathcal{S}}\gamma)_{\Omega_h}\end{aligned}$$

by Lemma 6.2 (6.29). Taking $\boldsymbol{\rho} = \boldsymbol{\mathcal{Q}}\mu$ in the definition of the local solvers (2.9a), we obtain

$$\begin{aligned}\langle \mu, \widehat{\mathcal{S}}\gamma \cdot \boldsymbol{n}\rangle_{\partial\Omega_h} &= \langle \gamma, \boldsymbol{\mathcal{Q}}\mu\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} - (\mathcal{Z}\gamma, \nabla\cdot\boldsymbol{\mathcal{Q}}\gamma)_{\Omega_h}\\ &= \langle \gamma, \boldsymbol{\mathcal{Q}}\mu\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}\end{aligned}$$

by Lemma 6.2 (6.30).

(v) By Lemma 6.2 (6.31) we have $\langle \mu, \widehat{\mathcal{S}}\mathsf{m}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} = 0$.

(vi) Using the definition of the local solvers (2.10c) by taking $\mathsf{m} = \mu$ and $\boldsymbol{v} = \boldsymbol{\mathcal{S}}f$, we get

$$\begin{aligned}\langle \mu, \widehat{\mathcal{S}}f\rangle_{\partial\Omega_h} &= \langle \mu, (\widehat{\mathcal{S}}f - \boldsymbol{\mathcal{S}}f)\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} + \langle \mu, \boldsymbol{\mathcal{S}}f\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}\\ &= \langle \mu, (\widehat{\mathcal{S}}f - \boldsymbol{\mathcal{S}}f)\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} + (\boldsymbol{\mathcal{Q}}\mu, \boldsymbol{\mathcal{S}}f)_{\Omega_h} + (\mathcal{U}\mu, \nabla\cdot\boldsymbol{\mathcal{S}}f)_{\Omega_h}.\end{aligned}$$

Using the definition of the local solvers (2.11a) by taking $\boldsymbol{\rho} = \boldsymbol{\mathcal{Q}}\mu$ and Lemma 6.2 (6.30), we get

$$(\boldsymbol{\mathcal{Q}}\mu, \boldsymbol{\mathcal{S}}f)_{\Omega_h} = (\mathcal{Z}f, \nabla\cdot\boldsymbol{\mathcal{Q}}\mu)_{\Omega_h} = 0.$$

Hence

$$\begin{aligned}\langle \mu, \widehat{\mathcal{S}}f\rangle_{\partial\Omega_h} &= \langle \mu, (\widehat{\mathcal{S}}f - \boldsymbol{\mathcal{S}}f)\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} + (\mathcal{U}\mu, \nabla\cdot\boldsymbol{\mathcal{S}}f)_{\Omega_h}\\ &= \langle \mu, (\widehat{\mathcal{S}}f - \boldsymbol{\mathcal{S}}f)\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} + \langle \mathcal{U}\mu, \boldsymbol{\mathcal{S}}f\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} - (\nabla\,\mathcal{U}\mu, \boldsymbol{\mathcal{S}}f)_{\Omega_h}\end{aligned}$$

by integrating by parts. Then taking $\eta = \mathcal{U}\mu$ in the definition of the local solvers (2.11b), we have

$$\begin{aligned}\langle \mu, \widehat{\mathcal{S}}f\rangle_{\partial\Omega_h} &= \langle \mu, (\widehat{\mathcal{S}}f - \boldsymbol{\mathcal{S}}f)\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} + \langle \mathcal{U}\mu, \boldsymbol{\mathcal{S}}f\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}\\ &\quad + (f, \mathcal{U}\mu)_{\Omega_h} - \langle \widehat{\mathcal{S}}f\cdot\boldsymbol{n}, \mathcal{U}\mu\rangle_{\partial\Omega_h}\\ &= (f, \mathcal{U}\mu)_{\Omega_h} + \langle \mu - \mathcal{U}\mu, (\widehat{\mathcal{S}}f - \boldsymbol{\mathcal{S}}f)\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}.\end{aligned}$$

Note that by Lemma 6.2 (6.30)

$$\langle \mu - \mathcal{U}\mu, (\widehat{\mathcal{S}}f - \boldsymbol{\mathcal{S}}f)\cdot\boldsymbol{n}\rangle_{\partial\Omega_h} = \langle \tau(\mu - \mathcal{U}\mu), \mathcal{Z}f\rangle_{\partial\Omega_h} = 0,$$

so we have

$$\langle \mu, \widehat{\mathcal{S}}f\rangle_{\partial\Omega_h} = (f, \mathcal{U}\mu)_{\Omega_h}.$$

This completes the proof of Lemma 6.1.

Appendix II: An L^∞ estimate for the SFH approximation of the flux

In this section we obtain pointwise bounds for the SFH approximation to the following problem.

$$\boldsymbol{p} + \nabla w = 0 \quad \text{in } \Omega, \tag{6.32a}$$

$$\nabla \cdot \boldsymbol{p} = f \quad \text{in } \Omega, \tag{6.32b}$$

$$w = g \quad \text{on } \partial\Omega. \tag{6.32c}$$

We assume that Ω is a convex polyhedral domain in dimension d and that the data f and g are sufficiently smooth. We note that the reason we assume Ω is convex is that we will use H^2 regularity results and estimates for the first derivative of the corresponding Green's function which hold in convex domains; see for example [16]. Moreover, we require that the family of meshes $\{\Omega_h\}$ be quasi-uniform.

We now state our main result.

Theorem 6.3. *Let $\boldsymbol{p}$ solve* (6.32) *and suppose that $\boldsymbol{p}_h$ is the SFH approximation to $\boldsymbol{p}$. Then, there exists a $C > 0$ independent of h such that*

$$\|\boldsymbol{p} - \boldsymbol{p}_h\|_{L^\infty(\Omega)} \le C(\|\boldsymbol{p} - \boldsymbol{\Pi p}\|_{L^\infty(\Omega)} + l(k)h\|\nabla \cdot (\boldsymbol{p} - \boldsymbol{\Pi p})\|_{L^\infty(\Omega_h)}),$$

where $l(k) = 1$ if $k \ge 1$ and $l(k) = \log(\frac{1}{h})$ if $k = 0$.

This result is very similar to the result contained in [17] for *conforming* mixed methods. In fact, we will follow very closely their techniques to prove the above theorem. Note that the estimates are quasi-optimal if $k = 0$ since a logarithmic factor appears in the estimate, which is the reason logarithmic factors appear in Theorem 4.2. Before proving the above theorem we gather some preliminary results.

7. Preliminary results

In this section we gather some preliminary results that were stated in [17]. We start defining the weight function σ which will allow us to convert $L^\infty(\Omega)$ estimates to weighted L^2 estimates.

$$\sigma(x) = (|x - x_0| + \theta^2)^{1/2}$$

where $\theta = C^* h$ and $C^* \ge 1$ and $x_0 \in \Omega$ is fixed.

Here we list some properties of σ.

Proposition 7.1. *Let $K \in \Omega_h$ then there exists a C independent of h and K such that*

$$\max_{x \in K} \sigma(x) \le C \min_{x \in K} \sigma(x) \tag{7.33}$$

$$|\partial^i \sigma^\alpha(x)| \le C|\sigma^{\alpha - i}(x)| \tag{7.34}$$

The weighted L^2 norm is given by

$$\|D^i v\|^2_{\sigma^\alpha} = \sum_{K \in \Omega_h} \sum_{\lambda = i} (\sigma^\alpha \partial^\lambda v, \partial^\lambda v).$$

Proposition 7.2. *Let $\omega \in H_h^{j+1}(\Omega_h)$ and $\boldsymbol{v} \in [H_h^{j+1}(\Omega_h)]^N$, then*

$$\|\omega - \mathbb{P}\omega\|_{\sigma^\alpha} \le h^{j+1}\|D^{j+1}\omega\|_{\sigma^\alpha}, \tag{7.35}$$

$$\|\boldsymbol{v}-\boldsymbol{\Pi}\boldsymbol{v}\|_{\sigma^\alpha}+h\|\nabla(\boldsymbol{v}-\boldsymbol{\Pi}\boldsymbol{v})\|_{\sigma^\alpha}\le h^{j+1}\|D^{j+1}\boldsymbol{v}\|_{\sigma^\alpha}. \tag{7.36}$$

If $\boldsymbol{v}\in\boldsymbol{V}_h$ and $\beta\in\mathbb{R}$, then

$$\|\sigma^\beta\boldsymbol{v}-\boldsymbol{\Pi}(\sigma^\beta\boldsymbol{v})\|_{\sigma^\alpha}\le C(\frac{h}{\theta})\|\boldsymbol{v}\|_{\sigma^{\alpha+2\beta}}. \tag{7.37}$$

The following results will be used to compare $L^\infty(\Omega)$ and weighted L^2 norms.

Proposition 7.3.

$$\|v\|_{\sigma^{-\alpha}}\le C\|v\|_{L^\infty(\Omega_h)}M,\quad \text{for } v\in L^\infty(\Omega_h) \tag{7.38}$$

where

$$M=\begin{cases}\theta^{(d-\alpha)/2} & \text{for } \alpha>d\\ |\log\theta|^{1/2} & \alpha=d.\end{cases}$$

If $\max_{x\in\Omega}|v(x)|=|v(x_0)|$, *then*

$$\|v\|_{L^\infty(\Omega_h)}\le C(\frac{\theta^\alpha}{h^d})^{1/2}\|v\|_{\sigma^{-\alpha}},\quad v\in W^h. \tag{7.39}$$

The following result is given in ([17], Lemma 3.1).

Proposition 7.4. *For every $\omega\in H_0^1(\Omega)\cap H^2(\Omega)$ we have*

$$\|D^2\omega\|_{\sigma^d}+\|D\omega\|_{\sigma^{d-2}}\le C(\frac{|\log\theta|^{1/2}}{\theta})\|\triangle\omega\|_{\sigma^{d+2}}$$

The following result can be found in ([17], Lemma 3.2).

Proposition 7.5. *Let $\beta\in H(div,\Omega)$ and suppose that $\omega\in H_0^1(\Omega)\cap H^2(\Omega)$ solve $-\triangle\omega=\nabla\cdot\beta$. Then, for $0<\alpha<2$*

$$\|D^2\omega\|_{\sigma^{d+\alpha}}\le C(\alpha)(\|\nabla\cdot\beta\|_{\sigma^{d+\alpha}}+\frac{1}{\theta}\|\beta\|_{\sigma^{d+\alpha}}).$$

If $\alpha=2$, then

$$\|D^2\omega\|_{\sigma^{d+2}}+\|D\phi\|_{\sigma^d}\le C(\|\nabla\cdot\beta\|_{\sigma^{d+2}}+\frac{|\log\theta|}{\theta}\|\beta\|_{\sigma^{d+2}}).$$

The proof of this result can be found in [17] and relies on H^2 regularity and estimates of the first derivative of the Greens function. These two properties hold if Ω is convex; see [16].

We end this section by writing the error equations that we will use.

$$(\boldsymbol{p}-\boldsymbol{p}_h,\boldsymbol{v})_{\Omega_h}-(\mathbb{P}w-w_h,\nabla\cdot\boldsymbol{v})_{\Omega_h}+\langle w-\widehat{w}_h,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\Omega_h}=0, \tag{7.40}$$

$$-(\boldsymbol{p}-\boldsymbol{p}_h,\nabla\omega)_{\Omega_h}+\langle(\boldsymbol{p}-\widehat{\boldsymbol{p}}_h)\cdot\boldsymbol{n},\omega\rangle_{\partial\Omega_h}=0, \tag{7.41}$$

for all $(\boldsymbol{v},\omega)\in\boldsymbol{V}_h\times W_h$.

8. Proof of Theorem 6.3

We let $0 < \alpha < 2$ be a fix number. Let x_0 be such that $|\boldsymbol{\Pi p} - \boldsymbol{p}_h|$ attains its maximum. Then, by (7.39) we get

$$\|\boldsymbol{\Pi p} - \boldsymbol{p}_h\|_{L^\infty(\Omega)} \le C\theta^{\alpha/2}\|\boldsymbol{\Pi p} - \boldsymbol{p}_h\|_{\sigma^{-(\alpha+d)}}, \tag{8.42}$$

where we used that $\theta \le h$. We first bound $\|\boldsymbol{\Pi p} - \boldsymbol{p}_h\|_{\sigma^{-(\alpha+d)}}$. Now set $\boldsymbol{\psi} := \sigma^{-(\alpha+d)}(\boldsymbol{\Pi p} - \boldsymbol{p}_h)$. We note that it follows from the analysis in [10] that $(\boldsymbol{\Pi p} - \boldsymbol{p}_h) \in H(\text{div}, \Omega)$, and, in fact, $\nabla \cdot (\boldsymbol{\Pi p} - \boldsymbol{p}_h)$=0. Therefore, $\boldsymbol{\psi} \in H(\text{div}, \Omega)$.

Then, we easily see that

$$\|\boldsymbol{\Pi p} - \boldsymbol{p}_h\|^2_{\sigma^{-(\alpha+d)}} = T_1 + T_2 + T_3.$$

where

$$\begin{aligned} T_1 =&(\boldsymbol{\Pi p} - \boldsymbol{p}, \boldsymbol{\psi})_{\Omega_h} \\ T_2 =&(\boldsymbol{p} - \boldsymbol{p}_h, \boldsymbol{\psi} - \boldsymbol{\Pi\psi})_{\Omega_h} \\ T_3 =&(\boldsymbol{p} - \boldsymbol{p}_h, \boldsymbol{\Pi\psi})_{\Omega_h}. \end{aligned}$$

We now proceed to bound each of these terms. The first estimate follows easily from the Cauchy-Schwarz inequality, the definition of $\boldsymbol{\psi}$ and Young's inequality.

$$T_1 \le \|\boldsymbol{\Pi p} - \boldsymbol{p}\|_{\sigma^{-(\alpha+d)}}\|\boldsymbol{\psi}\|_{\sigma^{\alpha+d}} \le \frac{1}{2}\|\boldsymbol{\Pi p} - \boldsymbol{p}_h\|^2_{\sigma^{-(\alpha+d)}} + C\|\boldsymbol{\Pi p} - \boldsymbol{p}\|^2_{\sigma^{-(\alpha+d)}}.$$

If we apply the Cauchy-Schwarz inequality and then (7.37) we get

$$\begin{aligned} T_2 \le& C\|\boldsymbol{p} - \boldsymbol{p}_h\|_{\sigma^{-(\alpha+d)}}\|\boldsymbol{\psi} - \boldsymbol{\Pi\psi}\|_{\sigma^{(\alpha+d)}} \\ \le& C(\frac{h}{\theta})\|\boldsymbol{p} - \boldsymbol{p}_h\|_{\sigma^{-(\alpha+d)}}\|\boldsymbol{\Pi p} - \boldsymbol{p}_h\|_{\sigma^{-(\alpha+d)}} \\ \le& C(\frac{h}{\theta})\|\boldsymbol{\Pi p} - \boldsymbol{p}_h\|^2_{\sigma^{-(\alpha+d)}} + C\|\boldsymbol{\Pi p} - \boldsymbol{p}\|^2_{\sigma^{-(\alpha+d)}}, \end{aligned}$$

where we used that $h \le \theta$.

In order to estimate T_3 we define $\phi \in H^1_0(\Omega) \cap H^2(\Omega)$ as the function that satisfies $\triangle\phi = \nabla \cdot \boldsymbol{\psi}$. If we define $\Gamma := \boldsymbol{\psi} - \nabla\phi$, then it is clear that $\nabla \cdot \Gamma = 0$. It then follows from estimate (vi) of Proposition 3.1 that $\boldsymbol{\Pi}\Gamma = \boldsymbol{\Pi}^{\rm RT}_k\Gamma$, where $\boldsymbol{\Pi}^{\rm RT}_k$ is the Raviart-Thomas projection of degree k. Therefore, $\boldsymbol{\Pi}\Gamma \in H(\text{div}, \Omega)$ and $\nabla\cdot\boldsymbol{\Pi}\Gamma = 0$. Hence,

$$\begin{aligned} T_3 =&(\boldsymbol{p} - \boldsymbol{p}_h, \boldsymbol{\Pi}\nabla\phi)_{\Omega_h} + (\boldsymbol{p} - \boldsymbol{p}_h, \boldsymbol{\Pi}\Gamma)_{\Omega_h} \\ =&(\boldsymbol{p} - \boldsymbol{p}_h, \boldsymbol{\Pi}\nabla\phi)_{\Omega_h}, \end{aligned}$$

where we used that $(\boldsymbol{p} - \boldsymbol{p}_h, \boldsymbol{\Pi}\Gamma)_{\Omega_h} = 0$ which follows from (7.40).

After a simple algebraic manipulations we have

$$\begin{aligned} T_3 =&(\boldsymbol{p} - \boldsymbol{p}_h, \boldsymbol{\Pi}\nabla\phi)_{\Omega_h} \\ =&(\boldsymbol{p} - \boldsymbol{p}_h, \nabla\phi)_{\Omega_h} + (\boldsymbol{p} - \boldsymbol{p}_h, \boldsymbol{\Pi}\nabla\phi - \nabla\phi)_{\Omega_h} \\ =&(\boldsymbol{\Pi p} - \boldsymbol{p}_h, \nabla\phi)_{\Omega_h} + (\boldsymbol{p} - \boldsymbol{\Pi p}, \nabla\phi)_{\Omega_h} \\ &+ (\boldsymbol{p} - \boldsymbol{p}_h, \boldsymbol{\Pi}\nabla\phi - \nabla\phi)_{\Omega_h} \\ =&(\boldsymbol{p} - \boldsymbol{\Pi p}, \nabla\phi)_{\Omega_h} + (\boldsymbol{p} - \boldsymbol{p}_h, \boldsymbol{\Pi}\nabla\phi - \nabla\phi)_{\Omega_h}. \end{aligned}$$

In the last equation we used integration by parts and used the fact that $\phi = 0$ on $\partial\Omega$ to show that

$$(\boldsymbol{\Pi p} - \boldsymbol{p}_h, \nabla\phi)_{\Omega_h} = -(\nabla \cdot (\boldsymbol{\Pi p} - \boldsymbol{p}_h), \phi)_{\Omega_h} = 0,$$

where we used that $\nabla\cdot(\boldsymbol{\Pi p}-\boldsymbol{p}_h)=0$. We can now simplify $(\boldsymbol{p}-\boldsymbol{\Pi p},\nabla\phi)_{\Omega_h}$.

If we use the property (3.16a) of $\boldsymbol{\Pi}$ we get

$$\begin{aligned}(\boldsymbol{p}-\boldsymbol{\Pi p},\nabla\phi)_{\Omega_h} =&(\boldsymbol{p}-\boldsymbol{\Pi p},\nabla(\phi-\mathbb{P}\phi))_{\Omega_h}\\ =&-(\nabla\cdot(\boldsymbol{p}-\boldsymbol{\Pi p}),\phi-\mathbb{P}\phi)_{\Omega_h}\\ &+\langle(\boldsymbol{p}-\boldsymbol{\Pi p})\cdot\boldsymbol{n},\phi-\mathbb{P}\phi\rangle_{\partial\Omega_h}\\ =&-(\nabla\cdot(\boldsymbol{p}-\boldsymbol{\Pi p}),\phi-\mathbb{P}\phi)_{\Omega_h}.\end{aligned}$$

In the last equation we used

$$\langle(\boldsymbol{p}-\boldsymbol{\Pi p})\cdot\boldsymbol{n},\phi-\mathbb{P}\phi\rangle_{\partial\Omega_h}=0,$$

which can be deduced by using Proposition 3.1 and the fact that $\phi=0$ on $\partial\Omega$. Hence,

$$T_3=T_4+T_5,$$

where

$$\begin{aligned}T_4:=&(\boldsymbol{p}-\boldsymbol{p}_h,\boldsymbol{\Pi}\nabla\phi-\nabla\phi)_{\Omega_h}\\ T_5:=&-(\nabla\cdot(\boldsymbol{p}-\boldsymbol{\Pi p}),\phi-\mathbb{P}\phi)_{\Omega_h}.\end{aligned}$$

If we apply (7.36) and Proposition 7.5 we get

$$\begin{aligned}T_4\le&\|\boldsymbol{p}-\boldsymbol{p}_h\|_{\sigma^{-(\alpha+d)}}\|\nabla\phi-\boldsymbol{\Pi}\nabla\phi\|_{\sigma^{\alpha+d}}\\ \le&Ch\|\boldsymbol{p}-\boldsymbol{p}_h\|_{\sigma^{-(\alpha+d)}}\|D^2\phi\|_{\sigma^{\alpha+d}}\\ \le&Ch\|\boldsymbol{p}-\boldsymbol{p}_h\|_{\sigma^{-(\alpha+d)}}(\|\nabla\cdot\boldsymbol{\psi}\|_{\sigma^{\alpha+d}}+\frac{1}{\theta}\|\boldsymbol{\psi}\|_{\sigma^{\alpha+d}}).\end{aligned}$$

We can easily show using the definitions of $\boldsymbol{\psi}$, σ and inequality (7.34) that

$$\|\nabla\cdot\boldsymbol{\psi}\|_{\sigma^{\alpha+d}}+\frac{1}{\theta}\|\boldsymbol{\psi}\|_{\sigma^{\alpha+d}}\le\frac{C}{\theta}\|\boldsymbol{\Pi p}-\boldsymbol{p}_h\|_{\sigma^{-(\alpha+d)}}.$$

Since we will need this estimate later we note here that we showed

$$\|D^2\phi\|_{\sigma^{\alpha+d}}\le C(\frac{h}{\theta})\|\boldsymbol{\Pi p}-\boldsymbol{p}_h\|_{\sigma^{-(\alpha+d)}}. \tag{8.43}$$

Therefore,

$$T_4\le C(\frac{h}{\theta})\|\boldsymbol{\Pi p}-\boldsymbol{p}_h\|^2_{\sigma^{-(\alpha+d)}}+\|\boldsymbol{\Pi p}-\boldsymbol{p}\|^2_{\sigma^{-(\alpha+d)}},$$

where we used that $\theta\le h$. By the estimates for T_1,T_2,T_3 and T_4 and taking $C^*=\frac{\theta}{h}$ sufficiently large we get

$$\|\boldsymbol{\Pi p}-\boldsymbol{p}_h\|^2_{\sigma^{-(\alpha+d)}}\le C\|\boldsymbol{\Pi p}-\boldsymbol{p}\|^2_{\sigma^{-(\alpha+d)}}+C\,T_5. \tag{8.44}$$

The estimate of T_5 will be different in the case of $k\ge1$ and $k=0$. We first assume that $k\ge1$. In this case, by using (7.35) and (8.43) we get

$$\begin{aligned}T_5\le&Ch^2\|\nabla\cdot(\boldsymbol{p}-\boldsymbol{\Pi p})\|_{\sigma^{-(\alpha+d)}}\|D^2\phi\|_{\sigma^{\alpha+d}}\\ \le&\frac{h^2}{\theta}\|\nabla\cdot(\boldsymbol{p}-\boldsymbol{\Pi p})\|_{\sigma^{-(\alpha+d)}}\|\boldsymbol{\Pi p}-\boldsymbol{p}_h\|_{\sigma^{-(\alpha+d)}}\\ \le&\delta\|\boldsymbol{\Pi p}-\boldsymbol{p}_h\|^2_{\sigma^{-(\alpha+d)}}+\frac{Ch^2}{\delta}\|\nabla\cdot(\boldsymbol{p}-\boldsymbol{\Pi p})\|^2_{\sigma^{-(\alpha+d)}}\end{aligned}$$

where $\delta>0$ is arbitrary. By using (8.44) and by taking δ sufficiently small we get

$$\|\boldsymbol{\Pi p}-\boldsymbol{p}_h\|^2_{\sigma^{-(\alpha+d)}}\le C\|\boldsymbol{\Pi p}-\boldsymbol{p}\|^2_{\sigma^{-(\alpha+d)}}+Ch^2\|\nabla\cdot(\boldsymbol{p}-\boldsymbol{\Pi p})\|^2_{\sigma^{-(\alpha+d)}}.$$

If we then apply (8.42) we get

$$\begin{aligned}\|\boldsymbol{\Pi p}-\boldsymbol{p}_h\|^2_{L^\infty(\Omega)} \leq& C\theta^\alpha\|\boldsymbol{\Pi p}-\boldsymbol{p}\|^2_{\sigma^{-(\alpha+d)}}+Ch^2\theta^\alpha\|\nabla\cdot(\boldsymbol{p}-\boldsymbol{\Pi p})\|^2_{\sigma^{-(\alpha+d)}}\\ \leq& C\|\boldsymbol{\Pi p}-\boldsymbol{p}\|^2_{L^\infty(\Omega)}+Ch^2\|\nabla\cdot(\boldsymbol{p}-\boldsymbol{\Pi p})\|^2_{L^\infty(\Omega_h)}.\end{aligned}$$

In the last inequality we used (7.38). This completes the proof in the case that $k \geq 1$. Now we assume that $k = 0$ and proceed to estimate T_5. In this case, we apply Proposition 7.5 to get

$$\begin{aligned}T_5 \leq& Ch\|\nabla\cdot(\boldsymbol{p}-\boldsymbol{\Pi p})\|_{\sigma^{-d}}\|D\phi\|_{\sigma^d}\\ \leq& Ch\|\nabla\cdot(\boldsymbol{p}-\boldsymbol{\Pi p})\|_{\sigma^{-d}}(\|\nabla\cdot\boldsymbol{\psi}\|_{\sigma^{2+d}}+\frac{|\log(\theta)|^{1/2}}{\theta}\|\boldsymbol{\psi}\|_{\sigma^{2+d}})\\ \leq& \frac{C|\log(\theta)|^{1/2}h}{\theta}\|\nabla\cdot(\boldsymbol{p}-\boldsymbol{\Pi p})\|_{\sigma^{-d}}\|\boldsymbol{\Pi p}-\boldsymbol{p}_h\|_{\sigma^{2-d-2\alpha}}.\end{aligned}$$

In the last inequality we used the definition of $\boldsymbol{\psi}$ and (7.34). If we let $\alpha > 1$ and apply (7.38) we get

$$T_5 \leq \frac{C|\log(\theta)|h}{\theta^\alpha}\|\nabla\cdot(\boldsymbol{p}-\boldsymbol{\Pi p})\|_{L^\infty(\Omega_h)}\|\boldsymbol{\Pi p}-\boldsymbol{p}_h\|_{L^\infty(\Omega)}.$$

If we now use (8.44), (8.42) and (7.38) we have

$$\begin{aligned}\|\boldsymbol{\Pi p}-\boldsymbol{p}_h\|^2_{L^\infty(\Omega)} \leq& C\|\boldsymbol{\Pi p}-\boldsymbol{p}\|^2_{L^\infty(\Omega)}\\ &+C|\log(h)|h\|\nabla\cdot(\boldsymbol{p}-\boldsymbol{\Pi p})\|_{L^\infty(\Omega_h)}\|\boldsymbol{\Pi p}-\boldsymbol{p}_h\|_{L^\infty(\Omega)}.\end{aligned}$$

The result follows if we use Young's inequality.

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