A FAMILY OF NON-CONFORMING ELEMENTS FOR THE BRINKMAN PROBLEM

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Abstract. We propose and analyze a new family of nonconforming elements for the Brinkman problem of porous media flow. The corresponding finite element methods are robust with respect to the limiting case of Darcy flow, and the discretely divergence-free functions are in fact divergence-free. Therefore, in the absence of sources and sinks, the method is strongly mass conservative. We also show how the proposed elements are part of a discrete de Rham complex.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded, connected, polyhedral domain. We consider the following Brinkman model of porous flow:

$$-\operatorname{div}(\nu \,\mathbf{grad}\, \boldsymbol{u}) + \alpha \boldsymbol{u} + \mathbf{grad}\, p = \boldsymbol{f} \qquad \text{in } \Omega, \tag{1.1a}$$

$$\operatorname{div} \boldsymbol{u} = g \qquad \text{in } \Omega, \tag{1.1b}$$

$$\boldsymbol{u} = 0 \qquad \text{on } \partial\Omega. \tag{1.1c}$$

Here, $\boldsymbol{u}$ is the velocity, p is the pressue, $\alpha > 0$ is the dynamic viscosity divided by the permeability, $\nu > 0$ is the effective viscosity, and $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega) := L^2(\Omega)^d$ and $g \in L^2(\Omega)$ are two forcing terms. Problem (1.1) models creeping flow in a highly porous media, and arise in various physical models, e.g., subsurface flow problems [16, 31], heat & mass transfer in pipes [24, 28], liquid composite molding [21], the behavior and influence of osteonal structures [30], and computational fuel cell dynamics [39].

To simplify the mathematical analysis, we assume that the coefficients in (1.1) are constant. Furthermore, we assume that g satisfies the following compatibility criterion throughout the paper:

$$\int_\Omega g \, dx = 0.$$

Defining the velocity space and pressure space, respectively, as

$$\boldsymbol{V} = \boldsymbol{H}_0^1(\Omega) := H_0^1(\Omega)^d \quad \text{and} \quad W = L_0^2(\Omega) := \{ w \in L^2(\Omega) : \ (w, 1) = 0 \},$$

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a pair of functions $(\boldsymbol{u}, p) \in \boldsymbol{V} \times W$ are defined to be a solution to (1.1) if for all $(\boldsymbol{v}, w) \in \boldsymbol{V} \times W$ there holds

$$a(\boldsymbol{u}, \boldsymbol{v}) - b(\boldsymbol{v}, p) = (\boldsymbol{f}, \boldsymbol{v}), \tag{1.2a}$$

$$b(\boldsymbol{u}, w) = (g, w). \tag{1.2b}$$

Here the bilinear forms $a(\cdot,\cdot) : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ and $b(\cdot,\cdot) : \boldsymbol{V} \times W \to \mathbb{R}$ are defined as

$$a(\boldsymbol{v}, \boldsymbol{w}) = (\nu\, \mathbf{grad}\, \boldsymbol{v}, \mathbf{grad}\, \boldsymbol{w}) + (\alpha \boldsymbol{v}, \boldsymbol{w}), \tag{1.3}$$

$$b(\boldsymbol{v}, w) = (\operatorname{div} \boldsymbol{v}, w), \tag{1.4}$$

and $(\cdot,\cdot) := (\cdot,\cdot)_\Omega$ denotes the L^2 inner product over Ω.

As the inf-sup condition

$$\sup_{\boldsymbol{v} \in \boldsymbol{V}} \frac{b(\boldsymbol{v}, w)}{\|\boldsymbol{v}\|_{H^1(\Omega)}} \geq C\|w\|_{L^2(\Omega)} \tag{1.5}$$

is known to hold [19], it follows from standard theory from saddle point problems [10] that there exists a unique solution $(\boldsymbol{u}, p) \in \boldsymbol{V} \times W$ to problem (1.2).

For ν of moderate size and $g \equiv 0$, equation (1.1) is a standard Stokes problem with an additional (non-harmful) positive zero–th order term. Thus, to compute the solution, it is quite natural to apply any of the standard Stokes finite elements (e.g. [14, 38, 1]) for problem (1.1). Unfortunately, as the effective viscosity ν tends to zero (the Darcy limit), and for $\boldsymbol{f} \equiv 0$, the model tends to a mixed formulation of Poisson's equation with homogeneous Neumann boundary conditions. As a result, many of the popular Stokes elements are not robust with respect to the parameter ν [29].

To make this last statement more precise, we state the framework given in [39] giving necessary and sufficient conditions to ensure robustness (with respect to ν) of finite element methods of the Brinkman problem (1.1) using stable finite element pairs. To this end, assume that the finite element method for (1.1) takes the following form: find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times W_h$ such that

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}) - b_h(\boldsymbol{v}, p_h) = (\boldsymbol{f}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{V}_h, \tag{1.6a}$$

$$b_h(\boldsymbol{u}_h, w) = (g, w) \qquad \forall w \in W_h. \tag{1.6b}$$

Here, $a_h(\cdot,\cdot)$ and $b_h(\cdot,\cdot)$ are the discrete analogues of the bilinear forms (1.3)–(1.4) given by

$$a_h(\boldsymbol{v}, \boldsymbol{w}) = \sum_{K \in \Omega_h} (\nu\, \mathbf{grad}\, \boldsymbol{v}, \mathbf{grad}\, \boldsymbol{w})_K + (\alpha \boldsymbol{v}, \boldsymbol{w}),$$

$$b_h(\boldsymbol{v}, w) = \sum_{K \in \Omega_h} (\operatorname{div} \boldsymbol{v}, w)_K,$$

and $\boldsymbol{V}_h \subset \boldsymbol{L}^2(\Omega)$ and $W_h \subset L^2_0(\Omega)$ are a pair of finite element spaces (not necessarily conforming) assumed to satisfy the following inf-sup condition:

$$\sup_{\boldsymbol{v} \in \boldsymbol{V}_h} \frac{b_h(\boldsymbol{v}, w)}{\|\boldsymbol{v}\|_{1,h}} \geq C\|w\|_{L^2(\Omega)} \qquad \forall w \in W_h, \tag{1.7}$$

where $\|\cdot\|^2_{1,h} = \sum_{K \in \Omega_h} \|\cdot\|^2_{H^1(K)}$ denotes the piecewise H^1 norm. The precise definition of the notation used is given below.

Defining the discretely divergence-free space $\boldsymbol{Z}_h$ as

$$\boldsymbol{Z}_h = \big\{\boldsymbol{v} \in \boldsymbol{V}_h : \ b_h(\boldsymbol{v}, w) = 0 \ \ \forall w \in W_h\big\}, \tag{1.8}$$

we have the following result given in [39] (also see [29]).

Theorem 1.1. (Theorem 3.1, [39]) *Define the norm*

$$|\!|\!|\boldsymbol{v}|\!|\!|_h^2 := a_h(\boldsymbol{v}, \boldsymbol{v}) + M \sum_{K\in\Omega_h} \|\operatorname{div} \boldsymbol{v}\|_{L^2(K)}^2, \tag{1.9}$$

where $M = \max\{\nu, \alpha\}$. Then finite element pairs $\boldsymbol{V}_h \times W_h$ satisfying the inf-sup condition (1.7) *are uniformly stable with respect to the norm* (1.9) *for the model problem* (1.1) *if and only if*

$$\boldsymbol{Z}_h = \big\{\boldsymbol{v} \in \boldsymbol{V}_h : \ \operatorname{div} \boldsymbol{v}\big|_K = 0 \quad \forall K \in \Omega_h\big\}. \tag{1.10}$$

Essentially Theorem 1.1 says that in order to obtain robust finite element methods, the discrete divergence-free velocities of a stable finite element pair must be divergence free almost everywhere. Note that (1.10) holds provided the following stronger (and easier to verify) condition is satisfied:

$$\operatorname{div} \boldsymbol{V}_h \subseteq W_h. \tag{1.11}$$

In light of Theorem 1.1, it is then reasonable to try to apply any family of $\boldsymbol{H}(\mathrm{div};\Omega)$ elements (e.g. BDM or RT [35, 32, 33, 8]) as these elements are known to satisfy both the inf-sup condition (1.7) as well as the inclusion (1.11). However, such a strategy does not lead to a convergent method as these are nonconforming approximations with no tangental continuity across interior edges. Again, to make this last statement more precise we state a theorem that is similar to a result stated in [37].

Theorem 1.2. *Let $\boldsymbol{V}_h \subset \boldsymbol{H}_0(\mathrm{div};\Omega)$ and assume that* (1.7) *and* (1.11) *are satisfied. Then* (1.6) *admits a unique solution $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times W_h$ such that*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{a,h} \le C\left[\inf_{\boldsymbol{v}\in \boldsymbol{Z}_h(g)} \|\boldsymbol{u} - \boldsymbol{v}\|_{a,h} + \sup_{\boldsymbol{w}\in \boldsymbol{V}_h\setminus\{0\}} \frac{E_h(\boldsymbol{u},\boldsymbol{w})}{\|\boldsymbol{w}\|_{a,h}}\right], \tag{1.12a}$$

$$\|p - p_h\|_{L^2(\Omega)} \le C\Bigg[\inf_{w\in W_h} \|p - w\|_{L^2(\Omega)} + M^{1/2} \inf_{\boldsymbol{v}\in \boldsymbol{Z}_h(g)} \|\boldsymbol{u} - \boldsymbol{v}\|_{a,h} + \sup_{\boldsymbol{w}\in \boldsymbol{V}_h\setminus\{0\}} \frac{E_h(\boldsymbol{u},\boldsymbol{w})}{\|\boldsymbol{w}\|_{1,h}}\Bigg], \tag{1.12b}$$

where M is defined in Theorem 1.1*,*

$$\boldsymbol{Z}_h(g) := \big\{\boldsymbol{v} \in \boldsymbol{V}_h : \ b_h(\boldsymbol{v}, w) = (g, w) \ \ \forall w \in W_h\big\}, \tag{1.13}$$

$$\|\boldsymbol{v}\|_{a,h}^2 := a_h(\boldsymbol{v}, \boldsymbol{v}), \tag{1.14}$$

and the consistency error E_h is given as

$$E_h(\boldsymbol{u}, \boldsymbol{v}) = \begin{cases} \sum_{F\in\mathcal{E}_h} \big\langle \nu\, \mathbf{curl}\, \boldsymbol{u}, [\boldsymbol{v}\times\boldsymbol{n}]\big\rangle_F & d = 3, \\ \sum_{F\in\mathcal{E}_h} \big\langle \nu \operatorname{curl} \boldsymbol{u}, [\boldsymbol{v}\cdot\boldsymbol{t}]\big\rangle_F & d = 2, \end{cases} \tag{1.15}$$

where $[\boldsymbol{v} \times \boldsymbol{n}]$ *(and* $[\boldsymbol{v} \cdot \boldsymbol{t}]$*) is the tangential jump of* $\boldsymbol{v}$ *across the face* F*.*

For completeness, we provide the proof of Theorem 1.2 in the appendix.

Taking Theorems 1.1–1.2 into account, we see that if finite element methods for the Brinkman problem take the form (1.6), then the finite element pairs must satisfy (1.7), (1.10) and have some sort of tangential continuity across edges of the mesh in order for the method to be stable, robust and convergent. The continuity requirement is essential to bound the consistency error (1.15). Although the more popular (and simpler) Stokes elements do not satisfy this requirement (notably (1.10)), there are a few elements that fall into this category. These include conforming elements (i.e. $\boldsymbol{V}_h \subset \boldsymbol{H}_0^1(\Omega)$) such as the $\boldsymbol{\mathcal{P}}^k - \mathcal{P}^{k-1}$ triangular elements for $k \geq 4$ on singular-vertex free meshes [36] and finite elements of $\boldsymbol{\mathcal{P}}^k - \mathcal{P}^{k-1}$ type on macro elements [4, 42, 41], as well as low-order nonconforming elements given in [29, 39, 37].

Knowing that $\boldsymbol{H}(\mathrm{div};\Omega)$ conforming elements satisfy (1.7) and (1.10), several authors [29, 37, 39] developed elements for the Brinkman problem by modifying $\boldsymbol{H}(\mathrm{div};\Omega)$ conforming finite elements to make them have some tangential continuity. To be more precise, their local space when restricted to the simplex K are of the form

$$\boldsymbol{M}(K) + \mathbf{curl}\,(b_K \boldsymbol{Q}(K)), \tag{1.16}$$

where here, $\boldsymbol{M}(K)$ is the local space corresponding to a low-order $\boldsymbol{H}(\mathrm{div};\Omega)$ space, b_K is the element bubble that vanishes on ∂K and the space $\boldsymbol{Q}(K)$ is a subset of linear functions. Since they are only adding divergence free functions, the resulting space will still satisfy (1.10) and (1.7). Also note that the normal component of functions in $\mathbf{curl}\,(b_K \boldsymbol{Q}(K))$ vanish on ∂K and hence, the resulting space is still $\boldsymbol{H}(\mathrm{div};\Omega)$ conforming. Thus, the only purpose of adding function $\mathbf{curl}\,(b_K \boldsymbol{Q}(K))$ is to enforce some tangential continuity. The result is low-order non-conforming elements for the Brinkman problem.

In this paper, we develop a family of elements (two for each $k \geq 1$) in two and three dimensions for the Brinkman problem, where our local spaces will also be of the form (1.16). In this case, $\boldsymbol{M}(K)$ is going to be the local space of an *arbitrary* order $\boldsymbol{H}(\mathrm{div};\Omega)$ conforming space. The novelty of our spaces is that $\boldsymbol{Q}(K)$ contains face/edge bubble functions in order to achieve some tangential continuity. In fact, our lowest order element does not coincide with any of the low-order elements presented in [29, 37, 39], although the dimension of our lowest-order element and the the degrees of freedom are the same as the smallest spaces in [29, 39].

Following the ideas developed by Tai and Winther [37], we also show that our Brinkman problem spaces are part of a discrete de Rham complex with extra smoothness. In order to do so, we define a family of spaces which approximate the space $\boldsymbol{H}_0^1(\mathbf{curl}\,;\Omega)$. We also need the recently introduced $H_0^2(\Omega)$ non-conforming spaces recently introduced in [20].

We should note that there are many convergent finite element methods for the Brinkman problem that do not fit into the framework above, i.e., methods that do not take the form (1.6). These include, but are not limited to, stabilization methods [12, 11, 5] and augmented Lagrangian methods [9, 18]. We also point out that the method in [23] uses the generalized MINI elements [1] and has the form (1.6). Although their finite element pairs do not satisfy the criteria set in Theorem 1.1 (i.e., condition (1.10)), their method is robust with respect to ν. This may seem contradictory to the discussion above, but Theorem 1.1 states that methods are robust with respect to the norm (1.9) if and only if (1.10) holds. In [23], the authors circumvented this problem by using a different norm, in particular, by using (1.9) without the divergence term. The price they pay is suboptimal convergence for the velocity in the Darcy regime. Numerical experiments in [29] verify this assertion.

Recently, penalty methods using standard $\boldsymbol{H}(\mathrm{div};\Omega)$ conforming elements have been used for the Brinkman problem [25, 26, 27]. In these papers, the authors penalize the tangential jumps in order to have convergent methods. Similar ideas had been previosly developed for the Stokes problem [40, 15]. As is common for many penalty methods, the penalty parameters have to be chosen sufficiently large to make the method stable. Nonetheless, those methods seem to be very competitive for the Brinkman problem. The error estimates derived here for our new elements are similar to the estimates derived in [25, 26, 27] for the penalty methods. In some sense, the present work and the papers [25, 26, 27] achieve the same goal of finding a family of robust methods with optimal convergence properties by both using standard $\boldsymbol{H}(\mathrm{div};\Omega)$ conforming spaces. Of course, we do this by adding local basis functions that provide some tangential continuity, and they accomplish this by penalizing the tangential components of the velocity.

The rest of the paper is outlined as follows. In the next section we introduce some notation. In Section 3, we introduce a family of nonconforming elements for the Brinkman problem in three dimensions. Here, we describe the space, its associated degrees of freedom, and unisolvency. We also define the canonical projection and study its stability and approximation properties. In Section 4 we describe the analogous two dimensional elements and study its properties. In Section 5 we study the convergence analysis of the Brinkman problem using the framework set in [39, 37] described above. Following [37, 29], in Section 6 we show how the proposed elements fit into a discrete de Rham complex with extra smoothness. As a byproduct of this discussion, we obtain new families of nonconforming methods in $H_0^1(\mathbf{curl}\,;\Omega)$. Finally, in Section 7 we show how to find local basis for our spaces.

2. Notation

Throughout the paper, we use $H^m(\Omega)$ $(m \geq 0)$ to denote the set of all $L^2(\Omega)$ functions whose distributional derivatives up to order m are in $L^2(\Omega)$, and $H_0^m(\Omega)$ to denote the set of $H^m(\Omega)$ functions whose traces vanish up to order $m-1$ on $\partial\Omega$. We also set $\boldsymbol{H}^m(\Omega) = H^m(\Omega)^d$ and

$$\begin{aligned}\boldsymbol{H}(\mathrm{div};\Omega) &= \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) :\ \mathrm{div}\,\boldsymbol{v} \in L^2(\Omega)\},\\ \boldsymbol{H}_0(\mathrm{div};\Omega) &= \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};\Omega) : \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\},\\ \boldsymbol{H}(\mathbf{curl}\,;\Omega) &= \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) :\ \mathbf{curl}\,\boldsymbol{v} \in \boldsymbol{L}^2(\Omega)\},\end{aligned}$$

where $\boldsymbol{n}$ denotes the outward unit normal of $\partial\Omega$. We recall that the curl of a three dimensional vector $\boldsymbol{v} = (v_1, v_2, v_3)^T$ is given by

$$\mathbf{curl}\,\boldsymbol{v} = \left(\frac{\partial v_2}{\partial x_3} - \frac{\partial v_3}{\partial x_2}, \frac{\partial v_3}{\partial x_1} - \frac{\partial v_1}{\partial x_3}, \frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1}\right)^T,$$

and the curl of a vector $\boldsymbol{v} = (v_1, v_2)^T$ in 2D is given by

$$\operatorname{curl} \boldsymbol{v} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}.$$

We also set the curl of a scalar in 2D to be

$$\mathbf{curl}\, v = \left(\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1} \right)^T.$$

In three dimensions we will also need the spaces

$$\begin{aligned}
\boldsymbol{H}_0(\mathbf{curl}\,; \Omega) &= \big\{ \boldsymbol{v} \in \boldsymbol{H}(\mathbf{curl}\,; \Omega) : \boldsymbol{v} \times \boldsymbol{n} = 0 \text{ on } \partial\Omega \big\}, \\
\boldsymbol{H}^1(\mathbf{curl}\,; \Omega) &= \big\{ \boldsymbol{v} \in \boldsymbol{H}^1(\Omega) : \ \mathbf{curl}\, \boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \big\}, \\
\boldsymbol{H}_0^1(\mathbf{curl}\,; \Omega) &= \big\{ \boldsymbol{v} \in \boldsymbol{H}^1(\mathbf{curl}\,; \Omega) \cap \boldsymbol{H}_0^1(\Omega) : \mathbf{curl}\, \boldsymbol{v} \times \boldsymbol{n} = 0 \text{ on } \partial\Omega \big\}.
\end{aligned}$$

Let Ω_h be a shape-regular simplical triangulation [6, 13] with $h_K = \operatorname{diam}(K)$ $\forall K \in \Omega_h$ and $h = \max_{K \in \Omega_h} h_K$. We denote by $\mathcal{E}_h$ the faces (3D) or edges (2D), by $\mathcal{E}_h^i$ the interior faces (3D) or edges (2D), and by $\mathcal{E}_h^b$ the boundary faces (3D) or edges (2D) in Ω_h. Given $K \in \Omega_h$, we denote by $\{\lambda_F\}$ to be the $(d+1)$ barycentric coordinates of K, labeled such that λ_F vanishes on the face (3D) or edge (2D) $F \subset \partial K$. The element bubble and face/edge bubbles are then given by

$$b_K = \prod_F \lambda_F, \quad b_F = \prod_{G \neq F} \lambda_G,$$

where the product runs over the faces/edges of K. We set $\omega(F)$ to be the patch of the edge/face of F defined as

$$\omega(F) = \big\{ K \in \Omega_h : \ F \subset \partial K \big\},$$

and use the convention

$$\|\boldsymbol{v}\|_{H^m(\omega(F))} = \sum_{K \in \omega(F)} \|\boldsymbol{v}\|_{H^m(K)}.$$

For a given simplex S in $\mathbb{R}^d$ and $m \geq 0$, the vector-valued polynomials are defined as $\boldsymbol{\mathcal{P}}^m(S) = [\mathcal{P}^m(S)]^d$, where $\mathcal{P}^m(S)$ is the space of polynomials defined on S of degree less than or equal to m. We also set $\mathcal{P}^m(S)$ and $\boldsymbol{\mathcal{P}}^m(S)$ to be the empty set for any negative m.

We will use the following notation for interior and boundary inner-products

$$(\boldsymbol{v}, \boldsymbol{\rho})_K = \int_K \boldsymbol{v} \cdot \boldsymbol{\rho}\, dx, \quad \big\langle m, \mu \big\rangle_F = \int_F m\mu\, ds,$$

and $\boldsymbol{n}_F$ denotes the unit outward pointing normal to a face F of K.

We will also need to define the tangential and normal jump operators. If $F \in \mathcal{E}_h^i$ is an interior face (3D)/edge (2D) with $F = K^+ \cap K^-$, then we set

$$\begin{aligned}
[\boldsymbol{v} \times \boldsymbol{n}]\big|_F &= (\boldsymbol{v}\big|_{K^+} \times \boldsymbol{n}_{K^+})\big|_F + (\boldsymbol{v}\big|_{K^-} \times \boldsymbol{n}_{K^-})\big|_F, \\
[\boldsymbol{v} \cdot \boldsymbol{t}]\big|_F &= (\boldsymbol{v}\big|_{K^+} \cdot \boldsymbol{t}_{K^+})\big|_F + (\boldsymbol{v}\big|_{K^-} \cdot \boldsymbol{t}_{K^-})\big|_F, \\
[\boldsymbol{v} \cdot \boldsymbol{n}]\big|_F &= (\boldsymbol{v}\big|_{K^+} \cdot \boldsymbol{n}_{K^+})\big|_F + (\boldsymbol{v}\big|_{K^-} \cdot \boldsymbol{n}_{K^-})\big|_F,
\end{aligned}$$

where $\boldsymbol{n}_{K^\pm}$ is the outward pointing unit normal to $\partial K^\pm$, and $\boldsymbol{t}_{K^\pm}$ is the unit tangent of $\partial K^\pm$.

If $F \in \mathcal{E}_h^b$ is a boundary face(3D)/edge (2D) with $F \subset \partial K$, then we set

$$[\boldsymbol{v} \times \boldsymbol{n}]\big|_F = (\boldsymbol{v}\big|_K \times \boldsymbol{n}_K)\big|_F,$$
$$[\boldsymbol{v} \cdot \boldsymbol{t}]\big|_F = (\boldsymbol{v}\big|_K \cdot \boldsymbol{t}_K)\big|_F,$$
$$[\boldsymbol{v} \cdot \boldsymbol{n}]\big|_F = (\boldsymbol{v}\big|_K \cdot \boldsymbol{n}_K)\big|_F.$$

Finally, we use C to denote a generic constant independent of h or the parameters ν and α.

3. Family of Finite Elements in Three Dimensions

3.1. **The Local Space.** Since our new elements are going to be based on $\boldsymbol{H}(\mathrm{div};\Omega)$ finite element spaces plus divergence free functions, we first review some well known elements. Let $K \in \Omega_h$, and let $\boldsymbol{M}^k(K)$ denote either the local Brezzi-Douglas-Marini (BDM) space of order k [7, 8, 33]

$$\boldsymbol{M}^k(K) = \boldsymbol{\mathcal{P}}^k(K) \qquad (k \geq 1), \tag{3.1}$$

or the Raviart-Thomas (RT) space [35, 32] of order k

$$\boldsymbol{M}^k(K) = \boldsymbol{\mathcal{P}}^k(K) + \mathcal{P}^{k+1}(K)\boldsymbol{x} \quad (k \geq 1). \tag{3.2}$$

We also define the space $\boldsymbol{A}^{k-1}(K)$ as follows: If $\boldsymbol{M}^k(K)$ is given by the RT space (3.2), then we set $\boldsymbol{A}^{k-1}(K) = \boldsymbol{\mathcal{P}}^{k-1}(K)$; if $\boldsymbol{M}^k(K)$ is given by the BDM space (3.1), then we define $\boldsymbol{A}^{k-1}(K) = \boldsymbol{N}^{k-1}(K)$, the Nedelec space of index $k-1$ [32]:

$$\boldsymbol{N}^{k-1}(K) = \boldsymbol{\mathcal{P}}^{k-2}(K) + \big\{\boldsymbol{v} \in \boldsymbol{\mathcal{P}}^{k-1}(K) : \boldsymbol{v} \cdot \boldsymbol{x} = 0\big\}. \tag{3.3}$$

It is well-known that a function $\boldsymbol{v} \in \boldsymbol{M}^k(K)$ is uniquely determined by the following degrees of freedom [10, 32, 33]:

$$(\boldsymbol{v}, \boldsymbol{\rho})_K \qquad \text{for all } \boldsymbol{\rho} \in \boldsymbol{A}^{k-1}(K), \tag{3.4a}$$

$$\big\langle \boldsymbol{v} \cdot \boldsymbol{n}_F, \mu \big\rangle_F \qquad \text{for all } \mu \in \mathcal{P}^k(F) \text{ and faces } F \text{ of } K. \tag{3.4b}$$

Here, $\boldsymbol{n}_F$ denotes a unit normal vector to the face F.

We then defined the local space for the three dimensional Brinkman problem as

$$\boldsymbol{V}^k(K) = \boldsymbol{M}^k(K) + \boldsymbol{U}^{k-1}(K),$$

where

$$\boldsymbol{U}^{k-1}(K) = \mathbf{curl}\,(b_K \boldsymbol{Q}^{k-1}(K)), \tag{3.5}$$

$$\boldsymbol{Q}^{k-1}(K) = \sum_F b_F \boldsymbol{Q}_F^{k-1}(K), \tag{3.6}$$

and

$$\boldsymbol{Q}_F^{k-1}(K) = \big\{\boldsymbol{q} \times \boldsymbol{n}_F \in \boldsymbol{\mathcal{P}}^{k-1}(K) \times \boldsymbol{n}_F : \big(\boldsymbol{q} \times \boldsymbol{n}_F, b_K b_F(\boldsymbol{w} \times \boldsymbol{n}_F)\big)_K = 0 \ \text{ for all } \boldsymbol{w} \in \boldsymbol{\mathcal{P}}^{k-2}(K)\big\}. \tag{3.7}$$

The following are the degrees of freedom that define a function $\boldsymbol{v} \in \boldsymbol{V}^k(K)$:

$$(\boldsymbol{v}, \boldsymbol{\rho})_K \qquad \text{for all } \boldsymbol{\rho} \in \boldsymbol{A}^{k-1}(K), \tag{3.8a}$$

$$\left\langle \boldsymbol{v} \cdot \boldsymbol{n}_F, \mu \right\rangle_F \qquad \text{for all } \mu \in \mathcal{P}^k(F) \text{ and faces } F \text{ of } K, \tag{3.8b}$$

$$\left\langle \boldsymbol{v} \times \boldsymbol{n}_F, \boldsymbol{\kappa} \right\rangle_F \qquad \text{for all } \boldsymbol{\kappa} \in \boldsymbol{\mathcal{P}}^{k-1}(F) \text{ and faces } F \text{ of } K. \tag{3.8c}$$

Before we prove unisolvency of the degrees of freedom, we first need some preliminary results. We start by listing some key properties of the space $\boldsymbol{U}^{k-1}(K)$.

Lemma 3.1. *Let $\boldsymbol{z} \in \boldsymbol{U}^{k-1}(K)$ with $\boldsymbol{z} = \mathbf{curl}\left(b_K \sum_F b_F(\boldsymbol{q}_F \times \boldsymbol{n}_F)\right)$ and $\boldsymbol{q}_F \in \boldsymbol{Q}_F^{k-1}(K)$. Then the following identities hold:*

$$\boldsymbol{z} \cdot \boldsymbol{n}|_{\partial K} = 0, \tag{3.9a}$$

$$\boldsymbol{z} \times \boldsymbol{n}_F|_F = -a_F b_F^2(\boldsymbol{q}_F \times \boldsymbol{n}_F) \qquad \text{for all faces } F \text{ of } K, \tag{3.9b}$$

$$(\boldsymbol{z}, \boldsymbol{w})_K = 0 \qquad \text{for all } \boldsymbol{w} \in \boldsymbol{\mathcal{P}}^{k-1}(K), \tag{3.9c}$$

where $a_F = |\mathbf{grad}\, \lambda_F|$.

Proof. Since b_K vanishes on ∂K, we can use the product rule to obtain for any $\boldsymbol{q} \in \boldsymbol{Q}^{k-1}(K)$,

$$\begin{aligned} (\mathbf{curl}\,(b_K \boldsymbol{q}) \cdot \boldsymbol{n})|_{\partial K} &= b_K \mathbf{curl}\, \boldsymbol{q} \cdot \boldsymbol{n}|_{\partial K} + \big((\mathbf{grad}\, b_K \times \boldsymbol{q}) \cdot \boldsymbol{n}\big)|_{\partial K} \\ &= \boldsymbol{q} \cdot (\mathbf{grad}\, b_K \times \boldsymbol{n})|_{\partial K} = 0, \end{aligned}$$

where we have used $(\mathbf{grad}\, b_K \times \boldsymbol{n})|_{\partial K} = 0$. Since $\boldsymbol{z} \in \boldsymbol{U}^{k-1}(K)$ is of the form $\boldsymbol{z} = \mathbf{curl}\,(b_K \boldsymbol{q})$, this proves (3.9a).

Next, using the product rule and the fact that b_K vanishes on ∂K, we have

$$\boldsymbol{z}|_{\partial K} = \sum_F b_F(\mathbf{grad}\, b_K) \times (\boldsymbol{q}_F \times \boldsymbol{n}_F).$$

Using the fact b_F vanishes on $\partial K \backslash F$ and that $\mathbf{grad}\, b_K = -a_F b_F \boldsymbol{n}_F$ we get

$$\boldsymbol{z}|_F = -a_F b_F^2 \boldsymbol{n}_F \times (\boldsymbol{q}_F \times \boldsymbol{n}_F).$$

The identity (3.9b) now easily follows.

Next for $\boldsymbol{w} \in \boldsymbol{\mathcal{P}}^{k-1}(K)$, integration by parts gives us

$$(\boldsymbol{z}, \boldsymbol{w})_K = -\sum_F (\boldsymbol{q}_F \times \boldsymbol{n}_F, b_K b_F \mathbf{curl}\, \boldsymbol{w})_K.$$

However, we can write $\mathbf{curl}\, \boldsymbol{w} = -(\mathbf{curl}\, \boldsymbol{w} \times \boldsymbol{n}_F) \times \boldsymbol{n}_F + (\mathbf{curl}\, \boldsymbol{w} \cdot \boldsymbol{n}_F)\boldsymbol{n}_F$, which gives

$$(\boldsymbol{z}, \boldsymbol{w})_K = \sum_F (\boldsymbol{q}_F \times \boldsymbol{n}_F, b_K b_F (\mathbf{curl}\, \boldsymbol{w} \times \boldsymbol{n}_F) \times \boldsymbol{n}_F)_K.$$

Hence, using the definition of $\boldsymbol{Q}_F^{k-1}(F)$, we have $(\boldsymbol{z}, \boldsymbol{w})_K = 0$. □

Lemma 3.2. *If $\boldsymbol{q}_F \times \boldsymbol{n}_F \in \boldsymbol{Q}_F^{k-1}(F)$ vanishes on F, then $\boldsymbol{q}_F \times \boldsymbol{n}_F$ vanishes on K. Also,*

$$\dim \boldsymbol{Q}_F^{k-1}(K) = \dim \boldsymbol{\mathcal{P}}^{k-1}(F), \tag{3.10}$$

$$\dim \boldsymbol{U}^{k-1}(K) = 4 \dim \boldsymbol{\mathcal{P}}^{k-1}(F). \tag{3.11}$$

Proof. If $\boldsymbol{q}_F \times \boldsymbol{n}_F$ vanishes on F, then we have $\boldsymbol{q}_F \times \boldsymbol{n}_F = \lambda_F \boldsymbol{p}$ for some $\boldsymbol{p} \in \boldsymbol{\mathcal{P}}^{k-2}(K)$. Noting

$$b_K(\boldsymbol{p} \times \boldsymbol{n}_F) = b_F(\boldsymbol{q}_F \times \boldsymbol{n}_F) \times \boldsymbol{n}_F,$$

it follows that

$$b_K(\boldsymbol{p} \times \boldsymbol{n}_F) \times \boldsymbol{n}_F = -b_F(\boldsymbol{q}_F \times \boldsymbol{n}_F). \tag{3.12}$$

Therefore, by the definition of $\boldsymbol{Q}_F^{k-1}(K)$ and (3.12), we have

$$0 = -\big(\boldsymbol{q}_F \times \boldsymbol{n}_F, b_F b_K(\boldsymbol{p} \times \boldsymbol{n}_F) \times \boldsymbol{n}_F\big)_K = \big(\boldsymbol{q}_F \times \boldsymbol{n}_F, b_F^2(\boldsymbol{q}_F \times \boldsymbol{n}_F)\big)_K,$$

and therefore $\boldsymbol{q}_F \times \boldsymbol{n}_F \equiv 0$.

In order to count the dimension we note that $\boldsymbol{\mathcal{P}}^{k-1}(K) \times \boldsymbol{n}_F = 2 \dim \mathcal{P}^{k-1}(K)$. Hence, we easily see from the definition of $\boldsymbol{Q}_F^{k-1}(K)$ that

$$\dim \boldsymbol{Q}_F^{k-1}(K) = 2\big(\dim \mathcal{P}^{k-1}(K) - \dim \mathcal{P}^{k-2}(K)\big) = \dim \boldsymbol{\mathcal{P}}^{k-1}(F).$$

In order to prove (3.11), we will show that if $0 = \boldsymbol{z} = \mathbf{curl}\,\big(b_K \sum_F b_F(\boldsymbol{q}_F \times \boldsymbol{n}_F)\big)$ for $\boldsymbol{q}_F \times \boldsymbol{n}_F \in \boldsymbol{Q}_F^{k-1}(F)$, then $\boldsymbol{q}_F \times \boldsymbol{n}_F = 0$ for all faces F. Consider an arbitrary face F of K. Then by (3.9b), we have

$$0 = \boldsymbol{z} \times \boldsymbol{n}_F\big|_F = -a_F b_F^2(\boldsymbol{q}_F \times \boldsymbol{n}_F)\big|_F,$$

which shows that $(\boldsymbol{q}_F \times \boldsymbol{n}_F)\big|_F = 0$, and therefore $\boldsymbol{q}_F \times \boldsymbol{n} = 0$ on K. This immediately shows that $\dim \boldsymbol{U}^{k-1}(K) = 4 \dim \boldsymbol{Q}_F^{k-1}(K)$, and therefore (3.11) follows from (3.10). ☐

Theorem 3.3. *We have*

$$\boldsymbol{V}^k(K) = \boldsymbol{M}^k(K) \oplus \boldsymbol{U}^{k-1}(K), \tag{3.13}$$

$$\dim \boldsymbol{V}^k(K) = \dim \boldsymbol{M}^k(K) + 4 \dim \boldsymbol{\mathcal{P}}^{k-1}(F). \tag{3.14}$$

Furthermore, any function $\boldsymbol{v} \in \boldsymbol{V}^k(K)$ is uniquely determined by the degrees of freedom (3.8).

Proof. Suppose that $\boldsymbol{v} \in \boldsymbol{M}^k(K) \cap \boldsymbol{U}^{k-1}(K)$. Then by Lemma 3.1 we have

$$(\boldsymbol{v}, \boldsymbol{\rho})_K = 0 \qquad \text{for all } \boldsymbol{\rho} \in \boldsymbol{\mathcal{P}}^{k-1}(K), \tag{3.15}$$

$$\big\langle \boldsymbol{v} \cdot \boldsymbol{n}, \mu \big\rangle_F = 0 \qquad \text{for all } \mu \in \mathcal{P}^k(F) \text{ and all faces } F \text{ of } K. \tag{3.16}$$

It follows that $\boldsymbol{v} \equiv 0$ since all the degrees of freedom (cf. (3.4)) of $\boldsymbol{v} \in \boldsymbol{M}^k(K)$ vanish. Therefore, (3.13) holds, and so by (3.11), the dimension count (3.14) holds as well.

Since $\dim \boldsymbol{M}^k(K) = \dim \boldsymbol{A}^{k-1}(K) + 4 \dim \mathcal{P}^k(K)$, it follows from Lemma 3.2 that $\dim \boldsymbol{V}^k(K)$ is exactly the number of degrees of freedom given by (3.8). Hence, we only need to show that if the degrees of freedom (3.8) vanish for $\boldsymbol{v} \in \boldsymbol{V}^k(K)$, then $\boldsymbol{v} \equiv 0$.

To this end, let $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{z}$ where $\boldsymbol{v}_0 \in \boldsymbol{M}^k(K)$ and $\boldsymbol{z} \in \boldsymbol{U}^{k-1}(K)$. Then by Lemma 3.1, we have

$$(\boldsymbol{v}_0, \boldsymbol{\rho})_K = 0 \qquad \text{for all } \boldsymbol{\rho} \in \boldsymbol{\mathcal{P}}^{k-1}(K), \tag{3.17}$$

$$\big\langle \boldsymbol{v}_0 \cdot \boldsymbol{n}_F, \mu \big\rangle_F = 0 \qquad \text{for all } \mu \in \mathcal{P}^k(F) \text{ and faces } F \text{ of } K, \tag{3.18}$$

and so $\boldsymbol{v}_0 = 0$ since all its degrees of freedom (3.4) vanish. Hence,

$$\boldsymbol{v} = \boldsymbol{z} = \mathbf{curl}\,\big(b_K \sum_F b_F(\boldsymbol{q}_F \times \boldsymbol{n}_F)\big),$$

for some $(\boldsymbol{q}_F \times \boldsymbol{n}_F) \in \boldsymbol{Q}_F^{k-1}(K)$.

Since the degrees of freedom (3.8c) vanish, we have by (3.9b) for any face F,

$$0 = \big\langle \boldsymbol{v} \times \boldsymbol{n}_F, \boldsymbol{q}_F \times \boldsymbol{n}_F \big\rangle_F = -a_F \big\langle b_F^2(\boldsymbol{q}_F \times \boldsymbol{n}_F), \boldsymbol{q}_F \times \boldsymbol{n}_F \big\rangle_F.$$

This of course shows that $(\boldsymbol{q}_F \times \boldsymbol{n}_F)$ vanishes on F, and by Lemma 3.2 we have that $(\boldsymbol{q}_F \times \boldsymbol{n}_F) = 0$ on K. This completes the proof. $\square$

3.2. **The Global Space and Projection.** Now that we have defined the local finite element spaces, we can naturally define the global space as

$$\begin{aligned} \boldsymbol{V}_h = &\big\{\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div};\Omega) : \ \boldsymbol{v}|_K \in \boldsymbol{V}^k(K) \text{ for all } K \in \Omega_h, \\ &\big\langle [\boldsymbol{v} \times \boldsymbol{n}], \boldsymbol{\mu}\big\rangle_F = 0 \text{ for all } \boldsymbol{\mu} \in \boldsymbol{\mathcal{P}}^{k-1}(F) \text{ and faces } F \text{ of } \Omega_h\big\}. \end{aligned} \tag{3.19}$$

We define the corresponding pressure space as

$$W_h = \big\{w \in L_0^2(\Omega) : w|_K \in \mathcal{P}^s(K), \text{ for all } K \in \Omega_h\big\}, \tag{3.20}$$

where $s = k$ if we use the RT space (3.2) or $s = k-1$ if we use BDM space (3.1).

We also define $\boldsymbol{M}_h$ (respectively, $\boldsymbol{U}_h$) to be the associated global space of $\boldsymbol{M}^k(K)$ (respectively, $\boldsymbol{U}^{k-1}(K)$) as

$$\boldsymbol{M}_h = \big\{\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div};\Omega) : \ \boldsymbol{v}\big|_K \in \boldsymbol{M}^k(K) \text{ for all } K \in \Omega_h\big\}, \tag{3.21}$$

$$\boldsymbol{U}_h = \big\{\boldsymbol{z} \in \boldsymbol{H}_0(\mathrm{div};\Omega) : \ \boldsymbol{z}\big|_K \in \boldsymbol{U}^{k-1}(K) \text{ for all } K \in \Omega_h\big\}. \tag{3.22}$$

The degrees of freedom (3.8) naturally lead us to define the projection $\boldsymbol{\Pi}_h : \boldsymbol{H}_0^1(\Omega) \to \boldsymbol{V}_h$ given locally as follows:

$$(\boldsymbol{\Pi}_h \boldsymbol{v} - \boldsymbol{v}, \boldsymbol{\rho})_K = 0 \qquad \text{for all } \boldsymbol{\rho} \in \boldsymbol{A}^{k-1}(K), \tag{3.23a}$$

$$\big\langle (\boldsymbol{\Pi}_h \boldsymbol{v} - \boldsymbol{v}) \cdot \boldsymbol{n}_F, \mu \big\rangle_F = 0 \qquad \text{for all } \mu \in \mathcal{P}^k(F) \text{ and faces } F \text{ of } K, \tag{3.23b}$$

$$\big\langle (\boldsymbol{\Pi}_h \boldsymbol{v} - \boldsymbol{v}) \times \boldsymbol{n}_F, \boldsymbol{\kappa} \big\rangle_F = 0 \qquad \text{for all } \boldsymbol{\kappa} \in \boldsymbol{\mathcal{P}}^{k-1}(F) \text{ and faces } F \text{ of } K. \tag{3.23c}$$

Using (3.23a) and (3.23b) we can easily show that the commutative property

$$\mathrm{div}\,\boldsymbol{\Pi}_h \boldsymbol{v} = P_h \mathrm{div}\,\boldsymbol{v} \qquad \text{for all } \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega) \tag{3.24}$$

holds, where P_h is the L^2 projection onto W_h. In particular, we have $\boldsymbol{\Pi}_h \boldsymbol{u} \in \boldsymbol{Z}(g)$, where $\boldsymbol{Z}(g)$ is defined by (1.13).

We now discuss the approximation properties of $\boldsymbol{\Pi}_h$. First, we denote by $\boldsymbol{\Pi}_M : \boldsymbol{H}^1(\Omega) \to \boldsymbol{M}_h$ either the BDM projection or RT projection, i.e.,

$$(\boldsymbol{\Pi}_M \boldsymbol{v} - \boldsymbol{v}, \boldsymbol{\rho})_K = 0 \qquad \text{for all } \boldsymbol{\rho} \in \boldsymbol{A}^{k-1}(K), \tag{3.25a}$$

$$\big\langle (\boldsymbol{\Pi}_M \boldsymbol{v} - \boldsymbol{v}) \cdot \boldsymbol{n}_F, \mu \big\rangle_F = 0 \qquad \text{for all } \mu \in \mathcal{P}^k(F) \text{ and faces } F \text{ of } K. \tag{3.25b}$$

We also define $\boldsymbol{\Pi}_U : \boldsymbol{H}^1(\Omega) \to \boldsymbol{U}_h$ locally as

$$\big\langle (\boldsymbol{\Pi}_U \boldsymbol{v} - \boldsymbol{v}) \times \boldsymbol{n}_F, \boldsymbol{\kappa} \big\rangle_F \qquad \text{for all } \boldsymbol{\kappa} \in \boldsymbol{\mathcal{P}}^{k-1}(F) \text{ and faces } F \text{ of } K. \tag{3.26}$$

It is then straightforward to verify the identity

$$\boldsymbol{I} - \boldsymbol{\Pi}_h = (\boldsymbol{I} - \boldsymbol{\Pi}_U)(\boldsymbol{I} - \boldsymbol{\Pi}_M), \tag{3.27}$$

where $\boldsymbol{I}$ denotes the identity operator on $\boldsymbol{H}^1(\Omega)$.

We now derive some stability estimates for the projection $\boldsymbol{\Pi}_U$. To this end, for $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ we write

$$\boldsymbol{\Pi}_U \boldsymbol{v}\big|_K = \mathbf{curl}\,\big(b_K \sum_F b_F(\boldsymbol{q}_F \times \boldsymbol{n}_F)\big)\big|_K \quad \text{with } \boldsymbol{q}_F \in \boldsymbol{Q}_F^{k-1}(K).$$

Note that by (3.9b), we have

$$\boldsymbol{\Pi}_U \boldsymbol{v} \times \boldsymbol{n}_F\big|_F = -a_F b_F^2(\boldsymbol{q}_F \times \boldsymbol{n}_F)\big|_F, \tag{3.28}$$

and therefore by (3.28) and (3.26),

$$\begin{aligned} a_F\big\langle b_F^2(\boldsymbol{q}_F \times \boldsymbol{n}_F), (\boldsymbol{q} \times \boldsymbol{n}_F)\big\rangle_F &= -\big\langle \boldsymbol{\Pi}_U \boldsymbol{v} \times \boldsymbol{n}_F, \boldsymbol{q}_F \times \boldsymbol{n}_F\big\rangle_F \\ &= -\big\langle \boldsymbol{v} \times \boldsymbol{n}_F, \boldsymbol{q}_F \times \boldsymbol{n}_F\big\rangle_F \\ &\le \|\boldsymbol{v} \times \boldsymbol{n}_F\|_{L^2(F)} \|\boldsymbol{q}_F \times \boldsymbol{n}_F\|_{L^2(F)}. \end{aligned}$$

It then follows that

$$a_F \|\boldsymbol{q}_F \times \boldsymbol{n}_F\|_{L^2(F)} \le C\|\boldsymbol{v} \times \boldsymbol{n}_F\|_{L^2(F)}. \tag{3.29}$$

Hence by a scaling argument, (3.28) and (3.29), we obtain

$$\begin{aligned} \|\boldsymbol{\Pi}_U \boldsymbol{v}\|_{L^2(K)} &\le Ch_K^{1/2} \sum_F \|\boldsymbol{\Pi}_U \boldsymbol{v} \times \boldsymbol{n}_F\|_{L^2(F)} = Ch_K^{1/2} \sum_F a_F \|b_F^2(\boldsymbol{q}_F \times \boldsymbol{n}_F)\|_{L^2(F)} \\ &\le Ch_K^{1/2} \sum_F a_F \|\boldsymbol{q}_F \times \boldsymbol{n}_F\|_{L^2(F)} \le Ch_K^{1/2} \|\boldsymbol{v} \times \boldsymbol{n}\|_{L^2(\partial K)}. \end{aligned}$$

Using this last estimate in (3.27) , we have

$$\begin{aligned} \|\boldsymbol{v} - \boldsymbol{\Pi}_h \boldsymbol{v}\|_{L^2(K)} &\le \|\boldsymbol{v} - \boldsymbol{\Pi}_M \boldsymbol{v}\|_{L^2(K)} + \|\boldsymbol{\Pi}_U(\boldsymbol{v} - \boldsymbol{\Pi}_M \boldsymbol{v})\|_{L^2(K)} \\ &\le \|\boldsymbol{v} - \boldsymbol{\Pi}_M \boldsymbol{v}\|_{L^2(K)} + Ch_K^{1/2} \|\boldsymbol{v} - \boldsymbol{\Pi}_M \boldsymbol{v}\|_{L^2(\partial K)}. \end{aligned}$$

Thus, by standard approximation results of the projection $\boldsymbol{\Pi}_M$, we have

Theorem 3.4. *Let m and s be two integers satisfying $0 \le m \le s \le k+1$ and $s \ge 1$. Then for any $\boldsymbol{v} \in H^s(K)$, there holds*

$$\|\boldsymbol{v} - \boldsymbol{\Pi}_h \boldsymbol{v}\|_{H^m(K)} \le Ch_K^{s-m} |\boldsymbol{v}|_{H^s(K)}. \tag{3.30}$$

In particular, if $\boldsymbol{v} \in H^s(\Omega)$ we have (cf. (1.14)*)*

$$\|\boldsymbol{v} - \boldsymbol{\Pi}_h \boldsymbol{v}\|_{a,h} \le C\big(\nu^{1/2} h^{s-1} + \alpha^{1/2} h^s\big) |\boldsymbol{v}|_{H^s(\Omega)}. \tag{3.31}$$

4. Family of finite elements in two dimensions

The two dimension finite elements for the Brinkman problem are similar in nature and and in their construction to that of the 3D case, so we only sketch the main points. In the two dimensional case, we define the local space as

$$\boldsymbol{V}^k(K) = \boldsymbol{M}^k(K) + \boldsymbol{U}^{k-1}(K), \tag{4.1}$$

where

$$\boldsymbol{U}^{k-1}(K) = \mathbf{curl}\,(b_K Q^{k-1}(K)), \tag{4.2a}$$

$$Q^{k-1}(K) = \sum_F b_F Q_F^{k-1}, \tag{4.2b}$$

and

$$Q_F^{k-1}(K) = \big\{q \in \mathcal{P}^{k-1}(K) :\ (q, b_K b_F w)_K = 0,\ \text{for all } w \in \mathcal{P}^{k-2}(K)\big\}. \tag{4.2c}$$

The corresponding degrees of freedom that define a function $\boldsymbol{v} \in \boldsymbol{V}^k(K)$ are defined as follows:

$$(\boldsymbol{v}, \boldsymbol{\rho})_K \qquad \text{for all } \boldsymbol{\rho} \in \boldsymbol{A}^{k-1}(K), \tag{4.3a}$$

$$\big\langle \boldsymbol{v} \cdot \boldsymbol{n}_F, \mu \big\rangle_F \qquad \text{for all } \mu \in \mathcal{P}^k(F) \text{ and all edges } F \text{ of } K, \tag{4.3b}$$

$$\big\langle \boldsymbol{v} \cdot \boldsymbol{t}_F, \kappa \big\rangle_F \qquad \text{for all } \kappa \in \mathcal{P}^{k-1}(F) \text{ and all edges } F \text{ of } K, \tag{4.3c}$$

where $\boldsymbol{t}_F$ is the unit tangental of the edge F obtained by rotating $\boldsymbol{n}_F$ 90 degrees counter-clockwise.

Lemma 4.1. *There holds*

$$\boldsymbol{V}^k(K) = \boldsymbol{M}^k(K) \oplus \boldsymbol{U}^{k-1}(K), \tag{4.4}$$

$$\dim \boldsymbol{V}^k(K) = \dim \boldsymbol{M}^k(K) + 3\dim(\mathcal{P}^{k-1}(F)). \tag{4.5}$$

Moreover, any function $\boldsymbol{v} \in \boldsymbol{V}^k(K)$ is uniquely determined by the degrees of freedom (4.3)*.*

Proof. Suppose that $\boldsymbol{v} \in \boldsymbol{M}^k(K) \cap \boldsymbol{U}^{k-1}(K)$. Then $\boldsymbol{v}$ can be written as $\boldsymbol{v} = \sum_F \mathbf{curl}\,(b_K b_F q_F)$ with $q_F \in Q_F^{k-1}(K)$. Therefore by (4.2c) and by integration by parts, we have for any $\boldsymbol{\rho} \in \boldsymbol{\mathcal{P}}^{k-1}(K)$,

$$(\boldsymbol{v}, \boldsymbol{\rho})_K = \sum_F (\mathbf{curl}\,(b_K b_F q_F), \boldsymbol{\rho})_K = -\sum_F (b_K b_F q_F, \mathbf{curl}\,\boldsymbol{\rho})_K = 0.$$

Furthermore since b_K vanishes on ∂K, we have for any $\mu \in \mathcal{P}^{k-1}(F)$,

$$\langle \boldsymbol{v} \cdot \boldsymbol{n}_F, \mu \rangle_F = \sum_F \big\langle \mathbf{grad}\,(b_K b_F q_F) \cdot \boldsymbol{t}_F, \mu \big\rangle_F = 0.$$

Since the degrees of freedom (4.3a)–(4.3b) uniquely define a function in $\boldsymbol{M}^k(K)$, we conclude $\boldsymbol{v} \equiv 0$, and the direct sum (4.4) follows.

To show (4.5), we first see that

$$\dim Q_F^{k-1}(K) = \dim \mathcal{P}^{k-1}(K) - \dim \mathcal{P}^{k-2}(K) = \dim \mathcal{P}^{k-1}(F).$$

Therefore, in light of (4.4) it suffices to show that if $0 = \boldsymbol{z} = \mathbf{curl}\,(b_K \sum_F b_F q_F)$ with $q_F \in Q_F^{k-1}(K)$, then $q_F = 0$ for all edges F. To this end, we note that on each edge,

$$0 = \boldsymbol{z} \cdot \boldsymbol{t}_F\big|_F = b_F q_F \mathbf{grad}\,(b_K) \cdot \boldsymbol{n}_F = -a_F b_F^2 q_F\big|_F.$$

It then follows that $q_F = \lambda_F p_F$ for some $p_F \in \mathcal{P}^{k-2}(K)$ for all edges F. But then by the definition of $Q_F^{k-1}(K)$, we have

$$0 = (q_F, b_K b_F p_F)_K = (p_F, b_K b_F \lambda_F p_F)_K,$$

and therefore $p_F = 0$. The dimension count (4.5) immediately follows.

To show that the degrees of freedom uniquely determine a function in $\boldsymbol{V}^k(K)$, we see that by (4.5), it suffices to show that if all the degrees of freedom vanish for some $\boldsymbol{v} \in \boldsymbol{V}^k(K)$, then $\boldsymbol{v} \equiv 0$.

We write $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{z}$ with $\boldsymbol{v}_0 \in \boldsymbol{M}^k(K)$ and $\boldsymbol{z} \in \boldsymbol{U}^{k-1}(K)$ with $\boldsymbol{z} = \mathbf{curl}\,(b_K \sum_F b_F q_F)$. Since a function in $\boldsymbol{M}^k(K)$ is uniquely determined by the degrees of freedom (4.3a)–(4.3b), and since functions in $\boldsymbol{U}^{k-1}(K)$ vanish at these degrees of freedom, we have $\boldsymbol{v}_0 \equiv 0$. Therefore by (4.3c), we have

$$0 = \big\langle \boldsymbol{v} \cdot \boldsymbol{t}_F, q_F \big\rangle_F = \big\langle b_F q_F \mathbf{grad}\,(b_K) \cdot \boldsymbol{n}_F, q_F \big\rangle_F = -a_F \big\langle b_F^2 q_F, q_F \big\rangle_F.$$

Hence, we have $q_F\big|_F = 0$ for all edges F. However, by the same argument given above, we conclude that $q_F \equiv 0$ and therefore $\boldsymbol{v} \equiv 0$. □

Analogous to the three dimensional case (3.19), the two dimensional global vector space for the Brinkman problem is defined as

$$\begin{aligned} \boldsymbol{V}_h = \big\{ &\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div}; \Omega) : \ \boldsymbol{v}\big|_K \in \boldsymbol{V}^k(K) \text{ for all } K \in \Omega_h, \\ &\big\langle [\boldsymbol{v} \cdot \boldsymbol{t}], \mu \big\rangle_F = 0 \text{ for all } \mu \in \mathcal{P}^{k-1}(F) \text{ and edges } F \text{ in } \Omega_h \big\}. \end{aligned} \tag{4.6}$$

The pressure space W_h is the same as the three dimension case, that is, W_h is defined by (3.20).

Similar to the three dimensional case we can define the canonical projection, given locally by

$$\begin{aligned} (\boldsymbol{\Pi}_h \boldsymbol{v} - \boldsymbol{v}, \boldsymbol{\rho})_K &= 0 \quad \text{for all } \boldsymbol{\rho} \in \boldsymbol{A}^{k-1}(K) \\ \big\langle (\boldsymbol{\Pi}_h \boldsymbol{v} - \boldsymbol{v}) \cdot \boldsymbol{n}_F, \mu \big\rangle_F &= 0 \quad \text{for all } \mu \in \mathcal{P}^k(F) \text{ and edges } F \text{ of } K, \\ \big\langle (\boldsymbol{\Pi}_h \boldsymbol{v} - \boldsymbol{v}) \cdot \boldsymbol{t}_F, \kappa \big\rangle_F &= 0 \quad \text{for all } \kappa \in \mathcal{P}^{k-1}(F) \text{ and edges } F \text{ of } K. \end{aligned}$$

It is easy to see that the commutative property (3.24) holds for the two dimensional projection. Furthermore, by using similar techniques as in the previous section, it can be shown that the estimates (3.30)–(3.31) hold as well. We omit the details.

5. Convergence Analysis of the Brinkman Problem

We now turn our attention to the convergence analysis of the finite element method for the Brinkman problem (1.1). In light of the discussion in the introduction, it suffices to verify the inf-sup condition (1.7) as well as show that the discretely divergence-free velocities are in fact divergence free, i.e., that (1.10) holds.

To show the inf-sup condition (1.7), we first note that by (3.30) there holds

$$\|\boldsymbol{\Pi}_h \boldsymbol{v}\|_{1,h} \le C\|\boldsymbol{v}\|_{H^1(\Omega)} \qquad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{5.1}$$

where we recall that $\|\cdot\|_{1,h}$ is a piecewise H^1 norm defined as $\|\boldsymbol{v}\|_{1,h}^2 = \sum_{K\in\Omega_h} \|\boldsymbol{v}\|_{H^1(K)}^2$. By (1.5), for any $w \in W_h$ there exists a $\boldsymbol{v} \in \boldsymbol{V}$ such that

$$C\|w\|_{L^2(\Omega)} \le \frac{b_h(\boldsymbol{v}, w)}{\|\boldsymbol{v}\|_{H^1(\Omega)}}. \tag{5.2}$$

It then follows from (5.2), (3.24) and (5.1) that

$$C\|w\|_{L^2(\Omega)} \le \frac{b_h(\boldsymbol{v}, w)}{\|\boldsymbol{v}\|_{H^1(\Omega)}} = \frac{b_h(\boldsymbol{\Pi}_h\boldsymbol{v}, w)}{\|\boldsymbol{v}\|_{H^1(\Omega)}} \le C\frac{b_h(\boldsymbol{\Pi}_h\boldsymbol{v}, w)}{\|\boldsymbol{\Pi}_h\boldsymbol{v}\|_{1,h}} \le C \sup_{\boldsymbol{v}_h\in\boldsymbol{V}_h} \frac{b_h(\boldsymbol{v}_h, w)}{\|\boldsymbol{v}_h\|_{1,h}}.$$

Furthermore, there holds

$$\operatorname{div} \boldsymbol{V}_h \subseteq W_h$$

due to the construction of $\boldsymbol{V}_h$ and by the properties of $\boldsymbol{M}_h$. It then follows from Theorem 1.2 that that the estimates (1.12) hold. Therefore, in light of (3.31) it remains to estimate the consistency error defined by (1.15).

By the definition of the finite element space, in three dimensions there holds for any $\boldsymbol{\mu} \in \mathcal{P}^{k-1}(F)$ and $\boldsymbol{\kappa} \in \mathcal{P}^0(F)$,

$$\big\langle \nu\, \mathbf{curl}\, \boldsymbol{u}, [\boldsymbol{v}\times\boldsymbol{n}]\big\rangle_F = \big\langle \nu\, \mathbf{curl}\, \boldsymbol{u} - \boldsymbol{\mu}, [\boldsymbol{v}\times\boldsymbol{n} - \boldsymbol{\kappa}]\big\rangle_F.$$

It then follows that if $\boldsymbol{u} \in \boldsymbol{H}^s(\Omega)$ for some $2 \le s \le k+1$, then for all faces $F \in \mathcal{E}_h$,

$$\big\langle \nu\, \mathbf{curl}\, \boldsymbol{u}, [\boldsymbol{v}\times\boldsymbol{n}]\big\rangle_F \le C\nu h^{s-1}|\boldsymbol{u}|_{H^s(\omega(F))}|\boldsymbol{v}|_{H^1(\omega(F))},$$

and therefore by (1.15) and (1.14),

$$E_h(\boldsymbol{u}, \boldsymbol{v}_h) \le Ch^{s-1}\nu|\boldsymbol{u}|_{H^s(\Omega)}|\boldsymbol{v}|_{1,h} \le Ch^{s-1}\nu^{1/2}|\boldsymbol{u}|_{H^s(\Omega)}\|\boldsymbol{v}\|_{a,h}. \tag{5.3}$$

A similar estimate holds in the two dimensional case.

Combining (5.3) with (1.12) and (3.31) we have the following result.

Theorem 5.1. *Let $(\boldsymbol{u}, p) \in \boldsymbol{V}\times W$ be the solution to the Brinkman problem* (1.1)*, and let $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times W_h$ solve* (1.6) *with the finite element spaces defined by* (3.19) *and* (3.20)*. Suppose that $\boldsymbol{u} \in \boldsymbol{H}^s(\Omega)\times H^{s-1}(\Omega)$ with $2 \le s \le k+1$. Then there holds*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{a,h} \le C\big(\nu^{1/2}h^{s-1} + \alpha^{1/2}h^s\big)|\boldsymbol{u}|_{H^s(\Omega)}.$$

If $p \in H^{s-1}(\Omega)$ and $\boldsymbol{M}_h$ is taken to be the BDM space of order k, then

$$\|p - p_h\|_{L^2(\Omega)} \le C\Big(h^{s-1}|p|_{H^{s-1}(\Omega)} + M^{1/2}\big(\nu^{1/2}h^{s-1} + \alpha^{1/2}h^s\big)|\boldsymbol{u}|_{H^s(\Omega)}\Big),$$

where $M = \max\{\nu, \alpha\}$. Otherwise, if $p \in H^s(\Omega)$ and $\boldsymbol{M}_h$ is taken to be the RT space of order k, we have

$$\|p - p_h\|_{L^2(\Omega)} \le C\Big(h^s|p|_{H^s(\Omega)} + M^{1/2}\big(\nu^{1/2}h^{s-1} + \alpha^{1/2}h^s\big)|\boldsymbol{u}|_{H^s(\Omega)}\Big).$$

6. Discrete de Rham Complexes

Following [37, 29], we show that the construction of the nonconforming elements for the Brinkman problem are closely related to discrete de Rham complexes with extra smoothness. In doing so, we obtain higher order nonconforming elements for a singular biharmonic problem [34, 37, 20] and singular problems posed in $H^1(\mathbf{curl}\,;\Omega)$. In particular, we show how the three dimensional finite element space V_h is part of the discrete analog of the complex

$$\mathbf{R} \xrightarrow{\subset} H_0^2 \xrightarrow{\mathbf{grad}} \boldsymbol{H}_0^1(\mathbf{curl}\,) \xrightarrow{\mathbf{curl}} \boldsymbol{H}_0^1 \xrightarrow{\operatorname{div}} L_0^2 \longrightarrow 0. \tag{6.1}$$

The sequence (6.1) is an exact complex provided that Ω is a convex polyhedral domain [19, 37], that is, the range of each map is the null space of the succeeding map. The statement that it is a complex just means that the composition of two consecutive maps is zero.

To define the discrete analogue of (6.1) we define the following local spaces:

$$X^{k+1}(K) = \mathcal{P}^{k+1}(K) + b_K Q^{k-1}(K), \tag{6.2}$$

$$\boldsymbol{Y}^{k+1}(K) = \boldsymbol{N}^{k+1}(K) + \mathbf{grad}\,\big(b_K Q^{k-1}(K)\big) + b_K \boldsymbol{Q}^{k-1}(K), \tag{6.3}$$

where we recall that $\boldsymbol{N}^{k+1}(K)$ denotes the Nedelec space of index $k+1$ (cf. (3.3)), $\boldsymbol{Q}^{k-1}(K)$ is defined by (3.6), and $Q^{k-1}(K)$ is the three dimensional analogue of (4.2b), i.e.,

$$Q^{k-1}(K) = \sum_F b_F Q_F^{k-1}(K), \tag{6.4}$$

$$Q_F^{k-1}(K) = \big\{q \in \mathcal{P}^{k-1}(K) :\ (q, b_K b_F p)_K = 0 \ \text{ for all } p \in \mathcal{P}^{k-2}(K)\big\}. \tag{6.5}$$

We note that the space $X^{k+1}(K)$ was recently introduced in [20]. In what follows, we shall show that the global space that uses $X^{k+1}(K)$ will take the place of H_0^2 in (6.1), and the global space that uses $\boldsymbol{Y}^{k+1}(K)$ will take the place of $\boldsymbol{H}_0^1(\mathbf{curl}\,)$, $\boldsymbol{V}_h$ will take the place of $\boldsymbol{H}_0^1$, while W_h will take the place of L_0^2 in (6.1).

Before proving this result, we first discuss the properties of the finite element spaces (6.2)–(6.3), their associated degrees of freedom, and unisolvency.

6.1. Properties of $X^{k+1}(K)$.

We define the following degrees of freedom for the local space $X^{k+1}(K)$:

(6.6a)	$w(a)$	for all vertices a,
(6.6b)	$\langle w, \mu\rangle_e$	for all $\mu \in \mathcal{P}^{k-1}(e)$ and edges e of K,
(6.6c)	$\langle w, \kappa\rangle_F$	for all $\kappa \in \mathcal{P}^{k-2}(F)$ and faces F of K,
(6.6d)	$(w, \rho)_K$	for all $\rho \in \mathcal{P}^{k-3}(K)$,
(6.6e)	$\langle \mathbf{grad}\, w \cdot \boldsymbol{n}_F, \omega\rangle_F$	for all $\omega \in \mathcal{P}^{k-1}(F)$ and faces F of K.

The following result can be found in [20].

Lemma 6.1. *There holds*

$$X^{k+1}(K) = \mathcal{P}^{k+1}(K) \oplus b_K Q^{k-1}(K), \tag{6.7}$$

$$\dim X^{k+1}(K) = \dim \mathcal{P}^{k+1}(K) + 4\mathcal{P}^{k-1}(F), \tag{6.8}$$

Furthermore, any function $w \in X^{k+1}(K)$ is uniquely determined by the degrees of freedom (6.6).

6.2. **Properties of $\boldsymbol{Y}^{k+1}(K)$.** We define the following degrees of freedom for the local space $\boldsymbol{Y}^{k+1}(K)$:

$$\langle \boldsymbol{y} \cdot \boldsymbol{t}_e, \kappa \rangle_e \qquad \text{for all } \kappa \in \mathcal{P}^k(e) \text{ and edges } e \text{ of } K, \tag{6.9a}$$

$$\langle \boldsymbol{y} \times \boldsymbol{n}_F, \boldsymbol{\mu} \rangle_F \qquad \text{for all } \boldsymbol{\mu} \in \boldsymbol{\mathcal{P}}^{k-1}(F) \text{ and faces } F \text{ of } K, \tag{6.9b}$$

$$(\boldsymbol{y}, \boldsymbol{\rho})_K \qquad \text{for all } \boldsymbol{\rho} \in \boldsymbol{\mathcal{P}}^{k-2}(K), \tag{6.9c}$$

$$\langle \boldsymbol{y} \cdot \boldsymbol{n}_F, \omega \rangle_F \qquad \text{for all } \omega \in \mathcal{P}^{k-1}(F) \text{ and faces } F \text{ of } K, \tag{6.9d}$$

$$\langle \mathbf{curl}\, \boldsymbol{y} \times \boldsymbol{n}_F, \boldsymbol{\chi} \rangle_F \qquad \text{for all } \boldsymbol{\chi} \in \boldsymbol{\mathcal{P}}^{k-1}(F) \text{ and faces } F \text{ of } K. \tag{6.9e}$$

Lemma 6.2. *There holds*

$$\boldsymbol{Y}^{k+1} = \boldsymbol{N}^{k+1}(K) \oplus \mathbf{grad}\,(b_K Q^{k-1}(K)) \oplus b_K \boldsymbol{Q}^{k-1}(K), \tag{6.10}$$

$$\dim \boldsymbol{Y}^{k+1}(K) = \dim \boldsymbol{N}^{k+1}(K) + 4 \dim \mathcal{P}^{k-1}(F) + 4 \dim \boldsymbol{\mathcal{P}}^{k-1}(F). \tag{6.11}$$

Moreover, any function $\boldsymbol{y} \in \boldsymbol{Y}^{k+1}(K)$ is uniquely determined by the degrees of freedom (6.9).

Proof. We first show that if all of the degrees of freedom vanish for $\boldsymbol{y} \in \boldsymbol{Y}^{k+1}(K)$, then $\boldsymbol{y} \equiv 0$. This fact along with (6.11) will show that the degrees of freedom uniquely determine a function in $\boldsymbol{Y}^{k+1}(K)$.

Suppose that $\boldsymbol{y} \in \boldsymbol{Y}^{k+1}(K)$ vanishes at all of the degrees of freedom (6.9). By the definition of $\boldsymbol{y}$, we can write $\boldsymbol{y} = \boldsymbol{y}_0 + \mathbf{grad}\,(b_K q) + b_K \boldsymbol{p}$ with $\boldsymbol{y}_0 \in \boldsymbol{N}^{k+1}(K)$, $q \in Q^{k-1}(K)$ and $\boldsymbol{p} \in \boldsymbol{Q}^{k-1}(K)$. Note that $\mathbf{grad}\,(b_K q) \cdot \boldsymbol{t}_e\big|_e = 0$ and $(b_K \boldsymbol{p}) \cdot \boldsymbol{t}_e\big|_e = 0$ for all edges e, and therefore

$$\big\langle \boldsymbol{y}_0 \cdot \boldsymbol{t}_e, \kappa \big\rangle_e \qquad \text{for all } \kappa \in \mathcal{P}^k(e) \text{ and edges } e \text{ of } K. \tag{6.12}$$

Furthermore, we have $\mathbf{grad}\,(b_K q) \times \boldsymbol{n}_F\big|_F = 0$ and $(b_K \boldsymbol{p}) \times \boldsymbol{n}_F\big|_F = 0$ for all faces F, and therefore

$$\big\langle \boldsymbol{y}_0 \times \boldsymbol{n}_F, \boldsymbol{\mu} \big\rangle_F \qquad \text{for all } \boldsymbol{\mu} \in \boldsymbol{\mathcal{P}}^{k-1}(F) \text{ and faces } F \text{ of } K. \tag{6.13}$$

Next we write $q = \sum_F b_F q_F$ for $q_F \in Q_F^{k-1}(K)$. Then by integration by parts and (6.5), we have for any $\boldsymbol{\rho} \in \boldsymbol{\mathcal{P}}^{k-2}(K)$

$$\big(\mathbf{grad}\,(b_K q), \boldsymbol{\rho}\big)_K = -\sum_F \big(q_F, b_K b_F \mathrm{div}\, \boldsymbol{\rho}\big)_K = 0. \tag{6.14}$$

Moreover, by writing $\boldsymbol{p} = \sum_F b_F(\boldsymbol{p}_F \times \boldsymbol{n}_F)$ with $(\boldsymbol{p} \times \boldsymbol{n}_F) \in \boldsymbol{Q}_F^{k-1}(K)$ and $\boldsymbol{\rho} = -(\boldsymbol{\rho} \times \boldsymbol{n}_F) \times \boldsymbol{n}_F + (\boldsymbol{\rho} \cdot \boldsymbol{n}_F)\boldsymbol{n}_F$, we have by (3.6),

$$(b_K \boldsymbol{p}, \boldsymbol{\rho})_K = -\sum_F \big((\boldsymbol{p}_F \times \boldsymbol{n}_F), b_K b_F (\boldsymbol{\rho} \times \boldsymbol{n}_F) \times \boldsymbol{n}_F\big)_K = 0.$$

Thus, we have

$$(\boldsymbol{y}_0, \boldsymbol{\rho})_K = 0 \qquad \text{for all } \boldsymbol{\rho} \in \boldsymbol{\mathcal{P}}^{k-2}(K). \tag{6.15}$$

Since the degrees of freedom (6.9a)–(6.9c) uniquely determine a function in $\boldsymbol{N}^{k+1}(K)$, we have by (6.12)–(6.15) that $\boldsymbol{y}_0 \equiv 0$ and so $\boldsymbol{y} = \mathbf{grad}\,(b_K q) + b_K \boldsymbol{p}$.

Next, since b_K vanishes on ∂K, we have by (6.9d),

$$0 = \big\langle \boldsymbol{y} \cdot \boldsymbol{n}, q_F \big\rangle_F = \big\langle \mathbf{grad}\,(b_K q) \cdot \boldsymbol{n}, q_F \big\rangle_F = \big\langle q_F b_F \mathbf{grad}\,(b_K) \cdot \boldsymbol{n}, q_F \big\rangle_F = -a_F \big\langle q_F b_F^2, q_F \big\rangle_F.$$

Thus, $q_F \equiv 0$ for each F, and hence $q_F \equiv 0$ on K which in turn shows that $\boldsymbol{y} = b_K \boldsymbol{p}$.

Finally, by (6.9e), we have by the product rule,

$$\begin{aligned} 0 = \big\langle \mathbf{curl}\,\boldsymbol{y} \times \boldsymbol{n}_F, \boldsymbol{p}_F \times \boldsymbol{n}_F \big\rangle_F &= \big\langle (\mathbf{grad}\,b_k \times \boldsymbol{p}) \times \boldsymbol{n}_F, \boldsymbol{p}_F \times \boldsymbol{n}_F \big\rangle_F \\ &= -a_F \big\langle b_F (\boldsymbol{n}_F \times \boldsymbol{p}) \times \boldsymbol{n}_F, \boldsymbol{p}_F \times \boldsymbol{n}_F \big\rangle_F \\ &= -a_F \big\langle b_F^2 (\boldsymbol{p}_F \times \boldsymbol{n}_F), \boldsymbol{p}_F \times \boldsymbol{n}_F \big\rangle_F. \end{aligned}$$

We conclude from the last identity that $\boldsymbol{p}_F \times \boldsymbol{n}_F\big|_F = 0$, and therefore, in light of Lemma 3.2, $\boldsymbol{p}_F \times \boldsymbol{n}_F \equiv 0$ on K. It then follows that $\boldsymbol{y} \equiv 0$.

We now show (6.10). Suppose that $\boldsymbol{y} \in \mathbf{grad}\,\big(b_K Q^{k-1}(K)\big) \cap \big(b_K \boldsymbol{Q}^{k-1}(K)\big)$. We then have $\boldsymbol{y} \cdot \boldsymbol{n}\big|_{\partial K} = 0$. Thus, by writing $\boldsymbol{y} = \mathbf{grad}\,(b_K q)$ and $q = \sum_F b_F q_F$, we have on each face F,

$$0 = \mathbf{grad}\,(b_K q) \cdot \boldsymbol{n}_F\big|_F = -a_F b_F^2 q_F\big|_F.$$

Therefore, $q_F\big|_F = 0$, from which we conclude $\boldsymbol{y} \equiv 0$.

Finally, since functions in $\mathbf{grad}\,(b_K Q^{k-1}(K))$ and $b_K \boldsymbol{Q}^{k-1}(K)$ both vanish at the degrees of freedom (6.9a)–(6.9c), and since functions in $\boldsymbol{N}^{k+1}(K)$ are uniquely determined by these degrees of freedom, we conclude that (6.10) holds.

The dimension count (6.11) then follows from (6.10) and noting

$$\dim \mathbf{grad}\,(b_K Q^{k-1}(K)) = 4\mathcal{P}^{k-1}(F) \qquad \text{and} \qquad \dim(b_K \boldsymbol{Q}^{k-1}(K)) = 4\boldsymbol{\mathcal{P}}^{k-1}(F).$$

□

6.3. **The Discrete Complex.** The degrees of freedom of the local spaces $X^{k+1}(K)$ and $\boldsymbol{Y}^{k+1}(K)$ given by (6.6) and (6.9) naturally leads us to define the global space as

$$\begin{aligned} X_h = \big\{ & w \in H_0^1(\Omega) :\ w\big|_K \in X^{k+1}(K) \text{ for all } K \in \Omega_h, \\ & \text{and } \big\langle [\mathbf{grad}\,w \cdot \boldsymbol{n}], \kappa \big\rangle_F = 0 \text{ for all } \kappa \in \mathcal{P}^{k-1}(F) \text{ and faces } F \big\}, \end{aligned} \tag{6.16}$$

$$\begin{aligned} \boldsymbol{Y}_h = \big\{ & \boldsymbol{y} \in \boldsymbol{H}_0(\mathbf{curl}\,; \Omega) :\ \boldsymbol{y}\big|_K \in \boldsymbol{Y}^{k+1}(K) \text{ for all } K \in \Omega_h, \\ & \text{and } \big\langle [\boldsymbol{y} \cdot \boldsymbol{n}], \omega \big\rangle_F = \big\langle [\mathbf{curl}\,\boldsymbol{y} \times \boldsymbol{n}], \boldsymbol{\chi} \big\rangle_F = 0 \\ & \text{for all } \omega \in \mathcal{P}^{k-1}(F),\ \boldsymbol{\chi} \in \boldsymbol{\mathcal{P}}^{k-1}(F), \text{ and faces } F \big\}. \end{aligned} \tag{6.17}$$

Note that X_h is a subspace of $H_0^1(\Omega)$ but not of $H_0^2(\Omega)$. However, since the normal derivatives of functions in X_h are weakly continuous, X_h can be used as non-conforming approximation to $H_0^2(\Omega)$; see [20]. Similarly $\boldsymbol{Y}^h$ is a non-conforming approximation to $\boldsymbol{H}_0^1(\mathbf{curl}\,;\Omega)$.

Theorem 6.3. *Let X_h, $\boldsymbol{V}_h$, $\boldsymbol{Y}_h$, and W_h be given by* (6.16), (3.19), (6.17) *and* (3.20), *respectively. Then the sequence*

$$\mathbf{R} \xrightarrow{\subset} X_h \xrightarrow{\mathbf{grad}} \boldsymbol{Y}_h \xrightarrow{\mathbf{curl}} \boldsymbol{V}_h \xrightarrow{\operatorname{div}} W_h \longrightarrow 0 \tag{6.18}$$

is an exact complex.

Proof. We will need the following well known result ([33, 32, 3, 2]) that says that the following is an exact discrete complex:

$$\mathbf{R} \xrightarrow{\subset} L_h \xrightarrow{\mathbf{grad}} \boldsymbol{N}_h \xrightarrow{\mathbf{curl}} \boldsymbol{M}_h \xrightarrow{\operatorname{div}} W_h \longrightarrow 0, \tag{6.19}$$

where

$$L_h = \{w \in H_0^1(\Omega) : w\big|_K \in \mathcal{P}^{k+1}(K) \text{ for all } K \in \Omega_h\},$$
$$\boldsymbol{N}_h = \{\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}\,;\Omega) : \boldsymbol{v}\big|_K \in \boldsymbol{N}^{k+1}(K) \text{ for all } K \in \Omega_h\},$$

$\boldsymbol{M}_h$ is given by (3.21) and W_h is given by (3.20).

Now suppose that $\mathbf{curl}\,\boldsymbol{y} = 0$ with $\boldsymbol{y} \in \boldsymbol{Y}_h$. By the definition of $\boldsymbol{Y}_h$ we know that $\boldsymbol{y}\big|_K = \boldsymbol{y}_0\big|_K + \mathbf{grad}\,(b_K q)\big|_K + b_K\boldsymbol{p}\big|_K$ with $\boldsymbol{y}_0 \in \boldsymbol{N}_h$, $q\big|_K \in Q^{k-1}(K)$ and $\boldsymbol{p}\big|_K \in \boldsymbol{Q}^{k-1}(K)$ for all $K \in \Omega_h$. Then clearly we have $\mathbf{curl}\,(\boldsymbol{y}_0)\big|_K + \mathbf{curl}\,(b_K\boldsymbol{p})\big|_K = 0$. Since $\mathbf{curl}\,\boldsymbol{y}_0\big|_K \in \boldsymbol{M}^k(K)$ and $\mathbf{curl}\,(b_K\boldsymbol{p})\big|_K \in \boldsymbol{U}^{k-1}(K)$ we have that $\mathbf{curl}\,\boldsymbol{y}_0\big|_K = 0$ and $\mathbf{curl}\,(b_K\boldsymbol{p})\big|_K = 0$. However, since functions of the form $b_K\boldsymbol{p}$ are uniquely determined by the degrees of freedom (6.9e), we must have $b_K\boldsymbol{p} = 0$. Therefore, by the exact complex (6.19), we have $\boldsymbol{y}_0 = \mathbf{grad}\,(w_0)$ for some $w_0 \in L_h$. Thus, $\boldsymbol{y}\big|_K = \mathbf{grad}\,w\big|_K$ where $w\big|_K = w_0\big|_K + b_K q\big|_K$. Since $\boldsymbol{y} \in \boldsymbol{Y}_h$ we have $\langle[\boldsymbol{y}\cdot\boldsymbol{n}], \omega\rangle_F = 0$ for all $\omega \in \mathcal{P}^{k-1}(F)$ and all faces F, and thus we also have $w \in X_h$.

Next, let $\boldsymbol{v} \in \boldsymbol{V}_h$ given by $\boldsymbol{v}\big|_K = \boldsymbol{v}_0\big|_K + \mathbf{curl}\,(b_K\boldsymbol{q})\big|_K$ with $\boldsymbol{v}_0 \in \boldsymbol{M}_h$ and $\boldsymbol{q}\big|_K \in \boldsymbol{Q}^{k-1}(K)$, and suppose that $\operatorname{div}\boldsymbol{v} = 0$. We then have $\operatorname{div}\boldsymbol{v}_0 = 0$, and therefore by (6.19), $\boldsymbol{v}_0 = \mathbf{curl}\,\boldsymbol{y}_0$ for some $\boldsymbol{y}_0 \in \boldsymbol{N}_h$. Therefore, we have $\boldsymbol{v}\big|_K = \mathbf{curl}\,\boldsymbol{y}\big|_K$, where $\boldsymbol{y}\big|_K = \boldsymbol{y}_0\big|_K + b_K\boldsymbol{q}\big|_K + \mathbf{grad}\,(b_K q)\big|_K$ and $q_K \in Q^{k-1}(K)$ is arbitrary. Since $\boldsymbol{v} \in \boldsymbol{V}_h$, we know that $\big\langle[\boldsymbol{v}\times\boldsymbol{n}], \boldsymbol{\mu}\big\rangle_F = 0$ for all $\boldsymbol{\mu} \in \boldsymbol{\mathcal{P}}^{k-1}(F)$ and all faces F. Therefore, we have $\big\langle[\mathbf{curl}\,\boldsymbol{y}\times\boldsymbol{n}], \boldsymbol{\mu}\big\rangle_F = 0$ for all $\boldsymbol{\mu} \in \mathcal{P}^{k-1}(F)$ and all faces F as well. We can also choose q so that $\langle[\boldsymbol{y}\cdot\boldsymbol{n}], \omega\rangle_F = 0$ for all $\omega \in \mathcal{P}^{k-1}(F)$ and all faces F. Therefore, we have shown that $\boldsymbol{y} \in \boldsymbol{Y}_h$. □

Remark 6.4. Above, we considered the discrete complex with zero boundary conditions. We would like to mention that we can easily obtain the discrete analogue of the complex

$$\mathbf{R} \xrightarrow{\subset} H^2 \xrightarrow{\mathbf{grad}} \boldsymbol{H}^1(\mathbf{curl}\,) \xrightarrow{\mathbf{curl}} \boldsymbol{H}^1 \xrightarrow{\operatorname{div}} L^2 \longrightarrow 0. \tag{6.20}$$

Indeed, to construct such a complex, we simply do not impose boundary conditions when defining the discrete spaces X_h, $\boldsymbol{Y}_h$ and $\boldsymbol{V}_h$.

6.4. **Remarks on the Two Dimensional Complex.** Analogous to (6.1), the two dimensional de Rham complex with extra smoothness is given by

$$\mathbf{R} \xrightarrow{\subset} H_0^2 \xrightarrow{\mathbf{curl}} \boldsymbol{H}_0^1 \xrightarrow{\text{div}} L_0^2 \longrightarrow 0. \tag{6.21}$$

The sequence is exact provided the domain Ω is simply connected.

To introduce the corresponding discrete de Rham complex, we define $X^{k+1}(K)$ in the two dimensional case as

$$X^{k+1}(K) = \mathcal{P}^{k+1}(K) + b_K Q^{k-1}(K), \tag{6.22}$$

with $Q^{k-1}(K)$ defined by (4.2b) and $k \geq 1$. The associated degrees of freedom of $X^{k+1}(K)$ are defined as follows:

$$w(a) \qquad \text{for all vertices } a \text{ of } K, \tag{6.23a}$$

$$\langle w, \mu\rangle_F \qquad \text{for all } \mu \in \mathcal{P}^{k-1}(F) \text{ and edges } F \text{ of } K, \tag{6.23b}$$

$$(w, \rho)_K \qquad \text{for all } \rho \in \mathcal{P}^{k-2}(K), \tag{6.23c}$$

$$\langle \mathbf{grad}\, w \cdot \boldsymbol{n}_F, \omega\rangle_F \qquad \text{for all } \omega \in \mathcal{P}^{k-1}(F) \text{ and edges } F \text{ of } K. \tag{6.23d}$$

The space $X^{k+1}(K)$ was introduced in [20], and the following result was proved there.

Lemma 6.5. *There holds*

$$X^{k+1}(K) = \mathcal{P}^{k+1}(K) \oplus b_K Q^{k-1}(K), \tag{6.24}$$

$$\dim X^{k+1}(K) = \dim \mathcal{P}^{k+1}(K) + 3\mathcal{P}^{k-1}(F). \tag{6.25}$$

Furthermore, any function $w \in X^{k+1}(K)$ is uniquely determined by the degrees of freedom (6.23)*.*

The two dimensional global space X_h is defined as

$$X_h = \big\{w \in H_0^1(\Omega) : \ w\big|_K \in X^{k+1}(K) \text{ for all } K \in \Omega_h, \\ \text{and } \big\langle [\mathbf{grad}\, w \cdot \boldsymbol{n}], \omega\big\rangle_F = 0 \text{ for all } \omega \in \mathcal{P}^{k-1}(F) \text{ and all edges } F\big\}. \tag{6.26}$$

Since the right-hand side of (6.22) is a direct sum, and by the definitions of the finite element spaces, we can easily see that

$$\mathbf{curl}\, X_h \subset \boldsymbol{V}_h, \qquad \text{div}\, \boldsymbol{V}_h \subset W_h.$$

Therefore, the following is a discrete de Rham complex:

$$\mathbf{R} \xrightarrow{\subset} X_h \xrightarrow{\mathbf{curl}} \boldsymbol{V}_h \xrightarrow{\text{div}} W_h \longrightarrow 0. \tag{6.27}$$

The following theorem shows that (6.27) is exact.

Theorem 6.6. *Let $\boldsymbol{V}_h$, X_h and W_h be defined by* (4.6), (6.26) *and* (3.20)*, respectively. Then* (6.27) *is an exact complex.*

Proof. It suffices to show that if $\boldsymbol{v} \in V_h$ with $\text{div}\, \boldsymbol{v} = 0$, then $\boldsymbol{v} = \mathbf{curl}\, w$ for some $w \in W_h$. To prove this, we use the face that the sequence given by

$$\mathbf{R} \xrightarrow{\subset} L_h \xrightarrow{\mathbf{curl}} \boldsymbol{M}_h \xrightarrow{\text{div}} W_h \longrightarrow 0 \tag{6.28}$$

is exact; see for example [3].

By the definition of $\boldsymbol{V}_h$, we may write $\boldsymbol{v}\big|_K = \boldsymbol{v}_0\big|_K + \mathbf{curl}\,(b_K q)\big|_K$ for each $K \in \Omega_h$, with $\boldsymbol{v}_0 \in \boldsymbol{M}_h$ and $q\big|_K \in Q^{k-1}(K)$. Clearly, we have $0 = \operatorname{div} \boldsymbol{v} = \operatorname{div} \boldsymbol{v}_0$. Thus, since the complex (6.28) is exact, we may write $\boldsymbol{v}_0 = \mathbf{curl}\, w_0$ for some $w_0 \in L_h$. It can then be readily checked that w defined by $w\big|_K = w_0\big|_K + b_K q\big|_K$ is in X_h. Thus, $\boldsymbol{v} = \mathbf{curl}\, w$ with $w \in X_h$, and therefore, the sequence (6.27) is an exact complex. □

7. Local basis for $\boldsymbol{Q}_F^k(K)$

To implement the new elements for the Brinkman problem, we need to calculate a local basis for the space $\boldsymbol{Q}_F^k(K)$. Here, we give an outline on how this can be easily done. We start with the more difficult case of three dimensions. As a first step, we show how to find a basis of $Q_F^k(K)$ (see (6.5)) in terms of the barycentric coordinates of K. To this end, we let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be the barycentric coordinates of K, and without loss of generality, we assume that λ_1 is the barycentric coordinate that vanishes on face F. Hence, $b_F = \lambda_2\lambda_3\lambda_4$ and $b_K = \lambda_1\lambda_2\lambda_3\lambda_4$.

For each $i = 1, 2, \ldots, (k+1)(k+2)/2$ let $p_i \in \mathcal{P}^{k-1}(K)$ be the unique solution to

$$(p_i, q\, b_F\, b_K)_K = -(r_i, q\, b_F\, b_K)_K \qquad \forall q \in \mathcal{P}^{k-1}(K), \tag{7.1}$$

where r_i is a function of the form $\lambda_2^{\ell}\lambda_3^{m}\lambda_4^{n}$ with $\ell + m + n = k$ (note that there are exactly $(k+1)(k+2)/2$ functions of this form). Clearly, p_i is well defined since (7.1) leads to a positive definite system for p_i. Moreover, we can easily solve for p_i in terms of barycentric coordinates. If we let $\phi_i = p_i + r_i$, we see that $\{\phi_1, \phi_2, \ldots, \phi_{(k+1)(k+2)/2}\}$ is a basis for $Q_F^k(K)$.

We now give some examples. Using the formula (cf. [17]),

$$\int_K \lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\ldots\lambda_{d+1}^{\alpha_{d+1}}\, dx = d!|K|\frac{\alpha_1!\alpha_2!\ldots\alpha_{d+1}!}{(|\alpha| + d)!},$$

we can easily solve for p_i for any k. Of course in the case $k = 0$, ϕ_1 is just a constant. In the case $k = 1$ we have

$$\phi_1 = -3/11 + \lambda_2, \quad \phi_2 = -3/11 + \lambda_3, \quad \phi_3 = -3/11 + \lambda_4.$$

In the case $k = 2$, we obtain

$$\begin{aligned}
\phi_1 &= \frac{4}{52} - \frac{3}{13}(\lambda_2 + \lambda_3) + \lambda_2\lambda_3, & \phi_2 &= \frac{4}{52} - \frac{3}{13}(\lambda_2 + \lambda_4) + \lambda_2\lambda_4,\\
\phi_3 &= \frac{4}{52} - \frac{3}{13}(\lambda_3 + \lambda_4) + \lambda_3\lambda_4, & \phi_4 &= \frac{1}{13} - \frac{8}{13}\lambda_2 + \lambda_2^2,\\
\phi_5 &= \frac{1}{13} - \frac{8}{13}\lambda_3 + \lambda_3^2, & \phi_6 &= \frac{1}{13} - \frac{8}{13}\lambda_4 + \lambda_4^2.
\end{aligned}$$

In order to calculate a basis $\boldsymbol{Q}_F^k(K)$, we use the fact that

$$\boldsymbol{Q}_F^k(K) = \{m_t \boldsymbol{t}_F + m_s \boldsymbol{s}_F : m_t, m_s \in Q_F^k(K)\},$$

where $\boldsymbol{t}_F, \boldsymbol{s}_F$ are orthonormal and tangent to the face F. One can prove this by using the definition of the space $\boldsymbol{Q}_F^k(K)$ given in (3.7). Thus, once we have a basis for $Q_F^k(K)$, we also have one for $\boldsymbol{Q}_F^k(K)$.

The two dimensional case is similar, and based on the discussion above we can easily find a basis for $Q_F^k(K)$. Below, we give a few examples (assuming λ_1 vanishes on the edge F): For $k = 1$,

$$\phi_1 = -3/8 + \lambda_2, \quad \phi_2 = -3/8 + \lambda_3,$$

and for $k = 2$,

$$\phi_1 = -\frac{1}{10} + \frac{3}{10}(\lambda_2 + \lambda_3) + \lambda_2\lambda_3, \quad \phi_2 = -\frac{2}{15} + \frac{4}{5}\lambda_2 + \lambda_2^2, \quad \phi_3 = -\frac{2}{15} + \frac{4}{5}\lambda_3 + \lambda_3^2.$$

References

[1] D. N. Arnold, F. Brezzi, and M. Fortin, *A stable finite element for the Stokes equations*, Calcolo 21 (1984), pp. 337–344.

[2] D. N. Arnold, R. S. Falk, and R. Winther, *Multigrid in $H(\mathrm{div})$ and $H(\mathrm{curl})$*, Numer. Math. 85 (2000), pp. 197–217.

[3] D. N. Arnold, R. S. Falk, and R. Winther, *Finite element exterior calculus, homological techniques, and applications*, Acta Numer. 15 (2006), pp. 1–155.

[4] D. N. Arnold and J. Qin, *Quadratic velocity/linear pressure Stokes elements*, in Advances in Computer Methods for Partial Differential Equations-VII, R. Vichnevetsky, D. Knight, and G. Richter, eds., IMACS, 1992, pp. 28–34.

[5] S. Badia, Santiago and R. Codina *Unified stabilized finite element formulations for the Stokes and the Darcy problems*, SIAM J. Numer. Anal. 47 (2009), pp. 1971–2000.

[6] S.C. Brenner and L.R. Scott, *The Mathematical Theory of Finite Element Methods (Third edition)*, Springer, 2008.

[7] F. Brezzi, J. Douglas and L.D. Marini, *Two families of mixed finite elements for second order elliptic problems*, Numer. Math. 47 (1985), pp. 217–235.

[8] F. Brezzi, J. Douglas, R. Durán and M. Fortin, *Mixed finite elements for second order elliptic problems in three variables*, Numer. Math. 51 (1987), pp. 237–250.

[9] F. Brezzi, M. Fortin, and D. L. Marini, *Mixed finite element methods with continuous stresses*, Math. Models Methods Appl. Sci. 3 (1993), pp. 275–287.

[10] F. Brezzi and M. Fortin, *Mixed and hybrid finite element methods*, Springer Series in Computational Mathematics, 15. Springer-Verlag, New York, 1991.

[11] E. Burman and P. Hansbo *Stabilized Crouzeix-Raviart element for the Darcy-Stokes problem*, Numer. Methods Partial Differential Equations 21 (2005), pp. 986–997.

[12] E. Burman and P. Hansbo *A unified stabilized method for Stokes' and Darcy's equations*, J. Comput. Appl. Math. 198 (2007), pp. 35–51.

[13] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland Publishing Company, Amsterdam, 1978.

[14] M. Crouzeix and P.-A. Raviart, *Conforming and nonconforming finite element methods for solving the stationary Stokes equations*, R.A.I.R.O Anal. Numer., 7 (1973), pp. 33–76.

[15] B. Cockburn, G. Kanschat and D. Schötzau, *A note on discontinuous Galerkin divergence-free solutions of the Navier-Stokes equations*, J. Sci. Comput. 31 (2007), no. 1-2, 61-73.

[16] D. B. Das *Hydrodynamic modelling for groundwater flow through permeable reactive barriers* Hydrol. Process. 16 (2002), pp. 3393–3418.

[17] M. A. Eisenberg and L. E. Malvern, *On finite element integration in natural coordinates*, International Journal for Numerical Methods in Engineering 7 (1973), pp. 574–575.

[18] L. E. Figueroa, G. N. Gatica, and A. Márquez *Augmented mixed finite element methods for the stationary Stokes equations* SIAM J. Sci. Comput. 31 (2008/09), pp. 1082–1119.

[19] V. Girault and P.-A. Raviart, *Finite element methods for the Navier-Stokes equations*, Berlin, Springer-Verlag, 1986.

[20] J. Guzmán, D. Leykekhman, and M. Neilan, *A family of non-conforming elements and the analysis of Nitsche's method for a singular perturbed biharmonic problem*, preprint.

[21] M. Griebel and M. Klitz, *Homogenization and numerical simulation of flow in geometries with textile microstructures*—, Multiscale Modeling & Simulation 8 (2010), pp. 1439–1460.
[22] P. Hansbo, P and M. Juntunen, M, *Weakly imposed Dirichlet boundary conditions for the Brinkman model of porous media flow*, Appl. Numer. Math. 59 (2009), pp. 1274–1289.
[23] M. Juntunen, R. Stenberg, *Analysis of finite element methods for the Brinkman problem*, Calcolo 47 (2010), pp. 129–147.
[24] T. Kara and J. Goldak, *Three-dimensional numerical analysis of heat and mass transfer in heat pipes,* Heat Mass Transfer 43 (2007), pp. 775–785.
[25] J. Könnö and R. Stenberg, *Non-conforming finite element method for the Brinkman problem*, In G. Kreiss, P. Lötstedt, A. Malqvist, and M. Neytcheva, editors, Numerical Mathematics and Advanced Applications 2009, pages 515522. Springer Berlin Heidelberg, 2010.
[26] J. Könnö and R. Stenberg, *Analysis of H(div)-conforming finite elements for the Brinkman problem*, preprint.
[27] J. Könnö and R. Stenberg, *Numerical Computations with H(div)-Finite Elements for the Brinkman Problem*, preprint.
[28] C. Y. Liua, W. M. Yinga, and J. O. Tan, *Flow in the adiabatic section of a heat pipe*, International Communications in Heat and Mass Transfer 16 (1989), pp. 79–88.
[29] K. A. Mardal, X.-C. Tai, and R. Winther, *A robust finite element method for Darcy-Stokes flow*, SIAM J. Numer. Anal. 40 (2002), pp. 1605–1631.
[30] W. Miladi and M. Racila, *Mathematical model of fluid flow in an osteon influence of cardiac system*, arXiv:1005.4992v1.
[31] V. Nassehi, *Modelling of combined NavierStokes and Darcy ows in crossow membrane filtration*, Chem. Eng. Sci. 53 (1998), pp. 1253–1265.
[32] J.-C. Nédélec, *Mixed Finite Elements in* $\mathbb{R}^3$, Numer. Math., 35 (1980), pp. 315–341.
[33] J.-C. Nédélec, *A new family of mixed finite elements in* $\mathbb{R}^3$, Numer. Math. 50 (1986), pp. 57–81.
[34] T. K. Nilssen, X.-C. Tai, and R. Winther, *A robust nonconforming* H^2*-element*, Math. Comp. 70 (2001), pp. 489–505.
[35] P.-A. Raviart and J. M. Thomas, *A mixed finite element method for 2nd order elliptic problems*, in Mathematical aspects of finite element methods (Proc. Conf., Consiglio Naz. delle Ricerche (C.N.R.), Rome, 1975), Springer, Berlin, 1977, pp. 292–315. Lecture Notes in Math., Vol. 606.
[36] L. R. Scott and M. Vogelius, *Norm estimates for a maximal right inverse of the divergence operator in spaces of piecewise polynomials*, RAIRO Modél. Math. Anal. Numér., 19 (1985), pp. 111–143.
[37] X.-C. Tai and R. Winther, *A discrete de Rham complex with enhanced smoothness*, Calcolo 43 (2006), pp. 287–306.
[38] C. Taylor and P. Hood, *A numerical solution of the Navier-Stokes equations using the finite element technique*, Internat. J. Comput. & Fluids 1 (1973), pp. 73–100.
[39] X. Xie, J. Xu, and G. Xue, *Uniformly-stable finite element methods for Darcy-Stokes-Brinkman models*, J. Comput. Math. 26 (2008), pp. 437455.
[40] J. Wang and X. Ye, *New finite element methods in computational fluid dynamics by* $H(\mathrm{div})$ *elements*, SIAM J. Numer. Anal. 45 (2007), no. 3, 1269-1286.
[41] X. Xu and S. Zhang, *A new divergence-free interpolation operator with applications to the Darcy-Stokes-Brinkman equations,* SIAM J. Sci. Comput. 32 (2010), pp. 855–874.
[42] S. Zhang, *A family of* $Q_{k+1,k} \times Q_{k,k+1}$ *divergence-free finite elements on rectangular grids*, SIAM J. Numer. Anal. 47 (2009), pp. 2090–2107.

Appendix A. Proof of Theorem 1.2

First by (1.14), (1.9) and (1.10), we have for any $\boldsymbol{v} \in \boldsymbol{Z}_h$,

$$a_h(\boldsymbol{v},\boldsymbol{v}) = \|\boldsymbol{v}\|_{a,h}^2 = |||\boldsymbol{v}|||_h^2 .$$

Thus, in light of the inf-sup condition (1.7), it follows that there exists a unique pair $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times W_h$ satisfying (1.6) [10].

To derive the error estimates (1.12), we first note that due to the inf-sup condition (1.7), the space $\boldsymbol{Z}_h(g)$ defined by (1.13) is non-empty. Let $P_h p \in W_h$ be the L^2 projection of p onto W_h, let $\boldsymbol{v} \in \boldsymbol{Z}_h(g)$ be arbitrary, and set $\boldsymbol{e}_h = \boldsymbol{u}_h - \boldsymbol{v}$ and $w_h = p_h - P_h p$. Then by (1.6), we have

$$a_h(\boldsymbol{e}_h, \boldsymbol{e}_h) - b_h(\boldsymbol{e}_h, w_h) = (f, \boldsymbol{e}_h) - a_h(\boldsymbol{v}, \boldsymbol{e}_h), \tag{A.1a}$$

$$b_h(\boldsymbol{e}_h, w_h) = 0. \tag{A.1b}$$

Noting that the inclusion (1.11) implies $b_h(\boldsymbol{e}_h, p) = b_h(\boldsymbol{e}_h, P_h p) = 0$, we have by (A.1) and (1.14),

$$\begin{aligned} \|\boldsymbol{e}_h\|_{a,h}^2 &= (f, \boldsymbol{e}_h) - a_h(\boldsymbol{v}, \boldsymbol{e}_h) \\ &= a_h(\boldsymbol{u} - \boldsymbol{v}, \boldsymbol{e}_h) + (f, \boldsymbol{e}_h) - a_h(\boldsymbol{u}, \boldsymbol{e}_h) + b_h(\boldsymbol{e}_h, p) \\ &\le \|\boldsymbol{u} - \boldsymbol{v}\|_{a,h} \|\boldsymbol{e}_h\|_{a,h} + (f, \boldsymbol{e}_h) - a_h(\boldsymbol{u}, \boldsymbol{e}_h) + b_h(\boldsymbol{e}_h, p). \end{aligned} \tag{A.2}$$

By Green's formula and (1.1), we have

$$(f, \boldsymbol{e}_h) - a_h(\boldsymbol{u}, \boldsymbol{e}_h) + b_h(\boldsymbol{e}_h, p) = E_h(\boldsymbol{u}, \boldsymbol{e}_h), \tag{A.3}$$

with $E_h(\cdot, \cdot)$ defined by (1.15). Thus, the estimate (1.12a) follows from (A.2), (A.3) and the triangle inequality.

Next by (1.7), (1.6) and (A.3), we obtain

$$\begin{aligned} C\|p_h - P_h p\|_{L^2(\Omega)} &\le \sup_{\boldsymbol{v} \in \boldsymbol{V}_h} \frac{b_h(\boldsymbol{v}, p_h - P_h p)}{\|\boldsymbol{v}\|_{1,h}} \\ &= \sup_{\boldsymbol{v} \in \boldsymbol{V}_h} \frac{a_h(\boldsymbol{u}_h, \boldsymbol{v}) - b(\boldsymbol{v}, P_h p) - (f, \boldsymbol{v})}{\|\boldsymbol{v}\|_{1,h}} \\ &= \sup_{\boldsymbol{v} \in \boldsymbol{V}_h} \frac{a_h(\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{v}) + a_h(\boldsymbol{u}, \boldsymbol{v}) - b_h(\boldsymbol{v}, p) - (f, \boldsymbol{v})}{\|\boldsymbol{v}\|_{1,h}} \\ &= \sup_{\boldsymbol{v} \in \boldsymbol{V}_h} \frac{a_h(\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{v}) + E_h(\boldsymbol{u}, \boldsymbol{v})}{\|\boldsymbol{v}\|_{1,h}} \\ &\le \sup_{\boldsymbol{v} \in \boldsymbol{V}_h} \frac{\|\boldsymbol{u}_h - \boldsymbol{u}\|_{a,h} \|\boldsymbol{v}\|_{a,h} + E_h(\boldsymbol{u}, \boldsymbol{v})}{\|\boldsymbol{v}\|_{1,h}}. \end{aligned}$$

The estimate (1.12b) then follows from (1.12a), the inequality $\|\boldsymbol{v}\|_{a,h} \le M^{1/2} \|\boldsymbol{v}\|_{1,h}$, the triangle inequality, and the fact that

$$\|p - P_h p\|_{L^2(\Omega)} = \inf_{w \in W_h} \|p - w\|_{L^2(\Omega)}.$$

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