A NEW FAMILY OF MIXED METHODS FOR THE REISSNER-MINDLIN PLATE MODEL BASED ON A SYSTEM OF FIRST-ORDER EQUATIONS

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Abstract. The mixed method for the biharmonic problem introduced in [15] is extended to the Reissner-Mindlin plate model. The Reissner-Mindlin problem is written as a system of first order equations and all the resulting variables are approximated. However, the hybrid form of the method allows one to eliminate all the variables and have a final system only involving the Lagrange multipliers that approximate the transverse displacement and rotation at the edges of the triangulation. Mixed finite element spaces for elasticity with weakly imposed symmetry are used to approximate the bending moment matrix. Optimal estimates independent of the plate thickness are proved for the transverse displacement, rotations and bending moments. A post-processing technique is provided for the displacement and rotations variables and we show numerically that they converge faster than the original approximations.

1. Introduction

In [15] we developed a new mixed finite element method for the biharmonic problem. Here we develop a similar method for the more challenging hard clamped Reissner-Mindlin plate model:

$$-\nabla \cdot (\boldsymbol{C}\underline{\boldsymbol{\epsilon}}(\boldsymbol{r})) - \lambda t^{-2}(\nabla u - \boldsymbol{r}) = 0 \quad \text{in } \Omega, \tag{1.1a}$$

$$-\lambda t^{-2}\nabla \cdot (\nabla u - \boldsymbol{r}) = f \quad \text{in } \Omega, \tag{1.1b}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.1c}$$

$$\boldsymbol{r} = 0 \quad \text{on } \partial\Omega, \tag{1.1d}$$

where $\Omega \subset R^2$ is a polygonal domain and $f \in L^2(\Omega)$. Here t is the thickness of the plate and λ is fixed positive parameter. Moreover, the tensor $\boldsymbol{C}$ is defined to be

$$\boldsymbol{C}\underline{\boldsymbol{\tau}} = \frac{E}{12(1-\nu^2)}((1-\nu)\,\underline{\boldsymbol{\tau}} + \nu\,\mathrm{tr}(\underline{\boldsymbol{\tau}})\,\underline{\boldsymbol{I}}),$$

where ν is the Poisson ratio, $E = \frac{2(1+\nu)\lambda}{\kappa}$ is the Young's modulus and κ is the shear correction factor. The variable u is the transverse displacement and $\boldsymbol{r}$ the rotation.

Mixed finite elements for (1.1) typically approximate directly u and $\boldsymbol{r}$, and the shear stress $\boldsymbol{\sigma} = \lambda t^{-2}(\boldsymbol{r} - \nabla u)$; see for instance [4, 6, 12, 34, 5, 16, 25, 26, 28, 30] and [27] for a

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review. Instead, our method is based on the following formulation of the above problem

$$
\begin{aligned}
\boldsymbol{q} &= \nabla u, & \underline{\boldsymbol{\rho}} &= \frac{1}{2}(\nabla \boldsymbol{r} - (\nabla \boldsymbol{r})^T) && \text{in } \Omega,\\
\mathcal{A}\underline{\boldsymbol{z}} &= \nabla \boldsymbol{r} - \underline{\boldsymbol{\rho}}, & \boldsymbol{\sigma} &= \nabla \cdot \underline{\boldsymbol{z}} && \text{in } \Omega,\\
\boldsymbol{r} - \boldsymbol{q} - \hat{t}^2\boldsymbol{\sigma} &= 0, & \nabla \cdot \boldsymbol{\sigma} &= f && \text{in } \Omega,\\
u &= 0, & \boldsymbol{r} &= 0 && \text{on } \partial\Omega,
\end{aligned}
$$

where we define $\hat{t} := \frac{t}{\sqrt{\lambda}}$ and $\mathcal{A}$ denotes the inverse of $\boldsymbol{C}$. We use the following convention $(\nabla \boldsymbol{q})_{ij} = \partial_{x_j}(q_i)$ for $1 \le i, j \le 2$ where q_i is the i-th component of $\boldsymbol{q}$. Moreover, $(\nabla \cdot \underline{\boldsymbol{z}})_i = \sum_{j=1}^d \partial_{x_j} z_{ij}$ where the z_{ij} is the ij-entry of $\underline{\boldsymbol{z}}$. Although we introduced three new variables, we later will present a hybrid form of the mixed method that will allow us to eliminate all the interior variables locally to obtain a system for the Lagrange multipliers which approximate u and $\boldsymbol{r}$ on the edges of the triangulation. This makes the method computationally competitive. We would like to point out that Amara et al. [1] considered a low order method where they also approximate the bending moment directly which is, however, different from our lowest order method.

A desirable property for a method to have is that the approximations have provable error bounds independent of the plate thickness t. Indeed, all the methods considered in the review paper [27] have this property. Similarly, for our method we will prove optimal error estimates for the transverse displacement, rotation and bending moment independent of t.

The key idea of our method is that we formulate (1.1) such that $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div};\Omega)$ and each row of $\underline{\boldsymbol{z}}$ belongs to $\boldsymbol{H}(\mathrm{div};\Omega)$ and all the other variables will only be required to be in $L^2(\Omega)$. This will allow us to use the Raviart-Thomas spaces, and in fact this is what we did in [15]. However, for the Reissner-Mindlin problem, in contrast to the biharmonic problem, we need to deal with the symmetric gradient of $\boldsymbol{r}$. We deal with this issue by using weakly symmetric elements borrowed from elasticity (see [2, 10, 24]) and this is why we introduced the anti-symmetric gradient $\underline{\boldsymbol{\rho}}$ above. By doing this we can hybridize our method and eliminate all the interior variables and only get a formulation for the Lagrange multipliers. Hence, the final linear system that arises from our new method has exactly $3(k+1)$ (for $k \ge 1$) times the number of interior edges as unknowns if we consider Raviart-Thomas elements of index k.

We would like to mention that the analysis of the method we present in this paper for the Reissner-Mindlin problem will have many similarities with the analysis we performed for the biharmonic problem [15]. However, there are two main differences. First, here we have to prove estimates that are independent of t whereas for the biharmonic problem this is not an issue. Second, here we have to borrow some techniques for weakly symmetric methods for elasticity because of our choices of spaces which again did not arise in [15].

In [15] for the biharmonic we were able to prove that the projection of the error of the variable u superconverges with two orders higher than the optimal estimate. This allowed us to define a local post-processing procedure that produces a new approximation to u that converges with two orders more than the original approximation to u. Such estimates are based on a duality argument and certain regularity needs to be assumed. For the biharmonic problem such regularity estimates are known. However, for the Reissner-Mindlin problem the elliptic regularity results depend on the thickness t which does not allow us to prove such great results for the transverse displacement. Nonetheless, we define a post-processing procedure for the Reissner-Mindlin problem and we observe numerically that convergence rates are much faster than the original approximation. We do, however, prove that the post-processed approximation for the rotation converges faster than that of the original approximation.

2. The Method

We assume that $\mathcal{T}_h$ is a shape-regular triangulation Ω. Moreover, we define the following function spaces.

$$\begin{aligned}
W_h :=& \{w \in L^2(\Omega) : w|_K \in \mathcal{P}^k(K), \text{ for all } K \in \mathcal{T}_h\},\\
\boldsymbol{Q}_h :=& \{\boldsymbol{m} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{m}|_K \in \boldsymbol{\mathcal{P}}^k(K), \text{ for all } K \in \mathcal{T}_h\},\\
\boldsymbol{\Sigma}_h :=& \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}\,,\Omega) : \boldsymbol{v}|_K \in \boldsymbol{RT}^k(K) \text{ for all } K \in \mathcal{T}_h\},\\
\underline{\boldsymbol{Z}}_h :=& \{\underline{\boldsymbol{s}} \in \underline{\boldsymbol{H}}(\mathrm{div}\,,\Omega) : \underline{\boldsymbol{s}}|_K \in \underline{\boldsymbol{RT}}^k(K) + \hat{\underline{\boldsymbol{Z}}}^k(K) \text{ for all } K \in \mathcal{T}_h\},\\
\underline{\boldsymbol{A}}_h :=& \{\underline{\boldsymbol{\eta}} \in \underline{\boldsymbol{L}}^2(\Omega) : \underline{\boldsymbol{\eta}}|_K \in \underline{\boldsymbol{A}}^k(K), \text{ for all } K \in \mathcal{T}_h\}.
\end{aligned}$$

Here $\boldsymbol{L}^2(\Omega) = [L^2(\Omega)]^2$, and $\underline{\boldsymbol{L}}^2(\Omega) = [L^2(\Omega)]^{2\times 2}$. Moreover, $\underline{\boldsymbol{H}}(\mathrm{div}\,,\Omega)$ are 2×2 matrix-valued functions such that each row belongs to the space $\boldsymbol{H}(\mathrm{div}\,,\Omega)$. The space of polynomials of degree less than or equal to k is denoted by $\mathcal{P}^k(K)$ and $\boldsymbol{\mathcal{P}}^k(K) = [\mathcal{P}^k(K)]^2$ and $\underline{\boldsymbol{\mathcal{P}}}^k(K) = [\mathcal{P}^k(K)]^{2\times 2}$. The space $\boldsymbol{RT}^k(K) = \boldsymbol{\mathcal{P}}^k(K) + \mathcal{P}^k(K)\boldsymbol{x}$ is the Raviart-Thomas space of index k. Also, $\underline{\boldsymbol{A}}^k(K) := \{\underline{\boldsymbol{\eta}} : \underline{\boldsymbol{\eta}} + \underline{\boldsymbol{\eta}}^T = 0 \text{ and } \underline{\boldsymbol{\eta}} \in \underline{\boldsymbol{\mathcal{P}}}^k(K)\}$ and $\tilde{\underline{\boldsymbol{A}}}^k(K) := \{\underline{\boldsymbol{\eta}} \in \underline{\boldsymbol{A}}^k(K) : (\underline{\boldsymbol{\eta}}, \underline{\boldsymbol{v}})_K = 0 \text{ for all } \underline{\boldsymbol{v}} \in \underline{\boldsymbol{\mathcal{P}}}^{k-1}(K)\}$. Finally, for $k \geq 1$

$$\hat{\underline{\boldsymbol{Z}}}^k(K) = \mathbf{curl}\,(\mathrm{curl}\,(\tilde{\underline{\boldsymbol{A}}}^k(K))b_K),$$

where $b_K = \lambda_1\lambda_2\lambda_3$ is the bubble function of K with $\lambda_i's$ the barycentric coordinates of K. Here we used the following notation

$$\mathrm{curl}\,(\underline{\boldsymbol{\eta}}) := \begin{pmatrix} \partial_1\eta_{12} - \partial_2\eta_{11} \\ \partial_1\eta_{22} - \partial_2\eta_{21} \end{pmatrix} \quad \text{and} \quad \mathbf{curl}\,(\boldsymbol{w}) := \begin{pmatrix} \partial_2 w_1 & -\partial_1 w_1 \\ \partial_2 w_2 & -\partial_1 w_2 \end{pmatrix},$$

for a matrix $\underline{\boldsymbol{\eta}}$ and for a vector $\boldsymbol{w}$. Note that $\mathrm{curl}\,(\underline{\boldsymbol{\eta}})$ is a vector whereas $\mathbf{curl}\,(\boldsymbol{w})$ is a matrix.

For $k = 0$ we define

$$\hat{\underline{\boldsymbol{Z}}}(K) := \{\underline{\boldsymbol{s}} \in \underline{\boldsymbol{\mathcal{P}}}^1(K) : \underline{\boldsymbol{s}}\boldsymbol{n}\cdot\boldsymbol{t}|_F \text{ is constant on each edge } F \text{ of } K\},$$

where $\boldsymbol{t}$ is a unit tangent vector of F. We note that $\dim(\underline{\boldsymbol{RT}}^k(K) + \hat{\underline{\boldsymbol{Z}}}^k(K)) = \dim(\underline{\boldsymbol{RT}}^k(K)) + (k+1)$ for $k \geq 1$, but $\dim(\underline{\boldsymbol{RT}}^0(K) + \hat{\underline{\boldsymbol{Z}}}^0(K)) = \dim(\underline{\boldsymbol{RT}}^0(K)) + 3$ for $k = 0$. We further note that the pair of spaces $\boldsymbol{Q}_h \times \underline{\boldsymbol{A}}_h \times \underline{\boldsymbol{Z}}_h$ for $k \geq 1$ are exactly the ones used in [2, 24] for elasticity with weakly imposed symmetry. For $k = 0$ we use the reduced lowest order element $\boldsymbol{Q}_h \times \underline{\boldsymbol{A}}_h \times \underline{\boldsymbol{Z}}_h$ given in [10]. Finally, note that $W_h \times \boldsymbol{\Sigma}_h$ are the Raviart-Thomas spaces for Poisson's problem.

Now we can define our method.

It finds $(u_h, \boldsymbol{q}_h, \boldsymbol{r}_h, \underline{\boldsymbol{\rho}}_h, \underline{\boldsymbol{z}}_h, \boldsymbol{\sigma}_h) \in W_h \times \boldsymbol{Q}_h \times \boldsymbol{Q}_h \times \underline{\boldsymbol{A}}_h \times \underline{\boldsymbol{Z}}_h \times \boldsymbol{\Sigma}_h$ that satisfy

$$(\boldsymbol{q}_h, \boldsymbol{v}) + (u_h, \nabla \cdot \boldsymbol{v}) = 0, \tag{2.2a}$$

$$(\mathcal{A}\, \underline{\boldsymbol{z}}_h, \underline{\boldsymbol{s}}) + (\boldsymbol{r}_h, \nabla \cdot \underline{\boldsymbol{s}}) + (\underline{\boldsymbol{\rho}}_h, \underline{\boldsymbol{s}}) = 0, \tag{2.2b}$$

$$-(\boldsymbol{\sigma}_h, \boldsymbol{m}) + (\boldsymbol{m}, \nabla \cdot \underline{\boldsymbol{z}}_h) = 0, \tag{2.2c}$$

$$(\boldsymbol{r}_h - \boldsymbol{q}_h, \boldsymbol{d}) - \hat{t}^2(\boldsymbol{\sigma}_h, \boldsymbol{d}) = 0, \tag{2.2d}$$

$$(w, \nabla \cdot \boldsymbol{\sigma}_h) = (f, w), \tag{2.2e}$$

$$(\underline{\boldsymbol{z}}_h, \underline{\boldsymbol{\eta}}) = 0, \tag{2.2f}$$

for all $(w, \boldsymbol{m}, \boldsymbol{d}, \underline{\boldsymbol{\eta}}, \underline{\boldsymbol{s}}, \boldsymbol{v}) \in W_h \times \boldsymbol{Q}_h \times \boldsymbol{Q}_h \times \underline{\boldsymbol{A}}_h \times \underline{\boldsymbol{Z}}_h \times \boldsymbol{\Sigma}_h$.

For matrix-valued functions we used the notation

$$(\underline{\boldsymbol{z}}, \underline{\boldsymbol{s}}) := \sum_{K \in \mathcal{T}_h} (\underline{\boldsymbol{z}}, \underline{\boldsymbol{s}})_K, \text{ where } (\underline{\boldsymbol{z}}, \underline{\boldsymbol{s}})_K := \int_K \underline{\boldsymbol{z}}(\boldsymbol{x}) : \underline{\boldsymbol{s}}(\boldsymbol{x}) d\boldsymbol{x},$$

where $:$ is the Froebenius inner product. For vector-valued and scalar-valued functions we use similar notation.

We prove that the method is well defined, but first we state a standard result that can be found in [17].

Proposition 2.1. *If $\boldsymbol{v} \in \boldsymbol{\Sigma}_h$ and $\nabla \cdot \boldsymbol{v} = 0$ then $\boldsymbol{v} \in \boldsymbol{\Sigma}_h \cap \boldsymbol{Q}_h$.*

Moreover, we need the following proposition.

Proposition 2.2. *Given any $\boldsymbol{w}$ in $\boldsymbol{Q}_h$ and $\boldsymbol{\zeta}$ in $\underline{\boldsymbol{A}}_h$, there exists a $\underline{\boldsymbol{\tau}}$ in $\underline{\boldsymbol{Z}}_h$ satisfying*

$$\nabla \cdot \underline{\boldsymbol{\tau}} = \boldsymbol{w}, \tag{2.3a}$$

$$(\underline{\boldsymbol{\tau}}, \underline{\boldsymbol{\eta}}) = (\boldsymbol{\zeta}, \underline{\boldsymbol{\eta}}), \qquad \forall \underline{\boldsymbol{\eta}} \in \underline{\boldsymbol{A}}^h, \text{ and} \tag{2.3b}$$

$$\|\underline{\boldsymbol{\tau}}\|_{L^2(\Omega)} \leq C\big(\|\boldsymbol{w}\|_{L^2(\Omega)} + \|\boldsymbol{\zeta}\|_{L^2(\Omega)}\big), \tag{2.3c}$$

where C only depends on the shape regularity of $\mathcal{T}_h$.

For the proof of this proposition see [2, 24] for $k \geq 1$ and [10] for $k = 0$.

We can now prove that the method is well defined.

Theorem 2.3. *The mixed method* (2.2) *is well defined.*

Proof. Since (2.2) is a square linear system it is enough to prove uniqueness. To this end, we assume that $f \equiv 0$ and we define the norm

$$\|\underline{\boldsymbol{s}}\|^2_{L^2(\Omega;\mathcal{A})} := (\mathcal{A}\underline{\boldsymbol{s}}, \underline{\boldsymbol{s}}).$$

Note that by the fact that $f = 0$, (2.2e) and by Proposition 2.1 we have that

$$\nabla \cdot \boldsymbol{\sigma}_h \in \boldsymbol{\Sigma}_h \cap \boldsymbol{Q}_h. \tag{2.4}$$

Then, we see that

$$\begin{aligned}
\|\underline{\boldsymbol{z}}_h\|^2_{L^2(\Omega;\mathcal{A})} &= -(\boldsymbol{r}_h, \nabla \cdot \underline{\boldsymbol{z}}_h) - (\underline{\boldsymbol{\rho}}_h, \underline{\boldsymbol{z}}_h) && \text{by (2.2b)} \\
&= -(\boldsymbol{\sigma}_h, \boldsymbol{r}_h) && \text{by (2.2c) and (2.2f)} \\
&= -(\boldsymbol{\sigma}_h, \boldsymbol{q}_h) - (\boldsymbol{\sigma}_h, \boldsymbol{r}_h - \boldsymbol{q}_h) && \\
&= (u_h, \nabla \cdot \boldsymbol{\sigma}_h) - \hat{t}^2(\boldsymbol{\sigma}_h, \boldsymbol{\sigma}_h) && \text{by (2.2a), (2.4) and (2.2d)} \\
&= -\hat{t}^2(\boldsymbol{\sigma}_h, \boldsymbol{\sigma}_h). && \text{by (2.2e)}
\end{aligned}$$

This shows that $\underline{\boldsymbol{z}}_h = 0$ and $\boldsymbol{\sigma}_h = 0$. Moreover, (2.2d) shows that $\boldsymbol{r}_h = \boldsymbol{q}_h$.

From (2.2b) we get

$$(\boldsymbol{r}_h, \nabla \cdot \underline{\boldsymbol{s}}) + (\underline{\boldsymbol{\rho}}_h, \underline{\boldsymbol{s}}) = 0, \quad \text{for all } \underline{\boldsymbol{s}} \in \underline{\boldsymbol{Z}}_h .$$

Applying Proposition 2.2 we have that $\boldsymbol{r}_h = 0$ and $\underline{\boldsymbol{\rho}}_h = 0$. Hence, $\boldsymbol{q}_h = 0$. By (2.2a) we have

$$(u_h, \nabla \cdot \boldsymbol{v}) = 0, \quad \text{for all } \boldsymbol{v} \in \boldsymbol{\Sigma}_h.$$

Since the divergence operator is onto from $\boldsymbol{\Sigma}_h$ to W_h we have that $u_h = 0$. □

3. Hybrid Form

We introduce the hybrid form of the method. This will allow us to remove all the interior degrees of freedom and have a formulation for only the Lagrange multipliers that approximate u and $\boldsymbol{r}$ on the edges of the triangulation. However, the other variable can be recovered element-by-element once we have solved for the Lagrange multipliers. We follow closely our first paper [15] where we used the notation used in [23].

We need to define the following non-conforming versions of $\boldsymbol{\Sigma}_h$ and $\underline{\boldsymbol{Z}}_h$.

$$\begin{aligned}
\tilde{\boldsymbol{\Sigma}}_h :=& \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{v}|_K \in \boldsymbol{RT}^k(K) \text{ for all } K \in \mathcal{T}_h\},\\
\tilde{\underline{\boldsymbol{Z}}}_h :=& \{\underline{\boldsymbol{s}} \in \underline{\boldsymbol{L}}^2(\Omega) : \underline{\boldsymbol{s}}|_K \in \boldsymbol{RT}^k(K) + \hat{\underline{\boldsymbol{Z}}}^k(K) \text{ for all } K \in \mathcal{T}_h\}.
\end{aligned}$$

We also need to define the Lagrange multiplier spaces

$$M_h := \{\mu : \mu|_F \in \mathcal{P}^k(F) \text{ for all faces of } F \text{ of } \mathcal{T}_h, \mu = 0 \text{ on } \partial\Omega\}.$$

For $k \geq 1$ we define

$$\boldsymbol{M}_h := \{\boldsymbol{\mu} : \boldsymbol{\mu}|_F \in \boldsymbol{\mathcal{P}}^k(F) \text{ for all faces of } F \text{ of } \mathcal{T}_h, \boldsymbol{\mu} = 0 \text{ on } \partial\Omega\},$$

and for $k = 0$ we define

$$\boldsymbol{M}_h := \{\boldsymbol{\mu} : \boldsymbol{\mu}|_F \in \boldsymbol{\mathcal{P}}^1(F) \text{ and } \boldsymbol{\mu} \cdot \boldsymbol{t}|_F \text{ is constant on all faces of } F \text{ of } \mathcal{T}_h, \boldsymbol{\mu} = 0 \text{ on } \partial\Omega\},$$

where here $\boldsymbol{t}$ is a unit tangent vector to F.

The hybrid method finds $(u_h, \boldsymbol{q}_h, \boldsymbol{r}_h, \underline{\boldsymbol{\rho}}_h, \underline{\boldsymbol{z}}_h, \boldsymbol{\sigma}_h, \lambda_h, \boldsymbol{\alpha}_h) \in W_h \times \boldsymbol{Q}_h \times \boldsymbol{Q}_h \times \underline{\boldsymbol{A}}_h \times \tilde{\underline{\boldsymbol{Z}}}_h \times \tilde{\boldsymbol{\Sigma}}_h \times M_h \times \boldsymbol{M}_h$ that satisfy

$$(\boldsymbol{q}_h, \boldsymbol{v}) + (u_h, \nabla \cdot \boldsymbol{v}) - \langle \lambda_h, \boldsymbol{v} \cdot \boldsymbol{n} \rangle = 0 \tag{3.5a}$$

$$(\mathcal{A}\, \underline{\boldsymbol{z}}_h, \underline{\boldsymbol{s}}) + (\boldsymbol{r}_h, \nabla \cdot \underline{\boldsymbol{s}}) + (\underline{\boldsymbol{\rho}}_h, \underline{\boldsymbol{s}}) - \langle \boldsymbol{\alpha}_h, \underline{\boldsymbol{s}}\, \boldsymbol{n} \rangle = 0 \tag{3.5b}$$

$$-(\boldsymbol{\sigma}_h, \boldsymbol{m}) + (\boldsymbol{m}, \nabla \cdot \underline{\boldsymbol{z}}_h) = 0 \tag{3.5c}$$

$$(\boldsymbol{r}_h - \boldsymbol{q}_h, \boldsymbol{d}) - \hat{t}^2(\boldsymbol{\sigma}_h, \boldsymbol{d}) = 0 \tag{3.5d}$$

$$(w, \nabla \cdot \boldsymbol{\sigma}_h) = (f, w) \tag{3.5e}$$

$$(\underline{\boldsymbol{z}}_h, \underline{\boldsymbol{\eta}}) = 0, \tag{3.5f}$$

$$\langle \boldsymbol{\sigma}_h \cdot \boldsymbol{n}, \mu \rangle = 0, \tag{3.5g}$$

$$\langle \underline{\boldsymbol{z}}_h \boldsymbol{n}, \boldsymbol{\mu} \rangle = 0, \tag{3.5h}$$

for all $(w, \boldsymbol{m}, \boldsymbol{d}, \underline{\boldsymbol{\eta}}, \underline{\boldsymbol{s}}, \boldsymbol{v}, \mu, \boldsymbol{\mu}) \in W_h \times \boldsymbol{Q}_h \times \boldsymbol{Q}_h \times \underline{\boldsymbol{A}}_h \times \tilde{\underline{\boldsymbol{Z}}}_h \times \tilde{\boldsymbol{\Sigma}}_h \times M_h \times \boldsymbol{M}_h$.

Here we used the notation

$$\langle \mu, \lambda \rangle := \sum_{K \in \mathcal{T}_h} \int_{\partial K} \mu(s)\, \lambda(s)\, ds.$$

Note that equation (3.5g) implies that $\boldsymbol{\sigma}_h \cdot \boldsymbol{n}$ is single valued across edges. Similarly, $\underline{\boldsymbol{z}}_h \boldsymbol{n}$ is single-valued across edges by (3.5h). Note that by our definition of $\tilde{\underline{\boldsymbol{Z}}}_h$ for $k = 0$ we needed $\boldsymbol{M}_h$ to contain more than piecewise constants.

The following result easily follows.

Theorem 3.1. *The problem* (3.5) *is well defined. Moreover, let* $(u_h, \boldsymbol{q}_h, \boldsymbol{r}_h, \underline{\boldsymbol{\rho}}_h, \underline{\boldsymbol{z}}_h, \boldsymbol{\sigma}_h, \lambda_h, \boldsymbol{\alpha}_h) \in W_h \times \boldsymbol{Q}_h \times \boldsymbol{Q}_h \times \underline{\boldsymbol{A}}_h \times \tilde{\underline{\boldsymbol{Z}}}_h \times \tilde{\boldsymbol{\Sigma}}_h \times M_h \times \boldsymbol{M}_h$ *be the solution to* (3.5), *then* $(u_h, \boldsymbol{q}_h, \boldsymbol{r}_h, \underline{\boldsymbol{\rho}}_h, \underline{\boldsymbol{z}}_h, \boldsymbol{\sigma}_h)$ *is the solution to* (2.2).

The hybrid form will allow us to eliminate locally $(u_h, \boldsymbol{q}_h, \boldsymbol{r}_h, \underline{\boldsymbol{\rho}}_h, \underline{\boldsymbol{z}}_h, \boldsymbol{\sigma}_h)$ to get a final coupled system for $(\lambda_h, \boldsymbol{\alpha}_h)$. In order to describe the result we introduce local solvers. First for $m \in M_h$ let $(\mathsf{u}_1(m), \mathbf{Q}_1(m), \mathbf{R}_1(m), \underline{\mathbf{L}}_1(m), \underline{\mathbf{Z}}_1(m), \mathbf{S}_1(m)) \in W_h \times \boldsymbol{Q}_h \times \boldsymbol{Q}_h \times \underline{\boldsymbol{A}}_h \times \tilde{\underline{\boldsymbol{Z}}}_h \times \tilde{\boldsymbol{\Sigma}}_h$ solve

$$(\mathbf{Q}_1(m), \boldsymbol{v}) + (\mathsf{u}_1(m), \nabla \cdot \boldsymbol{v}) = \langle m, \boldsymbol{v} \cdot \boldsymbol{n} \rangle, \tag{3.6a}$$

$$(\underline{\mathbf{Z}}_1(m), \underline{\boldsymbol{s}}) + (\mathbf{R}_1(m), \nabla \cdot \underline{\boldsymbol{s}}) + (\underline{\mathbf{L}}_1(m), \underline{\boldsymbol{s}}) = 0, \tag{3.6b}$$

$$-(\mathbf{S}_1(m), \boldsymbol{m}) + (\boldsymbol{m}, \nabla \cdot \underline{\mathbf{Z}}_1(m)) = 0, \tag{3.6c}$$

$$(\mathbf{R}_1(m) - \mathbf{Q}_1(m), \boldsymbol{d}) - \hat{t}^2 (\mathbf{S}_1(m), \boldsymbol{d}) = 0 \tag{3.6d}$$

$$(w, \nabla \cdot \mathbf{S}_1(m)) = 0, \tag{3.6e}$$

$$(\underline{\mathbf{Z}}_1(m), \underline{\boldsymbol{\eta}}) = 0, \tag{3.6f}$$

for all $(w, \boldsymbol{m}, \boldsymbol{d}, \underline{\boldsymbol{\eta}}, \underline{\boldsymbol{s}}, \boldsymbol{v}) \in W_h \times \boldsymbol{Q}_h \times \boldsymbol{Q}_h \times \underline{\boldsymbol{A}}_h \times \tilde{\underline{\boldsymbol{Z}}}_h \times \tilde{\boldsymbol{\Sigma}}_h$.

Similarly, for $\boldsymbol{\mu} \in \boldsymbol{M}_h$ let $(\mathsf{u}_2(m), \mathbf{Q}_2(m), \mathbf{R}_2(m), \underline{\mathbf{L}}_2(m), \underline{\mathbf{Z}}_2(m), \mathbf{S}_2(m)) \in W_h \times \boldsymbol{Q}_h \times \boldsymbol{Q}_h \times \underline{\boldsymbol{A}}_h \times \tilde{\underline{\boldsymbol{Z}}}_h \times \tilde{\boldsymbol{\Sigma}}_h$ solve

$$(\mathbf{Q}_2(\boldsymbol{\mu}), \boldsymbol{v}) + (\mathsf{u}_2(\boldsymbol{\mu}), \nabla \cdot \boldsymbol{v}) = 0, \tag{3.7a}$$

$$(\underline{\mathbf{Z}}_2(\boldsymbol{\mu}), \underline{\boldsymbol{s}}) + (\mathbf{R}_2(\boldsymbol{\mu}), \nabla \cdot \underline{\boldsymbol{s}}) + (\underline{\mathbf{L}}_2(\boldsymbol{\mu}), \underline{\boldsymbol{s}}) = \langle \boldsymbol{\mu}, \underline{\boldsymbol{s}}\, \boldsymbol{n} \rangle, \tag{3.7b}$$

$$-(\mathbf{S}_2(\boldsymbol{\mu}), \boldsymbol{m}) + (\boldsymbol{m}, \nabla \cdot \underline{\mathbf{Z}}_2(\boldsymbol{\mu})) = 0, \tag{3.7c}$$

$$(\mathbf{R}_2(\boldsymbol{\mu}) - \mathbf{Q}_2(\boldsymbol{\mu}), \boldsymbol{d}) - \hat{t}^2 (\mathbf{S}_2(m), \boldsymbol{d}) = 0, \tag{3.7d}$$

$$(w, \nabla \cdot \mathbf{S}_2(\boldsymbol{\mu})) = 0, \tag{3.7e}$$

$$(\underline{\mathbf{Z}}_2(\boldsymbol{\mu}), \underline{\boldsymbol{\eta}}) = 0, \tag{3.7f}$$

for all $(w, \boldsymbol{m}, \boldsymbol{d}, \underline{\boldsymbol{\eta}}, \underline{\boldsymbol{s}}, \boldsymbol{v}) \in W_h \times \boldsymbol{Q}_h \times \boldsymbol{Q}_h \times \underline{\boldsymbol{A}}_h \times \tilde{\underline{\boldsymbol{Z}}}_h \times \tilde{\boldsymbol{\Sigma}}_h$.

Finally, let $(\mathsf{u}_3(f), \mathbf{Q}_3(f), \mathbf{R}_3(f), \underline{\mathbf{L}}_3(f), \underline{\mathbf{Z}}_3(f), \mathbf{S}_3(f)) \in W_h \times \boldsymbol{Q}_h \times \boldsymbol{Q}_h \times \underline{\boldsymbol{A}}_h \times \tilde{\underline{\boldsymbol{Z}}}_h \times \tilde{\boldsymbol{\Sigma}}_h$ solve

$$
\begin{aligned}
(\mathbf{Q}_3(f), \boldsymbol{v}) + (\mathsf{u}_3(f), \nabla \cdot \boldsymbol{v}) =& 0, && (3.8a)\\
(\underline{\mathbf{Z}}_3(f), \underline{\boldsymbol{s}}) + (\mathbf{R}_3(f), \nabla \cdot \underline{\boldsymbol{s}}) + (\underline{\mathbf{L}}_3(f), \underline{\boldsymbol{s}}) =& 0, && (3.8b)\\
-(\mathbf{S}_3(f), \boldsymbol{m}) + (\boldsymbol{m}, \nabla \cdot \underline{\mathbf{Z}}_3(f)) =& 0, && (3.8c)\\
(\mathbf{R}_3(f) - \mathbf{Q}_3(f), \boldsymbol{d}) - \hat{t}^2(\mathbf{S}_3(f), \boldsymbol{d}) =& 0, && (3.8d)\\
(w, \nabla \cdot \mathbf{S}_3(f)) =& (f, w), && (3.8e)\\
(\underline{\mathbf{Z}}_3(f), \underline{\boldsymbol{\eta}}) =& 0, && (3.8f)
\end{aligned}
$$

for all $(w, \boldsymbol{m}, \boldsymbol{d}, \underline{\boldsymbol{\eta}}, \underline{\boldsymbol{s}}, \boldsymbol{v}) \in W_h \times \boldsymbol{Q}_h \times \boldsymbol{Q}_h \times \underline{\boldsymbol{A}}_h \times \tilde{\underline{\boldsymbol{Z}}}_h \times \tilde{\boldsymbol{\Sigma}}_h$.

Now that we have the local solvers we define three bilinear forms. For $m, \mu \in M_h$ and $\boldsymbol{\mu}, \boldsymbol{l} \in \boldsymbol{M}_h$ define

$$
\begin{aligned}
a(m, \mu) :=& (\mathcal{A}\, \underline{\mathbf{Z}}_1(m), \underline{\mathbf{Z}}_1(\mu)) + \hat{t}^2(\mathbf{S}_1(m), \mathbf{S}_1(\mu)),\\
c(\boldsymbol{\mu}, \boldsymbol{l}) :=& (\mathcal{A}\, \underline{\mathbf{Z}}_2(\boldsymbol{\mu}), \underline{\mathbf{Z}}_2(\boldsymbol{l})) + \hat{t}^2(\mathbf{S}_2(\boldsymbol{\mu}), \mathbf{S}_2(\boldsymbol{l})),\\
b(m, \boldsymbol{\mu}) :=& (\mathcal{A}\, \underline{\mathbf{Z}}_1(m), \underline{\mathbf{Z}}_2(\boldsymbol{\mu})) + \hat{t}^2(\mathbf{S}_1(m), \mathbf{S}_2(\boldsymbol{\mu})).
\end{aligned}
$$

The following problem allows us to find the Lagrange multipliers λ_h and $\boldsymbol{\alpha}_h$.

Let $(\lambda_h, \boldsymbol{\alpha}_h) \in M_h \times \boldsymbol{M}_h$ solve

$$
\begin{aligned}
a(\lambda_h, m) + b(m, \boldsymbol{\alpha}_h) =& (f, \mathsf{u}_1(m)), && (3.9a)\\
b(\lambda_h, \boldsymbol{\mu}) + c(\boldsymbol{\alpha}_h, \boldsymbol{\mu}) =& (f, \mathsf{u}_2(\boldsymbol{\mu})), && (3.9b)
\end{aligned}
$$

for all $(m, \boldsymbol{\mu}) \in M_h \times \boldsymbol{M}_h$.

It is clear that $\dim(M_h) = (k+1)\, N_e$ and $\dim(\boldsymbol{M}_h) = 2(k+1)\, N_e$ for $k \geq 1$ while $\dim(\boldsymbol{M}_h) = 3\, N_e$ for $k = 0$ where N_e denotes the number interior edges. Hence, (3.9) gives rise to a linear system with $3(k+1)\, N_e$ unknowns for $k \geq 1$ and $4\, N_e$ unknowns for $k = 0$.

Now we arrive at the main result of this section. The proof is found in the appendix.

Theorem 3.2. *The problem* (3.9) *is well defined. Moreover, if* $(\lambda_h, \boldsymbol{\alpha}_h) \in M_h \times \boldsymbol{M}_h$ *solves* (3.9) *and if we set*

$$
\begin{aligned}
u_h =& \mathsf{u}_1(\lambda_h) + \mathsf{u}_2(\boldsymbol{\alpha}_h) + \mathsf{u}_3(f), && (3.10a)\\
\boldsymbol{q}_h =& \mathbf{Q}_1(\lambda_h) + \mathbf{Q}_2(\boldsymbol{\alpha}_h) + \mathbf{Q}_3(f), && (3.10b)\\
\boldsymbol{r}_h =& \mathbf{R}_1(\lambda_h) + \mathbf{R}_2(\boldsymbol{\alpha}_h) + \mathbf{R}_3(f), && (3.10c)\\
\underline{\boldsymbol{\rho}}_h =& \underline{\mathbf{L}}_1(\lambda_h) + \underline{\mathbf{L}}_2(\boldsymbol{\alpha}_h) + \underline{\mathbf{L}}_3(f), && (3.10d)\\
\underline{\boldsymbol{z}}_h =& \underline{\mathbf{Z}}_1(\lambda_h) + \underline{\mathbf{Z}}_2(\boldsymbol{\alpha}_h) + \underline{\mathbf{Z}}_3(f), && (3.10e)\\
\boldsymbol{\sigma}_h =& \mathbf{S}_1(\lambda_h) + \mathbf{S}_2(\boldsymbol{\alpha}_h) + \mathbf{S}_3(f), && (3.10f)
\end{aligned}
$$

then $(u_h, \boldsymbol{q}_h, \boldsymbol{r}_h, \underline{\boldsymbol{\rho}}_h, \underline{\boldsymbol{z}}_h, \boldsymbol{\sigma}_h, \lambda_h, \boldsymbol{\alpha}_h) \in W_h \times \boldsymbol{Q}_h \times \boldsymbol{Q}_h \times \underline{\boldsymbol{A}}_h \times \tilde{\underline{\boldsymbol{Z}}}_h \times \tilde{\boldsymbol{\Sigma}}_h \times M_h \times \boldsymbol{M}_h$ *solves* (3.5)*.*

What this result says is that in order to solve our mixed method (2.2) we need only to solve the problem (3.9) for λ_h and $\boldsymbol{\alpha}_h$. Then, we can recover all the variables $(u_h, \boldsymbol{q}_h, \boldsymbol{r}_h, \underline{\boldsymbol{\rho}}_h, \underline{\boldsymbol{z}}_h, \boldsymbol{\sigma}_h)$ element-by-element which can be done in parallel.

4. Error Estimates

In this section we prove error estimates for all the variables. We start by writing the error equations

$$(\boldsymbol{q}-\boldsymbol{q}_h,\boldsymbol{v})+(u-u_h,\nabla\cdot\boldsymbol{v})=0, \tag{4.11a}$$
$$(\mathcal{A}(\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h),\underline{\boldsymbol{s}})+(\boldsymbol{r}-\boldsymbol{r}_h,\nabla\cdot\underline{\boldsymbol{s}})+(\underline{\boldsymbol{\rho}}-\underline{\boldsymbol{\rho}}_h,\underline{\boldsymbol{s}})=0, \tag{4.11b}$$
$$-(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\boldsymbol{m})+(\boldsymbol{m},\nabla\cdot(\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h))=0, \tag{4.11c}$$
$$(\boldsymbol{r}-\boldsymbol{q}-(\boldsymbol{r}_h-\boldsymbol{q}_h),\boldsymbol{d})-\hat{t}^2(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\boldsymbol{d})=0, \tag{4.11d}$$
$$(w,\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h))=0, \tag{4.11e}$$
$$(\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h,\underline{\boldsymbol{\eta}})=0, \tag{4.11f}$$

for all $(w,\boldsymbol{m},\boldsymbol{d},\underline{\boldsymbol{\eta}},\boldsymbol{v},\underline{\boldsymbol{s}})\in W_h\times\boldsymbol{Q}_h\times\boldsymbol{Q}_h\times\underline{\boldsymbol{A}}_h\times\boldsymbol{\Sigma}_h\times\underline{\boldsymbol{Z}}_h$.

We will use the Raviart-Thomas projection (see [36, 35]) $\boldsymbol{\Pi}:\boldsymbol{H}(\text{div};\Omega)\cap\boldsymbol{L}^p(\Omega)\to\boldsymbol{\Sigma}_h$ for some $p>2$ which satisfies the commutative property

$$\nabla\cdot(\boldsymbol{\Pi}\boldsymbol{v})=P\,\nabla\cdot\boldsymbol{v}, \tag{4.12}$$

where P is the L^2-projection onto W_h.

Moreover, the following approximation property holds

$$\|\boldsymbol{v}-\boldsymbol{\Pi}\boldsymbol{v}\|_{L^2(\Omega)}\le h^r\|\boldsymbol{v}\|_{H^r(\Omega)}, \tag{4.13}$$

for $1\le r\le k+1$.

We let $\underline{\boldsymbol{\Pi}}$ denote the matrix version of $\boldsymbol{\Pi}$ where $\boldsymbol{\Pi}$ acts row-wise. Moreover, we let $\boldsymbol{P}$ be the L^2-projection onto $\boldsymbol{Q}_h$ and $\underline{\boldsymbol{P}}$ is the L^2-projection onto $\underline{\boldsymbol{A}}_h$.

Since $\boldsymbol{\Pi}$ is not L^2-stable and derivatives of $\boldsymbol{\sigma}$ may not be bounded independent of t, we will use the recently introduced [19] smoothed projection $\boldsymbol{\Pi}^S:\boldsymbol{L}^2(\Omega)\to\boldsymbol{\Sigma}_h$; see also [11, 37, 38]. It satisfies the commutative property

$$\nabla\cdot(\boldsymbol{\Pi}^S\boldsymbol{v})=P^S\,\nabla\cdot\boldsymbol{v}, \tag{4.14}$$

where P^S is a projection onto W_h. It is important to note that $P^S\neq P$. These projections have the following invariance property $P^S w=w$ for all $w\in W_h$ and $\boldsymbol{\Pi}^S\boldsymbol{v}=\boldsymbol{v}$ for all $\boldsymbol{v}\in\boldsymbol{\Sigma}_h$. Finally, the following approximation properties hold for $\boldsymbol{v}\in\boldsymbol{H}^r(\Omega)$ and $w\in H^r(\Omega)$

$$\|\boldsymbol{v}-\boldsymbol{\Pi}^S\boldsymbol{v}\|_{L^2(\Omega)}\le C\,h^r\|\boldsymbol{v}\|_{H^r(\Omega)}, \tag{4.15}$$

and

$$\|w-P^S w\|_{L^2(\Omega)}\le C\,h^r\|w\|_{H^r(\Omega)},$$

for $0\le r\le k+1$.

The projection $\boldsymbol{\Pi}^S$, in contrast to the Raviart-Thomas projection $\boldsymbol{\Pi}$, is defined for functions in $\boldsymbol{L}^2(\Omega)$. However, $\boldsymbol{\Pi}^S$ is no longer defined locally on each element.

Before proving estimates for $\underline{\boldsymbol{z}}$ and $\underline{\boldsymbol{\rho}}$ we first need to prove some important lemmas. We start by proving an estimate for $\underline{\boldsymbol{\rho}}-\underline{\boldsymbol{\rho}}_h$ in terms of $\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h$.

Lemma 4.1. *We have*

$$\|\underline{\boldsymbol{\rho}}-\underline{\boldsymbol{\rho}}_h\|_{L^2(\Omega)}\le C(\|\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h\|_{L^2(\Omega)}+\|\underline{\boldsymbol{P}}\underline{\boldsymbol{\rho}}-\underline{\boldsymbol{\rho}}\|_{L^2(\Omega)}). \tag{4.16}$$

Proof. Let $\underline{\boldsymbol{\tau}}$ be from Proposition 2.2 with $\boldsymbol{w} = 0$ and $\underline{\boldsymbol{\zeta}} = \underline{\boldsymbol{P}}\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}_h$.

$$\begin{aligned}\|\underline{\boldsymbol{P}}\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}_h\|^2_{L^2(\Omega)} &= (\underline{\boldsymbol{P}}\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}_h, \underline{\boldsymbol{P}}\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}_h)\\ &= (\underline{\boldsymbol{P}}\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}_h, \underline{\boldsymbol{\tau}}) && \text{by (2.3b)}\\ &= (\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}_h, \underline{\boldsymbol{\tau}}) + (\underline{\boldsymbol{P}}\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}, \underline{\boldsymbol{\tau}})\\ &= -(\mathcal{A}(\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h), \underline{\boldsymbol{\tau}}) + (\underline{\boldsymbol{P}}\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}, \underline{\boldsymbol{\tau}}). && \text{by (2.3a) and (4.11b)}\end{aligned}$$

The result now follows from (2.3c), the fact that $\mathcal{A}$ is bounded and the triangle inequality. □

Since we now have a bound for $\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}_h$ in terms of $(\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h)$ we will not need to use a projection onto the full space of $\underline{\boldsymbol{Z}}_h$, but only to the Raviart-Thomas part of that space. This is the approach used in [32] for linear elasicity.

We now prove error estimates for the variable $\boldsymbol{r}$ in terms of the errors for $\underline{\boldsymbol{z}}$ and $\underline{\boldsymbol{\rho}}$.

The first result bounds the jumps of $\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h$.

Lemma 4.2. *Let F be the common edge of two elements $K, K' \in \mathcal{T}_h$. Then,*

$$\|(\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h)|_K - (\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h)|_{K'}\|_{L^2(F)} \le C\, h_F^{1/2}(\|\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h\|_{L^2(K\cup K')} + \|\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}_h\|_{L^2(K\cup K')}), \tag{4.17}$$

where h_F is the length of F. Moreover, if F is an edge of $K \in \mathcal{T}_h$ and F belongs to the boundary $\partial\Omega$ then

$$\|\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h\|_{L^2(F)} \le C\, h_F^{1/2}(\|\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h\|_{L^2(K)} + \|\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}_h\|_{L^2(K)}). \tag{4.18}$$

Proof. We only prove (4.17). In order to do so, set

$$\boldsymbol{\psi} = (\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h)|_K - (\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h)|_{K'}. \tag{4.19}$$

We will also need to define $\underline{\boldsymbol{s}} \in \underline{\boldsymbol{Z}}_h$ in the following way: First, $\underline{\boldsymbol{s}}|_K \in \underline{\boldsymbol{RT}}^k(K)$ solves

$$(\underline{\boldsymbol{s}}, \underline{\boldsymbol{v}})_K = 0 \qquad \text{for all } \underline{\boldsymbol{v}} \in \underline{\boldsymbol{\mathcal{P}}}^{k-1}(K), \tag{4.20a}$$

$$\langle \underline{\boldsymbol{s}}\boldsymbol{n}_K, \boldsymbol{\mu}\rangle_F = \langle \boldsymbol{\psi}, \boldsymbol{\mu}\rangle_F \quad \text{for all } \boldsymbol{\mu} \in \boldsymbol{\mathcal{P}}^k(F), \tag{4.20b}$$

$$\langle \underline{\boldsymbol{s}}\boldsymbol{n}_K, \boldsymbol{\mu}\rangle_G = 0, \qquad \text{for all } \boldsymbol{\mu} \in \boldsymbol{\mathcal{P}}^k(G), \text{ for all edges } G \text{ of } K \text{ and } G \ne F, \tag{4.20c}$$

where here $\boldsymbol{n}_K$ is the outward unit normal to K. Here $\underline{\boldsymbol{RT}}^k(K)$ is the set of matrix-valued functions such that each row belongs to $\boldsymbol{RT}^k(K)$.

Then, define $\underline{\boldsymbol{s}}|_{K'} \in \underline{\boldsymbol{RT}}^k(K')$ as follows

$$(\underline{\boldsymbol{s}}, \underline{\boldsymbol{v}})_{K'} = 0 \qquad \text{for all } \underline{\boldsymbol{v}} \in \underline{\boldsymbol{\mathcal{P}}}^{k-1}(K'), \tag{4.21a}$$

$$\langle \underline{\boldsymbol{s}}\boldsymbol{n}_{K'}, \boldsymbol{\mu}\rangle_F = -\langle \boldsymbol{\psi}, \boldsymbol{\mu}\rangle_F \quad \text{for all } \boldsymbol{\mu} \in \boldsymbol{\mathcal{P}}^k(F), \tag{4.21b}$$

$$\langle \underline{\boldsymbol{s}}\boldsymbol{n}_{K'}, \boldsymbol{\mu}\rangle_G = 0 \qquad \text{for all } \boldsymbol{\mu} \in \boldsymbol{\mathcal{P}}^k(G), \text{ for all edges } G \text{ of } K' \text{ and } G \ne F. \tag{4.21c}$$

Finally, set

$$\underline{\boldsymbol{s}}|_{\Omega\setminus K\cup K'} \equiv 0. \tag{4.22}$$

A standard scaling argument gives

$$\|\underline{\boldsymbol{s}}\|_{L^2(K\cup K')} \le C\, h_F^{1/2}\|\boldsymbol{\psi}\|_{L^2(F)}. \tag{4.23}$$

We then obtain

$$\begin{aligned}
\|\boldsymbol{\psi}\|_{L^2(F)}^2 =&\langle \boldsymbol{\psi}, \boldsymbol{\psi}\rangle_F \\
=&\langle \boldsymbol{\psi}, \underline{\boldsymbol{s}}\boldsymbol{n}_K\rangle_F && \text{by (4.20b)}\\
=&\langle (\boldsymbol{Pr}-\boldsymbol{r}_h)|_K, \underline{\boldsymbol{s}}\boldsymbol{n}_K\rangle_F + \langle (\boldsymbol{Pr}-\boldsymbol{r}_h)|_{K'}, \underline{\boldsymbol{s}}\boldsymbol{n}_{K'}\rangle_F \\
=&\int_{\partial K} (\boldsymbol{Pr}-\boldsymbol{r}_h)\cdot \underline{\boldsymbol{s}}\boldsymbol{n}_K + \int_{\partial K'} (\boldsymbol{Pr}-\boldsymbol{r}_h)\cdot \underline{\boldsymbol{s}}\boldsymbol{n}_{K'} && \text{by (4.20c),(4.21c)}\\
=&(\boldsymbol{Pr}-\boldsymbol{r}_h, \nabla\cdot\underline{\boldsymbol{s}})_K + (\boldsymbol{Pr}-\boldsymbol{r}_h, \nabla\cdot\underline{\boldsymbol{s}})_{K'} && \text{by (4.20a), (4.21a)}\\
=&(\boldsymbol{Pr}-\boldsymbol{r}_h, \nabla\cdot\underline{\boldsymbol{s}}) && \text{by (4.22)}\\
=&-(\mathcal{A}(\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h), \underline{\boldsymbol{s}}) - (\underline{\boldsymbol{\rho}}-\underline{\boldsymbol{\rho}}_h, \underline{\boldsymbol{s}}). && \text{by (4.11b)}
\end{aligned}$$

Therefore,

$$\|\boldsymbol{\psi}\|_{L^2(F)}^2 \leq (\|\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h\|_{L^2(K\cup K')} + \|\underline{\boldsymbol{\rho}}-\underline{\boldsymbol{\rho}}_h\|_{L^2(K\cup K')})\,\|\underline{\boldsymbol{s}}\|_{L^2(K\cup K')}.$$

The result now follows if we apply (4.23). □

The next result controls the piecewise gradient of $\boldsymbol{Pr}-\boldsymbol{r}_h$.

Lemma 4.3. *For every $K \in \mathcal{T}_h$ the following estimate holds*

$$\|\nabla(\boldsymbol{Pr}-\boldsymbol{r}_h)\|_{L^2(K)} \leq C(\|\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h\|_{L^2(K)} + \|\underline{\boldsymbol{\rho}}-\underline{\boldsymbol{\rho}}_h\|_{L^2(K)}).$$

Proof. Define $\underline{\boldsymbol{s}}|_K \in \underline{\boldsymbol{RT}}^k(K)$ as the solution to

$$(\underline{\boldsymbol{s}}, \underline{\boldsymbol{v}})_K = (\nabla(\boldsymbol{Pr}-\boldsymbol{r}_h), \underline{\boldsymbol{v}})_K \quad \text{for all } \underline{\boldsymbol{v}} \in \underline{\boldsymbol{\mathcal{P}}}^{k-1}(K), \tag{4.24a}$$

$$\langle \underline{\boldsymbol{s}}\boldsymbol{n}_K, \boldsymbol{\mu}\rangle_F = 0 \quad \text{for all } \boldsymbol{\mu} \in \boldsymbol{\mathcal{P}}^k(F), \text{ for all edges } F \text{ of } K. \tag{4.24b}$$

Also, set

$$\underline{\boldsymbol{s}}|_{\Omega\setminus K} \equiv 0. \tag{4.25}$$

Clearly, in this way $\underline{\boldsymbol{s}} \in \underline{\boldsymbol{Z}}_h$.

$$\begin{aligned}
\|\nabla(\boldsymbol{Pr}-\boldsymbol{r}_h)\|_{L^2(K)}^2 =&(\nabla(\boldsymbol{Pr}-\boldsymbol{r}_h), \underline{\boldsymbol{s}}) && \text{by (4.24a) and (4.25)}\\
=&-(\boldsymbol{Pr}-\boldsymbol{r}_h, \nabla\cdot\underline{\boldsymbol{s}}) && \text{by (4.24b) and integration by parts}\\
=&(\mathcal{A}(\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h), \underline{\boldsymbol{s}}) + (\underline{\boldsymbol{\rho}}-\underline{\boldsymbol{\rho}}_h, \underline{\boldsymbol{s}}). && \text{by (4.11b)}
\end{aligned}$$

Therefore,

$$\|\nabla(\boldsymbol{Pr}-\boldsymbol{r}_h)\|_{L^2(K)}^2 \leq C(\|\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h\|_{L^2(K)} + \|\underline{\boldsymbol{\rho}}-\underline{\boldsymbol{\rho}}_h\|_{L^2(K)})\|\underline{\boldsymbol{s}}\|_{L^2(K)}.$$

The result now follows from $\|\underline{\boldsymbol{s}}\|_{L^2(K)} \leq C\,\|\nabla(\boldsymbol{Pr}-\boldsymbol{r}_h)\|_{L^2(K)}$ which in turn follows from a standard scaling argument. □

We will also need the following trivial bound.

Lemma 4.4. *We have*

$$\|\boldsymbol{Pr}-\boldsymbol{r}_h\|_{L^2(\Omega)} \leq C(\|\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h\|_{L^2(\Omega)} + \|\underline{\boldsymbol{\rho}}-\underline{\boldsymbol{\rho}}_h\|_{L^2(\Omega)}). \tag{4.26}$$

Proof. There exists $\underline{\boldsymbol{s}} \in H^1(\Omega)$ that satisfies

$$\nabla \cdot \underline{\boldsymbol{s}} = \boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h \qquad \text{on } \Omega, \tag{4.27}$$

with

$$\|\underline{\boldsymbol{s}}\|_{H^1(\Omega)} \leq C\, \|\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h\|_{L^2(\Omega)}. \tag{4.28}$$

Hence, using

$$\begin{aligned} \|\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h\|^2_{L^2(\Omega)} =& (\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h, \nabla \cdot \underline{\boldsymbol{s}}) && \text{by (4.27)} \\ =& (\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h, \nabla \cdot \underline{\boldsymbol{\Pi}}\underline{\boldsymbol{s}}) && \text{by (4.12)} \\ =& -(\mathcal{A}(\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h), \underline{\boldsymbol{\Pi}}\underline{\boldsymbol{s}}) - (\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}_h, \underline{\boldsymbol{\Pi}}\underline{\boldsymbol{s}}) && \text{by (4.11b)} \end{aligned}$$

The result now follows after we use (4.13) with $r = 1$ and (4.28).

□

We also need the following Helmholtz decomposition; see for example [33].

Proposition 4.5. *Let $\boldsymbol{v} \in \boldsymbol{H}(div, \Omega)$. Then, there exists $\phi \in H^1_0(\Omega)$ and $p \in H^1(\Omega)$ such that*

$$\boldsymbol{v} = \nabla\phi + \mathbf{curl}\; p,$$

with

$$\|\phi\|_{H^1(\Omega)} \leq \|\nabla \cdot \boldsymbol{v}\|_{H^{-1}(\Omega)}, \tag{4.29}$$

and

$$\|p\|_{H^1(\Omega)} \leq C\, \|\boldsymbol{v}\|_{L^2(\Omega)}. \tag{4.30}$$

In fact, in the proposition above $\phi \in H^1_0(\Omega)$ solves

$$\triangle\phi = \nabla \cdot \boldsymbol{v} \quad \text{ on } \Omega.$$

From this equation we clearly see that the estimate (4.29) follows.

We will need to define an auxiliary function. By the above lemma there exist $\theta \in H^1_0(\Omega)$ and $q \in H^1(\Omega)$ such that

$$\boldsymbol{\sigma} = \nabla\theta + \mathbf{curl}\; q,$$

where $\triangle\theta = \nabla \cdot \boldsymbol{\sigma} = f$.

We define the auxiliary function $\tilde{\boldsymbol{\sigma}}$ as

$$\tilde{\boldsymbol{\sigma}} = \nabla\tilde{\theta} + \mathbf{curl}\; q,$$

where $\tilde{\theta} \in H^1_0(\Omega)$ is defined as the unique solution to $\triangle\tilde{\theta} = P\,f$. Note that as a consequence $\nabla \cdot \tilde{\boldsymbol{\sigma}} = P\,f$. Furthermore, we have

$$\|\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}\|_{L^2(\Omega)} = \|\nabla\theta - \nabla\tilde{\theta}\|_{L^2(\Omega)} \leq C\, \|f - Pf\|_{H^{-1}(\Omega)}, \tag{4.31}$$

where we used that $\triangle(\theta - \tilde{\theta}) = f - Pf$ and an energy argument.

Finally, we prove a simple but important lemma.

Lemma 4.6. *We have,*

$$\nabla \cdot (\mathbf{\Pi}^S\tilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}_h) = 0, \tag{4.32}$$

and

$$\mathbf{\Pi}^S\tilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}_h \in \mathbf{\Sigma}_h \cap \boldsymbol{Q}_h. \tag{4.33}$$

Proof. Using (2.2e) we have that $\nabla \cdot \boldsymbol{\sigma}_h = P\, f$. Using the commutative property (4.14) we have

$$\nabla \cdot \boldsymbol{\Pi}^S \tilde{\boldsymbol{\sigma}} = P^S \nabla \cdot \tilde{\boldsymbol{\sigma}} = P^S\, P\, f = P\, f,$$

where in the last equation we used that $P\, f \in W_h$. This proves (4.32), and (4.33) follows form Proposition 2.1. □

4.1. **Error estimates for $\underline{z}$ and $\underline{\rho}$.** In this section we prove optimal error estimates for $\underline{z}$ and $\underline{\rho}$ independent of the plate thickness t. We start this section by stating the main theorem of this section and a simple corollary.

Theorem 4.7. *We have*

$$\begin{aligned}&\|\underline{z} - \underline{z}_h\|_{L^2(\Omega)} + \frac{1}{\hat{t}} \|(\boldsymbol{P r} - \boldsymbol{r}_h) - (\boldsymbol{P q} - \boldsymbol{q}_h)\|_{L^2(\Omega)} + \|\underline{\rho} - \underline{\rho}_h\|_{L^2(\Omega)}\\ \leq& C\, (\|\underline{\boldsymbol{\Pi}} \underline{z} - \underline{z}\|_{L^2(\Omega)} + \|\underline{\rho} - \underline{\boldsymbol{P}} \underline{\rho}\|_{L^2(\Omega)} + (h + \hat{t})\, \|\boldsymbol{\sigma} - \boldsymbol{\Pi}^S \boldsymbol{\sigma}\|_{L^2(\Omega)} + \|f - P\, f\|_{H^{-1}(\Omega)}).\end{aligned} \tag{4.34}$$

The following corollary easily follows from this theorem.

Corollary 4.8. *For any $1 \leq r_0 \leq k+1$ and $0 \leq r_1, r_2, r_3 \leq k+1$ we have*

$$\begin{aligned}&\|\underline{z} - \underline{z}_h\|_{L^2(\Omega)} + \frac{1}{\hat{t}} \|(\boldsymbol{P r} - \boldsymbol{r}_h) - (\boldsymbol{P q} - \boldsymbol{q}_h)\|_{L^2(\Omega)} + \|\underline{\rho} - \underline{\rho}_h\|_{L^2(\Omega)}\\ \leq& C\, (h^{r_0}\, \|\boldsymbol{r}\|_{H^{1+r_0}(\Omega)} + \hat{t} h^{r_1}\, \|\boldsymbol{\sigma}\|_{H^{r_1}(\Omega)} + h^{1+r_2}\, \|\boldsymbol{\sigma}\|_{H^{r_2}(\Omega)} + h^{1+r_3}\, \|f\|_{H^{r_3}(\Omega)}).\end{aligned}$$

Here we used that $\|\underline{z}\|_{H^{r_0}(\Omega)} + \|\underline{\rho}\|_{H^{r_0}(\Omega)} \leq C\, \|\boldsymbol{r}\|_{H^{1+r_0}(\Omega)}$. Note that Corollary 4.8 provides optimal error estimates for $\underline{z}$ and $\underline{\rho}$. In general the norms on the right-hand side of the above corollary depend on t. However, in the case that Ω is convex the following regularity result holds (see [27])

$$\|\boldsymbol{\sigma}\|_{L^2(\Omega)} + t\, \|\boldsymbol{\sigma}\|_{H^1(\Omega)} + \|\boldsymbol{r}\|_{H^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}.$$

Hence, choosing $r_0 = r_1 = 1$ and $r_2 = r_3 = 0$ in Corollary 4.8 we get the first-order convergence result independent of the plate thickness

$$\|\underline{z} - \underline{z}_h\|_{L^2(\Omega)} + \|\underline{\rho} - \underline{\rho}_h\|_{L^2(\Omega)} \leq C\, h \|f\|_{L^2(\Omega)}.$$

In the remainder of this section we prove Theorem 4.7.

Proof. (Theorem 4.7)
We have

$$\begin{aligned}\|\underline{\boldsymbol{\Pi}} \underline{z} - \underline{z}_h\|^2_{L^2(\Omega;\mathcal{A})} =& (\mathcal{A}(\underline{\boldsymbol{\Pi}} \underline{z} - \underline{z}), \underline{\boldsymbol{\Pi}} \underline{z} - \underline{z}_h) - (\boldsymbol{P r} - \boldsymbol{r}_h, \nabla \cdot (\underline{\boldsymbol{\Pi}} \underline{z} - \underline{z}_h))\\ &- (\underline{\rho} - \underline{\rho}_h, \underline{\boldsymbol{\Pi}} \underline{z} - \underline{z}_h) && \text{by (4.11b)}\\ =& (\mathcal{A}(\underline{\boldsymbol{\Pi}} \underline{z} - \underline{z}), \underline{\boldsymbol{\Pi}} \underline{z} - \underline{z}_h) - (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{P r} - \boldsymbol{r}_h)\\ &- (\underline{\rho} - \underline{\rho}_h, \underline{\boldsymbol{\Pi}} \underline{z} - \underline{z}_h) && \text{by (4.11c)}\\ =& (\mathcal{A}(\underline{\boldsymbol{\Pi}} \underline{z} - \underline{z}), \underline{\boldsymbol{\Pi}} \underline{z} - \underline{z}_h) - (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{P r} - \boldsymbol{r}_h - (\boldsymbol{P q} - \boldsymbol{q}_h))\\ &- (\underline{\rho} - \underline{\rho}_h, \underline{\boldsymbol{\Pi}} \underline{z} - \underline{z}_h) - (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{P q} - \boldsymbol{q}_h)\\ =& (\mathcal{A}(\underline{\boldsymbol{\Pi}} \underline{z} - \underline{z}), \underline{\boldsymbol{\Pi}} \underline{z} - \underline{z}_h) - \frac{1}{\hat{t}^2} \|(\boldsymbol{P r} - \boldsymbol{r}_h) - (\boldsymbol{P q} - \boldsymbol{q}_h)\|^2_{L^2(\Omega)}\\ &- (\underline{\rho} - \underline{\rho}_h, \underline{\boldsymbol{\Pi}} \underline{z} - \underline{z}_h) - (\boldsymbol{\Pi}^S \tilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}_h, \boldsymbol{P q} - \boldsymbol{q}_h)\\ &- (\boldsymbol{\sigma} - \tilde{\boldsymbol{\sigma}}, \boldsymbol{P q} - \boldsymbol{q}_h) - (\tilde{\boldsymbol{\sigma}} - \boldsymbol{\Pi}^S \tilde{\boldsymbol{\sigma}}, \boldsymbol{P q} - \boldsymbol{q}_h). && \text{by (4.11d)}\end{aligned}$$

We simplify one of the terms above.

$$\begin{aligned}
-(\boldsymbol{\Pi}^S\tilde{\boldsymbol{\sigma}}-\boldsymbol{\sigma}_h,\boldsymbol{P}\boldsymbol{q}-\boldsymbol{q}_h) &= -(\boldsymbol{\Pi}^S\tilde{\boldsymbol{\sigma}}-\boldsymbol{\sigma}_h,\boldsymbol{q}-\boldsymbol{q}_h) && \text{by (4.33)}\\
&= (u-u_h,\nabla\cdot(\boldsymbol{\Pi}^S\tilde{\boldsymbol{\sigma}}-\boldsymbol{\sigma}_h)) && \text{by (4.11a)}\\
&= 0. && \text{by (4.32)}
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\|\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h\|^2_{L^2(\Omega;\mathcal{A})}+\frac{1}{\hat{t}^2}\|(\boldsymbol{P}\boldsymbol{r}-\boldsymbol{r}_h)-(\boldsymbol{P}\boldsymbol{q}-\boldsymbol{q}_h)\|^2_{L^2(\Omega)}\\
=&(\mathcal{A}(\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}),\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h)-(\underline{\boldsymbol{\rho}}-\underline{\boldsymbol{\rho}}_h,\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h)\\
&-(\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}},\boldsymbol{P}\boldsymbol{q}-\boldsymbol{q}_h)-(\tilde{\boldsymbol{\sigma}}-\boldsymbol{\Pi}^S\tilde{\boldsymbol{\sigma}},\boldsymbol{P}\boldsymbol{q}-\boldsymbol{q}_h).
\end{aligned}\tag{4.35}$$

Bounding the first term we get

$$(\mathcal{A}(\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}),\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h)\le C\|\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}\|_{L^2(\Omega)}\ \|\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h\|_{L^2(\Omega)}.\tag{4.36}$$

In order to bound the second term on the right of (4.35) we use (4.11f) to obtain

$$\begin{aligned}
-(\underline{\boldsymbol{\rho}}-\underline{\boldsymbol{\rho}}_h,\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h) &= -(\underline{\boldsymbol{\rho}}-\underline{\boldsymbol{P}}\underline{\boldsymbol{\rho}},\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h)-(\underline{\boldsymbol{P}}\underline{\boldsymbol{\rho}}-\underline{\boldsymbol{\rho}}_h,\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h)\\
&= -(\underline{\boldsymbol{\rho}}-\underline{\boldsymbol{P}}\underline{\boldsymbol{\rho}},\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h)-(\underline{\boldsymbol{P}}\underline{\boldsymbol{\rho}}-\underline{\boldsymbol{\rho}}_h,\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}).
\end{aligned}$$

We can then use (4.16) to arrive at the following bound

$$\begin{aligned}
-(\underline{\boldsymbol{\rho}}-\underline{\boldsymbol{\rho}}_h,\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h)\le& C\,(\|\underline{\boldsymbol{\rho}}-\underline{\boldsymbol{P}}\underline{\boldsymbol{\rho}}\|_{L^2(\Omega)}+\|\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}\|_{L^2(\Omega)})\|\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h\|_{L^2(\Omega)}\\
&+C\,(\|\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}\|_{L^2(\Omega)}+\|\underline{\boldsymbol{\rho}}-\underline{\boldsymbol{P}}\underline{\boldsymbol{\rho}}\|_{L^2(\Omega)})\|\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}\|_{L^2(\Omega)}.
\end{aligned}\tag{4.37}$$

To bound the third term on the right of (4.35) we use the triangle inequality (4.31) and (4.26) to get

$$\begin{aligned}
-(\boldsymbol{\sigma}-\tilde{\boldsymbol{\sigma}},\boldsymbol{P}\boldsymbol{q}-\boldsymbol{q}_h)\le&\hat{t}\|f-P\,f\|_{H^{-1}(\Omega)}\frac{1}{\hat{t}}\|(\boldsymbol{P}\boldsymbol{r}-\boldsymbol{r}_h)-(\boldsymbol{P}\boldsymbol{q}-\boldsymbol{q}_h)\|_{L^2(\Omega)}\\
&+C\,\|f-P\,f\|_{H^{-1}(\Omega)}(\|\boldsymbol{z}-\boldsymbol{z}_h\|_{L^2(\Omega)}+\|\underline{\boldsymbol{\rho}}-\underline{\boldsymbol{\rho}}_h\|_{L^2(\Omega)}).
\end{aligned}\tag{4.38}$$

We now bound the last term of (4.35). To this end, we apply Proposition 4.5 with $\boldsymbol{v}=\tilde{\boldsymbol{\sigma}}-\boldsymbol{\Pi}^S\tilde{\boldsymbol{\sigma}}$ and have $p\in H^1(\Omega)$ such that

$$\tilde{\boldsymbol{\sigma}}-\boldsymbol{\Pi}^S\tilde{\boldsymbol{\sigma}}=\mathbf{curl}\ p.$$

Note that we used that $\nabla\cdot(\tilde{\boldsymbol{\sigma}}-\boldsymbol{\Pi}^S\tilde{\boldsymbol{\sigma}})=P\,f-P^SP\,f=0$.

We let I be the Scott-Zhang [39] linear interpolant that has the following property

$$\|p-I\,p\|_{L^2(\Omega)}+h\|p-I\,p\|_{H^1(\Omega)}\le Ch^j\|p\|_{H^j(\Omega)},\tag{4.39}$$

for $j=1,2$.

We note that $\mathbf{curl}\ Ip\in\boldsymbol{\Sigma}_h\cap\boldsymbol{Q}_h$ and see that

$$\begin{aligned}
-(\mathbf{curl}\ p,\boldsymbol{P}\boldsymbol{q}-\boldsymbol{q}_h) &= -(\mathbf{curl}\ (p-Ip),\boldsymbol{P}\boldsymbol{q}-\boldsymbol{q}_h) && \text{by (4.11a)}\\
&= -(\mathbf{curl}\ (p-Ip),\boldsymbol{P}\boldsymbol{r}-\boldsymbol{r}_h)\\
&\quad-(\mathbf{curl}\ (p-Ip),\boldsymbol{P}\boldsymbol{q}-\boldsymbol{q}_h-(\boldsymbol{P}\boldsymbol{r}-\boldsymbol{r}_h)).
\end{aligned}$$

The last term is bounded in the following way

$$-(\mathbf{curl}\ (p-Ip),\boldsymbol{P}\boldsymbol{q}-\boldsymbol{q}_h-(\boldsymbol{P}\boldsymbol{r}-\boldsymbol{r}_h))\le C\,\hat{t}\,\|\tilde{\boldsymbol{\sigma}}-\boldsymbol{\Pi}^S\tilde{\boldsymbol{\sigma}}\|_{L^2(\Omega)}\ \frac{1}{\hat{t}}\|\boldsymbol{P}\boldsymbol{q}-\boldsymbol{q}_h-(\boldsymbol{P}\boldsymbol{r}-\boldsymbol{r}_h)\|_{L^2(\Omega)},$$

where we used (4.39) and (4.30).

To bound the other term we use integration by parts to get

$$-(\mathbf{curl}\ (p - Ip), \boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h) = -((p - Ip), \operatorname{curl}(\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h)) - \langle p - Ip, (\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h) \cdot \boldsymbol{t}\rangle,$$

where $\boldsymbol{t}$ when restricted to an element $K \in \mathcal{T}_h$ is the unit tangent vector of ∂K.

The first term above can be bounded using Lemma 4.3, (4.39) and (4.30)

$$-((p - Ip), \operatorname{curl}(\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h)) \leq C\, h\, \|\tilde{\boldsymbol{\sigma}} - \boldsymbol{\Pi}^S\tilde{\boldsymbol{\sigma}}\|_{L^2(\Omega)}(\|\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h\|_{L^2(\Omega)} + \|\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}_h\|_{L^2(\Omega)}).$$

The second term can be bounded using and Lemma 4.2, (4.39) and (4.30)

$$\begin{aligned}\langle p - Ip, (\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h) \cdot \boldsymbol{t}\rangle \leq & \sum_{F \in \mathcal{E}_h} \|p - I\, p\|_{L^2(F)} \|\boldsymbol{j}(\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h)\|_{L^2(F)} \\ \leq & C\, h\, \|\tilde{\boldsymbol{\sigma}} - \boldsymbol{\Pi}^S\tilde{\boldsymbol{\sigma}}\|_{L^2(\Omega)}\, (\|\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h\|_{L^2(\Omega)} + \|\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}_h\|_{L^2(\Omega)}).\end{aligned}$$

Here $\mathcal{E}_h$ is the collection of faces of the triangulation $\mathcal{T}_h$ and $\boldsymbol{j}$ is the jump operator on interior edges and the identity on boundary edges. More precisely, if F is an interior edge and $K, K' \in \mathcal{T}_h$ share F then

$$\boldsymbol{j}(\boldsymbol{v})|_F = |(\boldsymbol{v}|_K - \boldsymbol{v}|_{K'})|_F|,$$

and if F is a boundary edge then

$$\boldsymbol{j}(\boldsymbol{v})|_F = \boldsymbol{v}|_F.$$

Hence, we obtain

$$\begin{aligned}-(\tilde{\boldsymbol{\sigma}} - \boldsymbol{\Pi}\tilde{\boldsymbol{\sigma}}, \boldsymbol{P}\boldsymbol{q} - \boldsymbol{q}_h) \leq\ & C\, h\|\tilde{\boldsymbol{\sigma}} - \boldsymbol{\Pi}^S\tilde{\boldsymbol{\sigma}}\|_{L^2(\Omega)}(\|\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h\|_{L^2(\Omega)} + \|\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}_h\|_{L^2(\Omega)}) \\ & + C\, \hat{t}\, \|\tilde{\boldsymbol{\sigma}} - \boldsymbol{\Pi}^S\tilde{\boldsymbol{\sigma}}\|_{L^2(\Omega)}\, \frac{1}{\hat{t}}\, \|(\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h) - (\boldsymbol{P}\boldsymbol{q} - \boldsymbol{q}_h)\|_{L^2(\Omega)}.\end{aligned}$$

If we use (4.16) we have

$$\begin{aligned}-(\tilde{\boldsymbol{\sigma}} - \boldsymbol{\Pi}^S\tilde{\boldsymbol{\sigma}}, \boldsymbol{P}\boldsymbol{q} - \boldsymbol{q}_h) \leq\ & C\, h\|\tilde{\boldsymbol{\sigma}} - \boldsymbol{\Pi}^S\tilde{\boldsymbol{\sigma}}\|_{L^2(\Omega)}\|\underline{\boldsymbol{\mathit{\Pi}}}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h\|_{L^2(\Omega)} \\ & + C\, h\|\tilde{\boldsymbol{\sigma}} - \boldsymbol{\Pi}^S\tilde{\boldsymbol{\sigma}}\|_{L^2(\Omega)}(\|\underline{\boldsymbol{\mathit{\Pi}}}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}\|_{L^2(\Omega)} + \|\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{P}}\underline{\boldsymbol{\rho}}\|_{L^2(\Omega)}) \\ & + C\, \hat{t}\, \|\tilde{\boldsymbol{\sigma}} - \boldsymbol{\Pi}^S\tilde{\boldsymbol{\sigma}}\|_{L^2(\Omega)}\, \frac{1}{\hat{t}}\, \|(\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h) - (\boldsymbol{P}\boldsymbol{q} - \boldsymbol{q}_h)\|_{L^2(\Omega)}.\end{aligned} \tag{4.40}$$

Combining (4.36), (4.37), (4.38), (4.40) and (4.35) we arrive at

$$\begin{aligned}&\|\underline{\boldsymbol{\mathit{\Pi}}}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h\|_{L^2(\Omega)} + \frac{1}{\hat{t}}\|(\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h) - (\boldsymbol{P}\boldsymbol{q} - \boldsymbol{q}_h)\|_{L^2(\Omega)} \\ \leq & C\, (\|\underline{\boldsymbol{\mathit{\Pi}}}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}\|_{L^2(\Omega)} + \|\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{P}}\underline{\boldsymbol{\rho}}\|_{L^2(\Omega)} + \|f - P\, f\|_{H^{-1}(\Omega)} + (h + \hat{t})\, \|\tilde{\boldsymbol{\sigma}} - \boldsymbol{\Pi}^S\tilde{\boldsymbol{\sigma}}\|_{L^2(\Omega)}),\end{aligned}$$

where we used that $\hat{t}$ is bounded and that $\mathcal{A}$ is positive-definite.

Using the triangle inequality, (4.15), and (4.31) we have

$$\begin{aligned}\|\tilde{\boldsymbol{\sigma}} - \boldsymbol{\Pi}^S\tilde{\boldsymbol{\sigma}}\|_{L^2(\Omega)} \leq & \|\boldsymbol{\sigma} - \boldsymbol{\Pi}^S\boldsymbol{\sigma}\|_{L^2(\Omega)} + \|(\tilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}) - \boldsymbol{\Pi}^S(\tilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma})\|_{L^2(\Omega)} \\ \leq & \|\boldsymbol{\sigma} - \boldsymbol{\Pi}^S\boldsymbol{\sigma}\|_{L^2(\Omega)} + C\|\tilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}\|_{L^2(\Omega)} \\ \leq & \|\boldsymbol{\sigma} - \boldsymbol{\Pi}^S\boldsymbol{\sigma}\|_{L^2(\Omega)} + C\|f - P\, f\|_{H^{-1}(\Omega)}.\end{aligned}$$

Hence, using that $h + \hat{t}$ is bounded we have

$$\begin{aligned}
&\|\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h\|_{L^2(\Omega)} + \frac{1}{\hat{t}}\|(\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h) - (\boldsymbol{P}\boldsymbol{q} - \boldsymbol{q}_h)\|_{L^2(\Omega)} \\
\leq &C\,(\|\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}\|_{L^2(\Omega)} + \|\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{P}}\underline{\boldsymbol{\rho}}\|_{L^2(\Omega)} + \|f - P\,f\|_{H^{-1}(\Omega)} + (h+\hat{t})\,\|\boldsymbol{\sigma} - \boldsymbol{\Pi}^S\boldsymbol{\sigma}\|_{L^2(\Omega)}).
\end{aligned} \tag{4.41}$$

To complete the proof we note that by (4.16) and (4.41) we can also bound $\|\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}_h\|_{L^2(\Omega)}$ by the right-hand side of (4.41). □

4.2. **Error estimate for u, $\boldsymbol{r}$, $\boldsymbol{q}$ and $\boldsymbol{\sigma}$.** We will prove optimal error estimates for u, $\boldsymbol{r}$, $\boldsymbol{q}$ and sub-optimal error estimates for $\boldsymbol{\sigma}$. Our numerical experiments show that these estimates are sharp.

The next theorem is a consequence of Theorem 4.7.

Theorem 4.9. *For any $1 \leq r_0 \leq k+1$ and $0 \leq r_1, r_2, r_3, r_4 \leq k+1$, we have*

$$\begin{aligned}
&\|\boldsymbol{r} - \boldsymbol{r}_h\|_{L^2(\Omega)} \\
\leq &C\,h^{r_0}\,\|\boldsymbol{r}\|_{H^{1+r_0}(\Omega)} + C\,\hat{t}\,h^{r_1}\,\|\boldsymbol{\sigma}\|_{H^{r_1}(\Omega)} + C\,h^{1+r_2}\,\|\boldsymbol{\sigma}\|_{H^{r_2}(\Omega)} + C\,h^{1+r_3}\,\|f\|_{H^{r_3}(\Omega)}.
\end{aligned} \tag{4.42}$$

and

$$\begin{aligned}
&\|\boldsymbol{q} - \boldsymbol{q}_h\|_{L^2(\Omega)} + \|u - u_h\|_{L^2(\Omega)} \\
\leq &C\,h^{r_0}\,\|\boldsymbol{r}\|_{H^{1+r_0}(\Omega)} + C\,\hat{t}\,h^{r_1}\,\|\boldsymbol{\sigma}\|_{H^{r_1}(\Omega)} + C\,h^{1+r_2}\,\|\boldsymbol{\sigma}\|_{H^{r_2}(\Omega)} + C\,h^{1+r_3}\,\|f\|_{H^{r_3}(\Omega)} \\
&+ h^{r_4}\,\|u\|_{H^{1+r_4}(\Omega)}.
\end{aligned} \tag{4.43}$$

Also,

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2(\Omega)} \leq C\,(\|\nabla\cdot(\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h)\|_{L^2(\Omega)} + \|\boldsymbol{\Pi}^S\boldsymbol{\sigma} - \boldsymbol{\sigma}\|_{L^2(\Omega)} + \|f - P\,f\|_{H^{-1}(\Omega)}). \tag{4.44}$$

If the mesh is quasi-uniform, then

$$\begin{aligned}
&\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2(\Omega)} \leq \\
&C\,h^{r_0-1}\,\|\boldsymbol{r}\|_{H^{1+r_0}(\Omega)} + C\,\hat{t}\,h^{r_1-1}\,\|\boldsymbol{\sigma}\|_{H^{r_1-1}(\Omega)} + C\,h^{r_2}\,\|\boldsymbol{\sigma}\|_{H^{r_2}(\Omega)} + C\,h^{r_3}\,\|f\|_{H^{r_3}(\Omega)}.
\end{aligned} \tag{4.45}$$

Proof. (Theorem 4.9)
By Lemma 4.26 we have

$$\|\boldsymbol{r} - \boldsymbol{r}_h\|_{L^2(\Omega)} \leq C\,(\|\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h\|_{L^2(\Omega)} + \|\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}_h\|_{L^2(\Omega)} + \|\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}\|_{L^2(\Omega)}).$$

If we apply Corollary 4.8 we get (4.42).

Using (4.11a) we can easily prove that

$$\|Pu - u_h\|_{L^2(\Omega)} \leq C\,\|\boldsymbol{q} - \boldsymbol{q}_h\|_{L^2(\Omega)},$$

and since

$$\|\boldsymbol{q} - \boldsymbol{q}_h\|_{L^2(\Omega)} \leq \|\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h\|_{L^2(\Omega)} + \|\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h - (\boldsymbol{P}\boldsymbol{q} - \boldsymbol{q}_h)\|_{L^2(\Omega)},$$

we can prove (4.43) by using Corollary 4.8, (4.42) and using that $\|\boldsymbol{q}\|_{H^{r_4}(\Omega)} \leq C\|u\|_{H^{1+r_4}(\Omega)}$.

In order to prove (4.44) we use (4.33) and (4.11c)

$$\begin{aligned}\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{L^2(\Omega)}^2 =&(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\boldsymbol{\sigma}-\boldsymbol{\Pi}^S\tilde{\boldsymbol{\sigma}})+(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\boldsymbol{\Pi}^S\tilde{\boldsymbol{\sigma}}-\boldsymbol{\sigma}_h)\\ =&(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\boldsymbol{\sigma}-\boldsymbol{\Pi}^S\tilde{\boldsymbol{\sigma}})+(\nabla\cdot(\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h),\boldsymbol{\Pi}^S\tilde{\boldsymbol{\sigma}}-\boldsymbol{\sigma}_h)\\ =&(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\boldsymbol{\sigma}-\boldsymbol{\Pi}^S\tilde{\boldsymbol{\sigma}})+(\nabla\cdot(\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h),\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\\ &+(\nabla\cdot(\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h),\boldsymbol{\Pi}^S\tilde{\boldsymbol{\sigma}}-\boldsymbol{\sigma})\end{aligned}$$

where we also used the commutative property (4.12). This provides the following estimate

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{L^2(\Omega)}\le C(\|\nabla\cdot(\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h)\|_{L^2(\Omega)}+\|\boldsymbol{\Pi}^S\tilde{\boldsymbol{\sigma}}-\boldsymbol{\sigma}\|_{L^2(\Omega)}).$$

Finally, if we use the triangle inequality, (4.15) and (4.31) we get

$$\begin{aligned}\|\boldsymbol{\Pi}^S\tilde{\boldsymbol{\sigma}}-\boldsymbol{\sigma}\|_{L^2(\Omega)} \le& C(\|\boldsymbol{\Pi}^S\boldsymbol{\sigma}-\boldsymbol{\sigma}\|_{L^2(\Omega)}+\|\boldsymbol{\Pi}^S(\tilde{\boldsymbol{\sigma}}-\boldsymbol{\sigma})\|_{L^2(\Omega)})\\ \le& C(\|\boldsymbol{\Pi}^S\boldsymbol{\sigma}-\boldsymbol{\sigma}\|_{L^2(\Omega)}+\|\tilde{\boldsymbol{\sigma}}-\boldsymbol{\sigma}\|_{L^2(\Omega)})\\ \le& C(\|\boldsymbol{\Pi}^S\boldsymbol{\sigma}-\boldsymbol{\sigma}\|_{L^2(\Omega)}+\|f-P\,f\|_{H^{-1}(\Omega)}),\end{aligned}\tag{4.46}$$

which in turn proves (4.44).

We can then use the inverse estimate

$$\|\nabla\cdot(\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h)\|_{L^2(\Omega)}\le\frac{C}{h}\|\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h\|_{L^2(\Omega)},$$

and apply Corollary 4.8. This proves (4.45). □

4.3. **Superconvergence of $\boldsymbol{P}\boldsymbol{r}-\boldsymbol{r}_h$.** In this section we prove a superconvergence result for $\boldsymbol{P}\boldsymbol{q}-\boldsymbol{q}_h$ under the assumption that Ω is convex. In order to do this we use a duality argument. We need to define the problem

$$\begin{aligned}-\nabla\cdot(\boldsymbol{C}\underline{\boldsymbol{\epsilon}}(\boldsymbol{\theta}))-\lambda t^{-2}(\nabla w-\boldsymbol{\theta})=&\boldsymbol{P}\boldsymbol{r}-\boldsymbol{r}_h &&\text{in }\Omega, &(4.47a)\\ -\lambda t^{-2}\nabla\cdot(\nabla w-\boldsymbol{\theta})=&0 &&\text{in }\Omega, &(4.47b)\\ \theta=&0 &&\text{on }\partial\Omega, &(4.47c)\\ \boldsymbol{\psi}=&0 &&\text{on }\partial\Omega. &(4.47d)\end{aligned}$$

We let

$$\boldsymbol{\gamma}=-\hat{t}^{-2}(\nabla w-\boldsymbol{\theta}).\tag{4.48}$$

Then the following regularity result holds (see for example [27])

$$\|\boldsymbol{\gamma}\|_{L^2(\Omega)}+t\,\|\boldsymbol{\gamma}\|_{H^1(\Omega)}+\|\boldsymbol{\theta}\|_{H^2(\Omega)}+\|w\|_{H^2(\Omega)}\le C\|\boldsymbol{P}\boldsymbol{r}-\boldsymbol{r}_h\|_{L^2(\Omega)}.\tag{4.49}$$

Moreover, since $\nabla\cdot\boldsymbol{\gamma}=0$ we also find in [27] that there exists $m\in H^1(\Omega)$ such that

$$\boldsymbol{\gamma}=\mathbf{curl}\,(m)$$

with

$$\|m\|_{H^1(\Omega)}+t\|m\|_{H^2(\Omega)}\le C\|\boldsymbol{P}\boldsymbol{r}-\boldsymbol{r}_h\|_{L^2(\Omega)}.\tag{4.50}$$

We will also need the following proposition.

Proposition 4.10. *If $\boldsymbol{\psi}\in\boldsymbol{\Sigma}_h$ then there exists $\boldsymbol{v}\in\boldsymbol{Q}_h$ such that for every $K\in\mathcal{T}_h$*

$$\|\boldsymbol{\psi}-\boldsymbol{v}\|_{L^2(K)}\le C\,h_K\,\|\nabla\cdot\boldsymbol{\psi}-P^{k-1}\nabla\cdot\boldsymbol{\psi}\|_{L^2(K)},$$

where P^m is the L^2 projection onto the space

$$W_h^m:=\{w\in L^2(\Omega):w|_K\in\mathcal{P}^m(K),\ \textit{for all } K\in\mathcal{T}_h\}\quad \textit{for } m\ge0,$$

and $W_h^{-1} := \{0\}$.

One can prove this result for example by letting $\boldsymbol{v} = \boldsymbol{\Pi}^{\rm SFH}\boldsymbol{\psi}$ where $\boldsymbol{\Pi}^{\rm SFH}$ is the projection introduced in [21], noting that $\boldsymbol{\psi} = \boldsymbol{\Pi}\boldsymbol{\psi}$ and then following the proof of Proposition 2.1 (vi) in [20].

Theorem 4.11. *For* Ω *convex we have*

$$\begin{aligned}\|\boldsymbol{P}\boldsymbol{r}-\boldsymbol{r}_h\|_{L^2(\Omega)} \leq& C\,h(\|\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}\|_{L^2(\Omega)} + \|\underline{\boldsymbol{\rho}}-\underline{\boldsymbol{P}}\underline{\boldsymbol{\rho}}\|_{L^2(\Omega)} + (h+\hat{t})\,\|\boldsymbol{\sigma}-\boldsymbol{\Pi}^S\boldsymbol{\sigma}\|_{L^2(\Omega)})\\ &+ C\,h\,(h^{\ell_k}\,\|f-P\,f\|_{L^2(\Omega)} + h^{1+\ell_k}\,\|f-P^{k-1}\,f\|_{L^2(\Omega)} + \|f-P\,f\|_{H^{-1}(\Omega)}),\end{aligned}$$

where $\ell_k = 0$ *if* $k=0$ *and* $\ell_k = 1$ *if* $k \geq 1$.

Before proving Theorem 4.11 we state a simple corollary.

Corollary 4.12. *Assuming* Ω *is convex we have for any* $1 \leq r_0 \leq k+1$ *and* $0 \leq r_1, r_2, r_3 \leq k+1$

$$\|\boldsymbol{P}\boldsymbol{r}-\boldsymbol{r}_h\|_{L^2(\Omega)} \leq C\,h(h^{r_0}\,\|\boldsymbol{r}\|_{H^{1+r_0}(\Omega)} + \hat{t}h^{r_1}\,\|\boldsymbol{\sigma}\|_{H^{r_1}(\Omega)} + h^{1+r_2}\,\|\boldsymbol{\sigma}\|_{H^{r_2}(\Omega)} + h^{\ell_k+r_3}\,\|f\|_{H^{r_3}(\Omega)}),$$

where $\ell_k = 0$ *if* $k=0$ *and* $\ell_k = 1$ *if* $k \geq 1$.

This result shows that $\boldsymbol{P}\boldsymbol{r}-\boldsymbol{r}_h$ converges with one order more than $\boldsymbol{r}-\boldsymbol{r}_h$. For smooth solutions we should expect that $\boldsymbol{P}\boldsymbol{r}-\boldsymbol{r}_h$ converges with order $k+2$. We exploit this later with post-processing.

Proof. (Theorem 4.11) Let

$$\mathcal{A}\underline{\boldsymbol{s}} = \underline{\boldsymbol{\epsilon}}(\boldsymbol{\theta}). \tag{4.51}$$

Then, we see by (4.47a)

$$-\nabla\cdot\underline{\boldsymbol{s}} + \boldsymbol{\gamma} = \boldsymbol{P}\boldsymbol{r}-\boldsymbol{r}_h. \tag{4.52}$$

Hence,

$$\begin{aligned}\|\boldsymbol{P}\boldsymbol{r}-\boldsymbol{r}_h\|^2_{L^2(\Omega)} =& -(\boldsymbol{P}\boldsymbol{r}-\boldsymbol{r}_h, \nabla\cdot\underline{\boldsymbol{s}}) + (\gamma, \boldsymbol{P}\boldsymbol{r}-\boldsymbol{r}_h) && \text{by (4.52)}\\ =& -(\boldsymbol{P}\boldsymbol{r}-\boldsymbol{r}_h, \nabla\cdot\underline{\boldsymbol{\Pi}}\underline{\boldsymbol{s}}) + (\gamma, \boldsymbol{P}\boldsymbol{r}-\boldsymbol{r}_h) && \text{by (4.12)}\\ =&(\underline{\boldsymbol{\rho}}-\underline{\boldsymbol{\rho}}_h, \underline{\boldsymbol{\Pi}}\underline{\boldsymbol{s}}) + (\mathcal{A}(\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h), \underline{\boldsymbol{\Pi}}\underline{\boldsymbol{s}}) + (\gamma, \boldsymbol{P}\boldsymbol{r}-\boldsymbol{r}_h) && \text{by (4.11b)}\\ =&(\underline{\boldsymbol{\rho}}-\underline{\boldsymbol{\rho}}_h, \underline{\boldsymbol{\Pi}}\underline{\boldsymbol{s}}) + (\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h, \mathcal{A}\underline{\boldsymbol{s}}) + (\mathcal{A}(\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h), \underline{\boldsymbol{\Pi}}\underline{\boldsymbol{s}}-\underline{\boldsymbol{s}})\\ &+ (\boldsymbol{\gamma}, \boldsymbol{P}\boldsymbol{r}-\boldsymbol{r}_h),\end{aligned}$$

and by (4.51) we have

$$\|\boldsymbol{P}\boldsymbol{r}-\boldsymbol{r}_h\|^2_{L^2(\Omega)} = T_0 + T_1 + T_2 + T_3, \tag{4.53}$$

where

$$\begin{aligned}T_0 :=&(\underline{\boldsymbol{\rho}}-\underline{\boldsymbol{\rho}}_h, \underline{\boldsymbol{\Pi}}\underline{\boldsymbol{s}}),\\ T_1 :=&(\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h, \underline{\boldsymbol{\epsilon}}(\boldsymbol{\theta})),\\ T_2 :=&(\mathcal{A}(\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h), \underline{\boldsymbol{\Pi}}\underline{\boldsymbol{s}}-\underline{\boldsymbol{s}}),\\ T_3 :=&(\boldsymbol{\gamma}, \boldsymbol{P}\boldsymbol{r}-\boldsymbol{r}_h).\end{aligned}$$

We first bound T_0. By using that $\underline{\boldsymbol{s}}$ is symmetric and $\underline{\boldsymbol{\rho}}-\underline{\boldsymbol{\rho}}_h$ is anti-symmetric we have

$$T_0 = (\underline{\boldsymbol{\rho}}-\underline{\boldsymbol{\rho}}_h, \underline{\boldsymbol{\Pi}}\underline{\boldsymbol{s}}-\underline{\boldsymbol{s}}) \leq C\,h\,\|\underline{\boldsymbol{\rho}}-\underline{\boldsymbol{\rho}}_h\|_{L^2(\Omega)}\|\underline{\boldsymbol{s}}\|_{H^1(\Omega)} \leq C\,h\,\|\underline{\boldsymbol{\rho}}-\underline{\boldsymbol{\rho}}_h\|_{L^2(\Omega)}\|\boldsymbol{P}\boldsymbol{r}-\boldsymbol{r}_h\|_{L^2(\Omega)}, \tag{4.54}$$

where we used $\|\underline{\boldsymbol{s}}\|_{H^1(\Omega)} \leq C\,\|\boldsymbol{\theta}\|_{H^2(\Omega)}$ and (4.49).

By using (4.13), the fact that $\|\underline{\boldsymbol{s}}\|_{H^1(\Omega)} \leq C\,\|\boldsymbol{\theta}\|_{H^2(\Omega)}$, and (4.49) we get

$$T_2 \leq C\,h\,\|\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h\|_{L^2(\Omega)}\|\underline{\boldsymbol{s}}\|_{H^1(\Omega)} \leq C\,h\,\|\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h\|_{L^2(\Omega)}\|\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h\|_{L^2(\Omega)}. \tag{4.55}$$

If we let $\underline{\boldsymbol{\eta}} = (\nabla\boldsymbol{\theta} - (\nabla\boldsymbol{\theta})^t)/2$ we see that

$$\begin{aligned}
T_1 =&(\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h, \nabla\boldsymbol{\theta}) - (\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h, \underline{\boldsymbol{\eta}}) \\
=& - (\nabla\cdot(\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h), \boldsymbol{\theta}) - (\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h, \underline{\boldsymbol{\eta}} - \underline{\boldsymbol{P}}\underline{\boldsymbol{\eta}}) && \text{by (4.11f)} \\
=& - (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\theta}) - (\boldsymbol{\sigma}_h - \nabla\cdot\underline{\boldsymbol{z}}_h, \boldsymbol{\theta}) - (\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h, \underline{\boldsymbol{\eta}} - \underline{\boldsymbol{P}}\underline{\boldsymbol{\eta}}) \\
=& - (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \nabla w) - \hat{t}^2(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\gamma}) \\
&- (\boldsymbol{\sigma}_h - \nabla\cdot\underline{\boldsymbol{z}}_h, \boldsymbol{\theta} - \boldsymbol{P}\boldsymbol{\theta}) - (\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h, \underline{\boldsymbol{\eta}} - \underline{\boldsymbol{P}}\underline{\boldsymbol{\eta}}) && \text{by (4.48) and (2.2c)} \\
=&(\nabla\cdot(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h), w) - ((\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h) - (\boldsymbol{P}\boldsymbol{q} - \boldsymbol{q}_h), \boldsymbol{P}\boldsymbol{\gamma}) \\
&- \hat{t}^2(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\gamma} - \boldsymbol{P}\boldsymbol{\gamma}) - (\boldsymbol{\sigma}_h - \nabla\cdot\underline{\boldsymbol{z}}_h, \boldsymbol{\theta} - \boldsymbol{P}\boldsymbol{\theta}) \\
&- (\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h, \underline{\boldsymbol{\eta}} - \underline{\boldsymbol{P}}\underline{\boldsymbol{\eta}}) && \text{by (4.11d)} \\
=&T_4 - T_3 + T_5 + T_6 + T_7 + T_8,
\end{aligned}$$

where

$$\begin{aligned}
T_4 :=&(\nabla\cdot(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h), w), \\
T_5 :=&(\boldsymbol{P}\boldsymbol{q} - \boldsymbol{q}_h, \boldsymbol{P}\boldsymbol{\gamma}), \\
T_6 :=& - \hat{t}^2(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\gamma} - \boldsymbol{P}\boldsymbol{\gamma}), \\
T_7 :=& - (\boldsymbol{\sigma}_h - \nabla\cdot\underline{\boldsymbol{z}}_h, \boldsymbol{\theta} - \boldsymbol{P}\boldsymbol{\theta}), \\
T_8 :=& - (\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h, \underline{\boldsymbol{\eta}} - \underline{\boldsymbol{P}}\underline{\boldsymbol{\eta}}).
\end{aligned}$$

Hence,

$$T_1 + T_3 = T_4 + T_5 + T_6 + T_7 + T_8. \tag{4.56}$$

We now bound T_i for $4 \leq i \leq 8$.

Since $\nabla\cdot(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) = f - Pf$ we have

$$\begin{aligned}
T_4 =&(f - Pf, w - Pw) \leq Ch^{1+\ell_k}\,\|f - Pf\|_{L^2(\Omega)}\,\|w\|_{H^2(\Omega)} \\
\leq&Ch^{1+\ell_k}\,\|f - Pf\|_{L^2(\Omega)}\,\|\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h\|_{L^2(\Omega)},
\end{aligned} \tag{4.57}$$

where $\ell_k = 0$ if $k = 0$ and $\ell_k = 1$ if $k \geq 1$. Here we used (4.49).

By the definition of $\boldsymbol{P}$ we have

$$T_5 = (\boldsymbol{P}\boldsymbol{q} - \boldsymbol{q}_h, \boldsymbol{\gamma}).$$

Therefore,

$$T_5 = (\boldsymbol{P}\boldsymbol{q} - \boldsymbol{q}_h, \mathbf{curl}\,(m)) = (\boldsymbol{P}\boldsymbol{q} - \boldsymbol{q}_h, \mathbf{curl}\,(m - I\,m)),$$

where we used $\mathbf{curl}\,(Im) \in \boldsymbol{\Sigma}_h \cap \boldsymbol{Q}_h$, the definition of $\boldsymbol{P}$ and (4.11a).

Then, using integration by parts we get

$$\begin{aligned}
T_5 =&(\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h, \mathbf{curl}\,(m - I\,m)) + ((\boldsymbol{P}\boldsymbol{q} - \boldsymbol{q}_h) - (\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h), \mathbf{curl}\,(m - I\,m)) \\
=&(\mathrm{curl}\,(\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h), m - I\,m) + \langle(\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h)\cdot\boldsymbol{t}, m - I\,m\rangle \\
&+ ((\boldsymbol{P}\boldsymbol{q} - \boldsymbol{q}_h) - (\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h), \mathbf{curl}\,(m - I\,m)).
\end{aligned}$$

Using Lemma 4.17, Lemma 4.3 and (4.39) we have

$$\begin{aligned}T_5 \leq & C\,h(\|\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h\|_{L^2(\Omega)} + \|\underline{\boldsymbol{s}} - \underline{\boldsymbol{s}}_h\|_{L^2(\Omega)})\|m\|_{H^1(\Omega)} \\ & + C\,h\,\frac{1}{\hat{t}}\|(\boldsymbol{P}\boldsymbol{q} - \boldsymbol{q}_h) - (\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h)\|_{L^2(\Omega)}\,\hat{t}\|m\|_{H^2(\Omega)},\end{aligned}$$

and using (4.50) we obtain

$$T_5 \leq Ch(\|\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h\|_{L^2(\Omega)} + \|\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}_h\|_{L^2(\Omega)} + \frac{1}{\hat{t}}\|(\boldsymbol{P}\boldsymbol{q} - \boldsymbol{q}_h) - (\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h)\|_{L^2(\Omega)})\|(\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h)\|_{L^2(\Omega)}. \tag{4.58}$$

By using (4.33) and the definition of $\boldsymbol{P}$ we have

$$T_6 = -\hat{t}^2(\boldsymbol{\sigma} - \boldsymbol{\Pi}^S\tilde{\boldsymbol{\sigma}}, \gamma - \boldsymbol{P}\boldsymbol{\gamma}) \leq C\,h\,\hat{t}\|\boldsymbol{\sigma} - \boldsymbol{\Pi}^S\tilde{\boldsymbol{\sigma}}\|_{L^2(\Omega)}\hat{t}\|\boldsymbol{\gamma}\|_{H^1(\Omega)}.$$

By (4.49), (4.46) and using that $\hat{t}$ is bounded we have

$$T_6 \leq C\,h\,(\hat{t}\|\boldsymbol{\sigma} - \boldsymbol{\Pi}^S\boldsymbol{\sigma}\|_{L^2(\Omega)} + \|f - P\,f\|_{H^{-1}(\Omega)})\|\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h\|_{L^2(\Omega)}. \tag{4.59}$$

To estimate T_7 we use that $\nabla\cdot\underline{\boldsymbol{z}}_h \in \boldsymbol{Q}_h$ and the definition of $\boldsymbol{P}$ to get

$$T_7 = -(\boldsymbol{\sigma}_h - \boldsymbol{v}, \boldsymbol{\theta} - \boldsymbol{P}\boldsymbol{\theta}) \leq C\,h^{1+\ell_k}\,\|\boldsymbol{\sigma}_h - \boldsymbol{v}\|_{L^2(\Omega)}\|\boldsymbol{\theta}\|_{H^2(\Omega)},$$

for any $\boldsymbol{v} \in \boldsymbol{Q}_h$. Hence, using Proposition 4.10 and (2.2e) we get

$$T_7 \leq C\,h^{2+\ell_k}\,\|Pf - P^{k-1}\,f\|_{L^2(\Omega)}\|\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h\|_{L^2(\Omega)} \leq C\,h^{2+\ell_k}\,\|f - P^{k-1}\,f\|_{L^2(\Omega)}\|\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h\|_{L^2(\Omega)}, \tag{4.60}$$

where we used that $P^{k-1} = PP^{k-1}$ and P is L^2 stable. Finally, since $\|\underline{\boldsymbol{\eta}}\|_{H^1(\Omega)} \leq \|\boldsymbol{\theta}\|_{H^2(\Omega)}$ and (4.49) we have

$$T_8 \leq C\,h\,\|\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h\|_{L^2(\Omega)}\|\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h\|_{L^2(\Omega)}. \tag{4.61}$$

Combining (4.57), (4.58), (4.59), (4.60), (4.61), and (4.56) we get

$$\begin{aligned}T_1 + T_3 \leq & C\,h\,(\|\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h\|_{L^2(\Omega)} + \|\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}_h\|_{L^2(\Omega)})\|\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h\|_{L^2(\Omega)} \\ & + C\,h\,\frac{1}{\hat{t}}\|(\boldsymbol{P}\boldsymbol{q} - \boldsymbol{q}_h) - (\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h)\|_{L^2(\Omega)}\,\|\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h\|_{L^2(\Omega)} \\ & + C\,h\,(h^{\ell_k}\,\|f - P\,f\|_{L^2(\Omega)} + h^{1+\ell_k}\,\|f - P^{k-1}\,f\|_{L^2(\Omega)})\|\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h\|_{L^2(\Omega)} \\ & + C\,h\,(\hat{t}\|\boldsymbol{\sigma} - \boldsymbol{\Pi}^S\boldsymbol{\sigma}\|_{L^2(\Omega)} + \|f - P\,f\|_{H^{-1}(\Omega)})\|\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h\|_{L^2(\Omega)}.\end{aligned}$$

Combining this inequality with (4.54), (4.55) and (4.53) we get

$$\begin{aligned}\|\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h\|_{L^2(\Omega)} \leq & C\,h\,(\|\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h\|_{L^2(\Omega)} + \|\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}_h\|_{L^2(\Omega)}) \\ & + C\,h\,\frac{1}{\hat{t}}\|(\boldsymbol{P}\boldsymbol{q} - \boldsymbol{q}_h) - (\boldsymbol{P}\boldsymbol{r} - \boldsymbol{r}_h)\|_{L^2(\Omega)} \\ & + C\,h\,(h^{\ell_k}\,\|f - P\,f\|_{L^2(\Omega)} + h^{1+\ell_k}\,\|f - P^{k-1}\,f\|_{L^2(\Omega)}) \\ & + C\,h\,(\hat{t}\|\boldsymbol{\sigma} - \boldsymbol{\Pi}^S\boldsymbol{\sigma}\|_{L^2(\Omega)} + \|f - P\,f\|_{H^{-1}(\Omega)}).\end{aligned}$$

We complete the proof if we use Theorem 4.7. □

5. Post-processing

Here we present a post-processing approximation for the transverse displacement and the rotation. They are motivated by the good results we obtained in [15] for the biharmonic problem. The main reason post-processing works well is that local averages (or projections) of some errors converge faster than the error itself for these methods; see Corollary 4.12. In the next section we present the numerical convergence properties of the new approximation.

We first need to define the following space

$$\mathcal{P}_{\perp}^{\ell,m}(K) := \{v \in \mathcal{P}^{\ell}(K) : (v, \omega)_K = 0 \text{ for all } \omega \in \mathcal{P}^m(K)\}.$$

We also need the vector-valued space $\boldsymbol{\mathcal{P}}_{\perp}^{\ell,m} = [\mathcal{P}_{\perp}^{\ell,m}(K)]^2$.

In order to define the post-processed approximation to u we need to define a post-processed approximation to $\boldsymbol{r}$. It is defined in the following way.

We define $\boldsymbol{r}_h^\star|_K \in \boldsymbol{\mathcal{P}}^{k+1}(K)$ as the solution to

$$(\nabla \boldsymbol{r}_h^\star, \nabla \boldsymbol{v})_K = (\mathcal{A}\,\underline{\boldsymbol{z}}_h + \underline{\boldsymbol{\rho}}_h, \nabla \boldsymbol{v})_K \quad \text{for all } \boldsymbol{v} \in \boldsymbol{\mathcal{P}}_{\perp}^{k+1,0}(K), \tag{5.62a}$$

$$(\boldsymbol{r}_h^\star, \boldsymbol{w})_K = (\boldsymbol{r}_h, \boldsymbol{w})_K \quad \text{for all } \boldsymbol{w} \in \boldsymbol{\mathcal{P}}^0(K), \tag{5.62b}$$

for all $K \in \mathcal{T}_h$.

Finally, we can define $u_h^\star$, the post-processed approximation to u. We define $u_h^\star|_K \in \mathcal{P}^{k+2}(K)$ locally by

$$(\nabla u_h^\star, \nabla v)_K = (\boldsymbol{r}_h^\star - \hat{t}^2 \boldsymbol{\sigma}_h, \nabla v)_K \quad \text{for all } v \in \mathcal{P}_{\perp}^{k+2,0}(K), \tag{5.63a}$$

$$(u_h^\star, w)_K = (u_h, w)_K \quad \text{for all } w \in \mathcal{P}^0(K), \tag{5.63b}$$

for all $K \in \mathcal{T}_h$.

Note that $\nabla u = \boldsymbol{r} - \hat{t}^2 \boldsymbol{\sigma}$, and therefore it makes sense to define $u_h^\star$ as above. It is easy to prove that $\boldsymbol{r}_h^\star$ and $u_h^\star$ are well defined; see for example [15]. Moreover, following for example [15], we can prove the following result if we use Corollaries 4.8 and 4.12.

Theorem 5.1. *If Ω is convex we have for any $1 \le r_0 \le k+1$ and $0 \le r_1, r_2, r_3 \le k+1$*

$$\|\boldsymbol{r} - \boldsymbol{r}_h^\star\|_{L^2(\Omega)} \le C\, h(h^{r_0} \|\boldsymbol{r}\|_{H^{1+r_0}(\Omega)} + \hat{t} h^{r_1} \|\boldsymbol{\sigma}\|_{H^{r_1}(\Omega)} + h^{1+r_2} \|\boldsymbol{\sigma}\|_{H^{r_2}(\Omega)} + h^{\ell_k + r_3} \|f\|_{H^{r_3}(\Omega)}),$$

where $\ell_k = 0$ if $k = 0$ and $\ell_k = 1$ if $k \ge 1$.

6. Numerical Experiments

In this section we provide numerical experiments in the case of $k = 1$ of our method to test the robustness of the method for t small. We take the semi-infinite plate example given in [7]. However, we only consider the solution on $\Omega = [0,1] \times [0,1]$. Hence, we will not have zero boundary conditions. Our method can easily be adapted to handle non-homogeneous boundary conditions. We choose $\nu = 0$, $E = 12$, $\lambda = 6$, $t = 10^{-6}$ and the right-hand side will be $f = \cos x$. The solution $u, \boldsymbol{r} = (r_1, r_2)$ are given by

$$u(x,y) = \left(1 + \lambda^{-1} t^2 - e^{-y} + l_1(2\lambda^{-1}t^2 + y)e^{-y} - l_2 \lambda^{-1} t^2 e^{-y}\right) \cos x$$

$$r_1(x,y) = \left(-1 + e^{-y} - l_1 y e^{-y} + l_2 \lambda^{-1} t^2 e^{-y} - l_3 \lambda^{-1} t \sqrt{12 + t^2}\, e^{-\sqrt{12+t^2}\,y/t}\right) \sin x$$

$$r_2(x,y) = \left(e^{-y} + l_1(1-y)e^{-y} + l_2 \lambda^{-1} t^2 e^{-y} - l_3 \lambda^{-1} t^2 e^{-\sqrt{12+t^2}\,y/t}\right) \cos x$$

with

$$\begin{aligned} l_1 =& (-\sqrt{12+t^2}\,\lambda - \sqrt{12+t^2}\,t^2 + t^3)/m \\ l_2 =& -\sqrt{12+t^2}\,\lambda/m \\ l_3 =& -\lambda\, t/m \\ m =& \sqrt{12+t^2}\,\lambda + 2\,\beta\, t^2 - 2t^3. \end{aligned}$$

The other variables can be found by differentiating and algebraic manipulations. We note that the problem has a boundary layer along the line $y = 0$ for t small. A thorough discussion on boundary layers of the Reissner-Mindlin plate problem can be found in [8, 9]. The i-th mesh in our computation is a uniform mesh with mesh size $h = \frac{1}{2^i}$, see Figure 1 for an example.

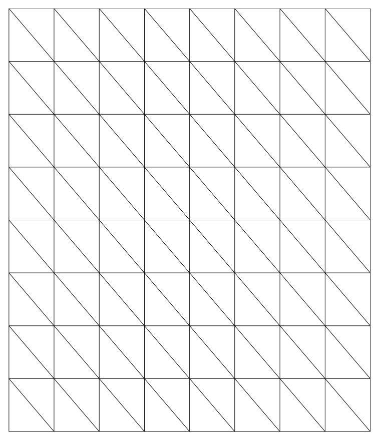

FIGURE 1. Mesh 3, $h = \frac{1}{2^3}$

As we see from Table 1 the L^2-norms of the errors converge with optimal order $k+1$ for all variables except $\boldsymbol{\sigma}$ which converges with order k. From Table 2 we see that $\|u-u_h^\star\|_{L^2(\Omega)}$ and $\|\boldsymbol{r}-\boldsymbol{r}_h^\star\|_{L^2(\Omega)}$ converge much faster than $\|u-u_h\|_{L^2(\Omega)}$ and $\|\boldsymbol{r}-\boldsymbol{r}_h\|_{L^2(\Omega)}$, respectively. This is exactly the behavior we observed for the biharmonic problem as well. Finally, in Table 3 we measure $\|\boldsymbol{I}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{L^\infty(\Omega)}$ where $\boldsymbol{I}$ is the Lagrange interpolant onto piece-wise linear functions. We see that the error measured in the max-norm does not decrease with the mesh size. The reason for this is that $\boldsymbol{\sigma}$ has a sharp layer near the boundary $y = 0$ since t is small. However, notice that the approximation in the L^2-norm of $\boldsymbol{\sigma}$ given Table 1 does not deteriorate.

7. DIFFERENT SPACES

Here we show that it is possible to use other spaces. The idea is to choose $\boldsymbol{Q}_h \times \underline{\boldsymbol{A}}_h \times \underline{\boldsymbol{Z}}_h$ as stable spaces for elasticity with weakly imposed symmetry. In other words, they need to satisfy the statement of Proposition 2.2. Then, one chooses $W_h \times \boldsymbol{\Sigma}_h$ as stable spaces for the Poisson problem. Finally, one needs to link them with the following condition

$$\{\boldsymbol{v} \in \boldsymbol{\Sigma}_h : \nabla\cdot\boldsymbol{v} = 0\} \subset \boldsymbol{Q}_h. \tag{7.64}$$

TABLE 1. History of convergence, $k = 1$

mesh	$\|u - u_h\|_{L^2(\Omega)}$		$\|\boldsymbol{q} - \boldsymbol{q}_h\|_{L^2(\Omega)}$		$\|\boldsymbol{r} - \boldsymbol{r}_h\|_{L^2(\Omega)}$		$\|\underline{\boldsymbol{\rho}} - \underline{\boldsymbol{\rho}}_h\|_{L^2(\Omega)}$		$\|\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h\|_{L^2(\Omega)}$		$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2(\Omega)}$	
i	error	order	error	order	error	order	error	order	error	order	error	order
1	.34e-2	0.00	.68e-2	0.00	.68e-2	0.00	.36e-2	0.00	.10e-1	0.00	.84e-1	0.00
2	.87e-3	1.96	.17e-2	1.96	.17e-2	1.96	.10e-2	1.85	.27e-2	1.92	.49e-1	0.77
3	.22e-3	1.99	.44e-3	1.99	.44e-3	1.99	.26e-3	1.97	.68e-3	1.96	.27e-1	0.89
4	.55e-4	2.00	.11e-3	2.00	.11e-3	2.00	.65e-4	1.99	.17e-3	1.98	.14e-1	0.94
5	.14e-4	2.00	.28e-4	2.00	.28e-4	2.00	.16e-4	2.00	.44e-4	1.99	.71e-2	0.97
6	.34e-5	2.00	.69e-5	2.00	.69e-5	2.00	.41e-5	2.00	.11e-4	2.00	.36e-2	0.98

TABLE 2. History of convergence of post-processed approximations

mesh	$\|u - u_h^\star\|_{L^2(\Omega)}$		$\|\boldsymbol{r} - \boldsymbol{r}_h^\star\|_{L^2(\Omega)}$	
i	error	order	error	order
1	.53e-4	0.00	.89e-3	0.00
2	.37e-5	3.86	.12e-3	2.93
3	.24e-6	3.95	.15e-4	2.97
4	.15e-7	3.98	.19e-5	2.99
5	.95e-9	3.99	.23e-6	2.99
6	.62e-10	3.93	.29e-7	3.00

TABLE 3. History of convergence of $\boldsymbol{\sigma}$ in the max-norm

mesh	$\|\boldsymbol{I\sigma} - \boldsymbol{\sigma}_h\|_{L^\infty(\Omega)}$
i	error
1	.84e+0
2	.84e+0
3	.84e+0
4	.84e+0
5	.84e+0
6	.84e+0

To be more precise the following result holds.

Theorem 7.1. *Suppose that $\boldsymbol{Q}_h \times \underline{\boldsymbol{A}}_h \times \underline{\boldsymbol{Z}}_h$ are stable spaces for elasticity with weakly imposed symmetry and suppose that $W_h \times \boldsymbol{\Sigma}_h$ are stable spaces for the Poisson problem. If* (7.64) *holds, then the method* (2.2) *with such spaces is well-defined.*

This result can be proved similar to Theorem 2.3.

7.1. **Other Examples.** Example of spaces that lead to a well-defined method are the following with $k \geq 1$

$$\begin{aligned}
W_h :=&\{w \in L^2(\Omega) : w|_K \in \mathcal{P}^{k-1}(K), \text{ for all } K \in \mathcal{T}_h\},\\
\boldsymbol{Q}_h :=&\{\boldsymbol{m} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{m}|_K \in \boldsymbol{\mathcal{P}}^{k-1}(K), \text{ for all } K \in \mathcal{T}_h\},\\
\boldsymbol{\Sigma}_h :=&\{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}\,,\Omega) : \boldsymbol{v}|_K \in \boldsymbol{RT}^{k-1}(K) \text{ for all } K \in \mathcal{T}_h\},\\
\underline{\boldsymbol{Z}}_h :=&\{\underline{\boldsymbol{s}} \in \underline{\boldsymbol{H}}(\mathrm{div}\,,\Omega) : \underline{\boldsymbol{s}}|_K \in \underline{\boldsymbol{\mathcal{P}}}^k(K) + \hat{\underline{\boldsymbol{Z}}}^k(K) \text{ for all } K \in \mathcal{T}_h\},\\
\underline{\boldsymbol{A}}_h :=&\{\underline{\boldsymbol{\eta}} \in \underline{\boldsymbol{L}}^2(\Omega) : \underline{\boldsymbol{\eta}}|_K \in \underline{\boldsymbol{A}}^k(K), \text{ for all } K \in \mathcal{T}_h\}.
\end{aligned}$$

The triple $\boldsymbol{Q}_h \times \underline{\boldsymbol{A}}_h \times \underline{\boldsymbol{Z}}_h$ are stable spaces for elasticity with weakly imposed symmetry as shown in [29] for the the case $k = 1$ and in general in [31]. Moreover, $W_h \times \boldsymbol{\Sigma}_h$ is simply the Raviart-Thomas spaces for Poisson's problem. By Proposition 2.1 we see that (7.64) holds.

Another set of spaces are ($k \geq 2$)

$$\begin{aligned}
W_h :=&\{w \in L^2(\Omega) : w|_K \in \mathcal{P}^{k-2}(K), \text{ for all } K \in \mathcal{T}_h\},\\
\boldsymbol{Q}_h :=&\{\boldsymbol{m} \in \boldsymbol{L}^2(\Omega) : \boldsymbol{m}|_K \in \boldsymbol{\mathcal{P}}^{k-1}(K), \text{ for all } K \in \mathcal{T}_h\},\\
\boldsymbol{\Sigma}_h :=&\{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div}\,,\Omega) : \boldsymbol{v}|_K \in \boldsymbol{\mathcal{P}}^{k-1}(K) \text{ for all } K \in \mathcal{T}_h\},\\
\underline{\boldsymbol{Z}}_h :=&\{\underline{\boldsymbol{s}} \in \underline{\boldsymbol{H}}(\mathrm{div}\,,\Omega) : \underline{\boldsymbol{s}}|_K \in \underline{\boldsymbol{\mathcal{P}}}^k(K) + \hat{\underline{\boldsymbol{Z}}}^k(K) \text{ for all } K \in \mathcal{T}_h\},\\
\underline{\boldsymbol{A}}_h :=&\{\underline{\boldsymbol{\eta}} \in \underline{\boldsymbol{L}}^2(\Omega) : \underline{\boldsymbol{\eta}}|_K \in \underline{\boldsymbol{A}}^k(K), \text{ for all } K \in \mathcal{T}_h\}.
\end{aligned}$$

In this example $W_h \times \boldsymbol{\Sigma}_h$ is the Brezzi-Douglas-Marini pair [14] for Poisson's problem and (7.64) holds trivially.

It is not difficult to carry out the error analysis of the resulting methods. For example, Theorem 4.7 holds for these methods. Of course, now the projections have changed according to the new spaces.

8. Concluding Remarks

We have developed a family of locking-free methods (one for each $k \geq 0$) for the Reissner-Mindlin problem. Based on the hybrid form, the only globally coupled degrees of freedom are those of the rotations and displacement on the edges of the triangulation; see (3.9). We should mention that the low-order non-conforming methods in [18, 22, 34] also only have edge degrees of freedom for the rotations and displacement. It would be interesting to see if there are connections between our family of methods and non-conforming methods in the way mixed methods for the Poisson problem have been related to non-conforming methods for Poisson's problem; see [3] for example.

9. Appendix

Here we prove Theorem 3.2.

Proof. (Theorem 3.2)

We first prove that (3.9) is a well-defined problem for $\lambda_h, \boldsymbol{\alpha}_h$. Since (3.9) is a square system we need to show uniqueness, so we let $f = 0$. If we let $m = \lambda_h$ and $\boldsymbol{\mu} = \boldsymbol{\alpha}_h$ and add the two equations (3.9a) and (3.9b) we get

$$a(\lambda_h, \lambda_h) + 2\, b(\lambda_h, \boldsymbol{\alpha}_h) + c(\boldsymbol{\alpha}_h, \boldsymbol{\alpha}_h) = 0.$$

Using the definition of the bilinear forms a, b, c, this gives exactly

$$\|\underline{\mathbf{Z}}_1(\lambda_h) + \underline{\mathbf{Z}}_2(\boldsymbol{\alpha}_h)\|^2_{L^2(\Omega;\mathcal{A})} + \hat{t}^2\|\mathbf{S}_1(\lambda_h) + \mathbf{S}_2(\boldsymbol{\alpha}_h)\|^2_{L^2(\Omega)} = 0,$$

which in turn gives us that

$$\underline{\mathbf{Z}}_1(\tilde{\lambda}_h) + \underline{\mathbf{Z}}_2(\tilde{\boldsymbol{\alpha}}_h) = 0. \tag{9.65}$$

and

$$\mathbf{S}_1(\tilde{\lambda}_h) + \mathbf{S}_2(\tilde{\boldsymbol{\alpha}}_h) = 0. \tag{9.66}$$

Next we show that $\mathbf{R}_1(\lambda_h) + \mathbf{R}_2(\boldsymbol{\alpha}_h)$ and $\underline{\mathbf{L}}_1(\lambda_h) + \underline{\mathbf{L}}_2(\boldsymbol{\alpha}_h)$ are identically zero. Indeed, by (3.6b), (3.7b) (9.65) we have that

$$(\mathbf{R}_1(\tilde{\lambda}_h) + \mathbf{R}_2(\tilde{\boldsymbol{\alpha}}_h), \nabla\cdot\underline{\boldsymbol{s}}) + (\underline{\mathbf{L}}_1(\tilde{\lambda}_h) + \underline{\mathbf{L}}_2(\tilde{\boldsymbol{\alpha}}_h), \underline{\boldsymbol{s}}) = \langle\tilde{\boldsymbol{\alpha}}_h, \underline{\boldsymbol{s}}\,\boldsymbol{n}\rangle, \tag{9.67}$$

for all $\underline{\boldsymbol{s}} \in \tilde{\underline{\boldsymbol{Z}}}_h$. In particular, for $\underline{\boldsymbol{s}} \in \underline{\boldsymbol{Z}}_h$ we have

$$(\mathbf{R}_1(\tilde{\lambda}_h) + \mathbf{R}_2(\tilde{\boldsymbol{\alpha}}_h), \nabla\cdot\underline{\boldsymbol{s}}) + (\underline{\mathbf{L}}_1(\tilde{\lambda}_h) + \underline{\mathbf{L}}_2(\tilde{\boldsymbol{\alpha}}_h), \underline{\boldsymbol{s}}) = 0.$$

Applying Proposition (2.2) with $\boldsymbol{w} = \mathbf{R}_1(\tilde{\lambda}_h) + \mathbf{R}_2(\tilde{\boldsymbol{\alpha}}_h)$ and $\boldsymbol{\zeta} = \underline{\mathbf{L}}_1(\tilde{\lambda}_h) + \underline{\mathbf{L}}_2(\tilde{\boldsymbol{\alpha}}_h)$ we have that $\mathbf{R}_1(\lambda_h) + \mathbf{R}_2(\boldsymbol{\alpha}_h)$ and $\underline{\mathbf{L}}_1(\lambda_h) + \underline{\mathbf{L}}_2(\boldsymbol{\alpha}_h)$ are identically zero. Therefore, we have

$$\langle\boldsymbol{\alpha}_h, \underline{\boldsymbol{s}}\,\boldsymbol{n}\rangle = 0,$$

for all $\underline{\boldsymbol{s}} \in \tilde{\underline{\boldsymbol{Z}}}_h$. Since $\tilde{\underline{\boldsymbol{Z}}}_h$ contains the Raviart-Thomas space we can use the degrees of freedom of the Raviart-Thomas space for $k \geq 1$ to show that $\boldsymbol{\alpha}_h = 0$. However, for $k = 0$ we need to use the degrees of freedom of all of $\tilde{\underline{\boldsymbol{Z}}}_h$. Let us describe them here. Let $\underline{\boldsymbol{s}} \in \underline{\boldsymbol{RT}}^0(K) + \hat{\underline{\boldsymbol{Z}}}^0(K)$. Then, $\underline{\boldsymbol{s}}$ is determined by the average of $\underline{\boldsymbol{s}}\boldsymbol{n}\cdot\boldsymbol{t}$ on each edge of K and by the average and first moment of $\underline{\boldsymbol{s}}\boldsymbol{n}\cdot\boldsymbol{n}$ on each edge of K; see [10]. Also, we can write

$$0 = \langle\boldsymbol{\alpha}_h, \underline{\boldsymbol{s}}\,\boldsymbol{n}\rangle = \langle\boldsymbol{\alpha}_h\cdot\boldsymbol{n}, \underline{\boldsymbol{s}}\boldsymbol{n}\cdot\boldsymbol{n}\rangle + \langle\boldsymbol{\alpha}_h\cdot\boldsymbol{t}, \underline{\boldsymbol{s}}\boldsymbol{n}\cdot\boldsymbol{t}\rangle.$$

By the definition of $\boldsymbol{M}^h$ (for $k = 0$) $\boldsymbol{\alpha}_h\cdot\boldsymbol{t}$ is constant on edges and $\boldsymbol{\alpha}_h\cdot\boldsymbol{n}$ is linear on edges, hence, we can choose $\underline{\boldsymbol{s}} \in \tilde{\underline{\boldsymbol{Z}}}_h$ appropriately to prove that $\boldsymbol{\alpha}_h = 0$.

We now only need to show that $\lambda_h = 0$. To this end, we first note that by (3.6d) and (3.7d)

$$(\mathbf{Q}_1(\lambda_h) + \mathbf{Q}_2(\boldsymbol{\alpha}_h), \boldsymbol{d}) = 0,$$

for all $\boldsymbol{d} \in \boldsymbol{Q}_h$. Here we used that $\mathbf{R}_1(\lambda_h) + \mathbf{R}_2(\boldsymbol{\alpha}_h) = 0$ and (9.66). This implies that $\mathbf{Q}_1(\lambda_h) + \mathbf{Q}_2(\boldsymbol{\alpha}_h) = 0$ which combined with (3.6a) and (3.7a) gives that

$$(\mathsf{u}_1(\lambda_h) + \mathsf{u}_2(\boldsymbol{\alpha}_h), \nabla\cdot\boldsymbol{v}) = \langle\lambda_h, \boldsymbol{v}\cdot\boldsymbol{n}\rangle,$$

for all $\boldsymbol{v} \in \tilde{\boldsymbol{\Sigma}}_h$. Using the degrees of freedom of the Raviart-Thomas space $\tilde{\boldsymbol{\Sigma}}_h$ we can easily show that $\lambda_h = 0$. Hence we have proved that (3.9).

Next we assume $\lambda_h, \boldsymbol{\alpha}_h$ solve (3.9) and that $u_h, \boldsymbol{q}_h, \boldsymbol{r}_h, \underline{\boldsymbol{\rho}}_h, \underline{\boldsymbol{z}}_h, \boldsymbol{\sigma}_h$ are given by (3.10). We then will show that $(u_h, \boldsymbol{q}_h, \boldsymbol{r}_h, \underline{\boldsymbol{\rho}}_h, \underline{\boldsymbol{z}}_h, \boldsymbol{\sigma}_h, \lambda_h, \boldsymbol{\alpha}_h)$ solves (3.5). To this end, using the definition of the local solvers, we easily can show that $(u_h, \boldsymbol{q}_h, \boldsymbol{r}_h, \underline{\boldsymbol{\rho}}_h, \underline{\boldsymbol{z}}_h, \boldsymbol{\sigma}_h, \lambda_h, \boldsymbol{\alpha}_h)$ satisfies (3.5a)-(3.5f). Hence, by the uniqueness of (3.5) it is enough to show that

$$\begin{aligned}\langle\boldsymbol{\sigma}_h\cdot\boldsymbol{n}, \mu\rangle &= 0,\\ \langle\underline{\boldsymbol{z}}_h\,\boldsymbol{n}, \boldsymbol{\mu}\rangle &= 0,\end{aligned}$$

for all $\mu, \boldsymbol{\mu} \in M_h \times \boldsymbol{M}_h$. Therefore, if $\lambda_h, \boldsymbol{\alpha}_h$ solves (3.9) we need to show that

$$\begin{aligned}\langle(\mathbf{S}_1(\lambda_h) + \mathbf{S}_2(\boldsymbol{\alpha}_h) + \mathbf{S}_3(f)) \cdot \boldsymbol{n}, \mu\rangle =& 0,\\ \langle(\underline{\mathbf{Z}}_1(\lambda_h) + \underline{\mathbf{Z}}_2(\boldsymbol{\alpha}_h) + \underline{\mathbf{Z}}_3(f))\ \boldsymbol{n}, \boldsymbol{\mu}\rangle =& 0.\end{aligned}$$

for all $\mu, \boldsymbol{\mu} \in M_h \times \boldsymbol{M}_h$.

This in turn follows from the following identities

$$\langle \mathbf{S}_3(f) \cdot \boldsymbol{n}, \mu\rangle = (f, \mathsf{u}_1(\mu)), \tag{9.68a}$$
$$\langle \underline{\mathbf{Z}}_3(f)\, \boldsymbol{n}, \boldsymbol{\mu}\rangle = -(f, \mathsf{u}_2(\boldsymbol{\mu})), \tag{9.68b}$$
$$\langle \mathbf{S}_1(m) \cdot \boldsymbol{n}, \mu\rangle = -(\underline{\mathbf{Z}}_1(m), \underline{\mathbf{Z}}_1(\mu)) - \hat{t}^2(\mathbf{S}_1(m), \mathbf{S}_1(\mu)), \tag{9.68c}$$
$$\langle \underline{\mathbf{Z}}_2(\boldsymbol{\mu})\, \boldsymbol{n}, \boldsymbol{l}\rangle = (\underline{\mathbf{Z}}_2(\boldsymbol{\mu}), \underline{\mathbf{Z}}_2(\boldsymbol{r})) + \hat{t}^2(\mathbf{S}_2(\boldsymbol{\mu}), \mathbf{S}_2(\boldsymbol{l})), \tag{9.68d}$$
$$\langle \mathbf{S}_2(\boldsymbol{\mu}) \cdot \boldsymbol{n}, \mu\rangle = -(\underline{\mathbf{Z}}_2(\boldsymbol{\mu}), \underline{\mathbf{Z}}_1(\mu)) - \hat{t}^2(\mathbf{S}_2(\boldsymbol{\mu}), \mathbf{S}_1(\mu)), \tag{9.68e}$$
$$\langle \underline{\mathbf{Z}}_1(\mu)\, \boldsymbol{n}, \boldsymbol{\mu}\rangle = (\underline{\mathbf{Z}}_2(\boldsymbol{\mu}), \underline{\mathbf{Z}}_1(\mu)) + \hat{t}^2(\mathbf{S}_2(\boldsymbol{\mu}), \mathbf{S}_1(\mu)), \tag{9.68f}$$

which hold for all $m, \mu \in M_h$ and $\boldsymbol{\mu}, \boldsymbol{r} \in \boldsymbol{M}_h$.

Since the proof of the above identities are similar we only prove (9.68a), (9.68c) and (9.68e). To this end, we first note that by Proposition 2.1 we have that

$$\mathbf{S}_1(m),\ \mathbf{S}_2(\boldsymbol{\mu}) \in \boldsymbol{\Sigma}_h \cap \boldsymbol{Q}_h. \tag{9.69}$$

Then,

$$\begin{aligned}
(f, \mathsf{u}_1(\mu)) =& (\mathsf{u}_1(\mu), \nabla \cdot \mathbf{S}_3(f)) && \text{by (3.8e)}\\
=& -(\mathbf{Q}_1(\mu), \mathbf{S}_3(f)) + \langle \mu, \mathbf{S}_3(f) \cdot \boldsymbol{n}\rangle && \text{by (3.6a)}\\
=& -(\mathbf{R}_1(\mu), \mathbf{S}_3(f)) - (\mathbf{Q}_1(\mu) - \mathbf{R}_1(\mu), \mathbf{S}_3(f)) + \langle \mu, \mathbf{S}_3(f) \cdot \boldsymbol{n}\rangle &&\\
=& -(\mathbf{R}_1(\mu), \nabla \cdot \underline{\mathbf{Z}}_3(f)) + \hat{t}^2(\mathbf{S}_1(\mu), \boldsymbol{P}\mathbf{S}_3(f)) + \langle \mu, \mathbf{S}_3(f) \cdot \boldsymbol{n}\rangle && \text{by (3.8c), (3.6d)}\\
=& (\mathcal{A}\, \underline{\mathbf{Z}}_1(\mu), \underline{\mathbf{Z}}_3(f)) + (\underline{\mathbf{L}}_1(\mu), \underline{\mathbf{Z}}_3(f)) &&\\
& + \hat{t}^2(\mathbf{S}_1(\mu), \mathbf{S}_3(f)) + \langle \mu, \mathbf{S}_3(f) \cdot \boldsymbol{n}\rangle && \text{by (3.6b), (9.69)}\\
=& -(\mathbf{R}_3(f), \nabla \cdot \underline{\mathbf{Z}}_1(\mu)) - (\underline{\mathbf{L}}_3(f), \underline{\mathbf{Z}}_1(\mu)) &&\\
& + \hat{t}^2(\mathbf{S}_1(\mu), \mathbf{S}_3(f)) + \langle \mu, \mathbf{S}_3(f) \cdot \boldsymbol{n}\rangle && \text{by (3.8b), (3.8f)}\\
=& -(\mathbf{R}_3(f), \mathbf{S}_1(\mu)) + \hat{t}^2(\mathbf{S}_1(\mu), \mathbf{S}_3(f)) + \langle \mu, \mathbf{S}_3(f) \cdot \boldsymbol{n}\rangle && \text{by (3.6c), (3.6f)}\\
=& -(\mathbf{Q}_3(f), \mathbf{S}_1(\mu)) + \langle \mu, \mathbf{S}_3(f) \cdot \boldsymbol{n}\rangle && \text{by (3.8d), (9.69)}\\
=& (\mathsf{u}_3(f), \nabla \cdot \mathbf{S}_1(\mu)) + \langle \mu, \mathbf{S}_3(f) \cdot \boldsymbol{n}\rangle && \text{by (3.6c)}\\
=& \langle \mu, \mathbf{S}_3(f) \cdot \boldsymbol{n}\rangle. && \text{by (3.6e)}
\end{aligned}$$

This proves (9.68a).

Also,

$$\begin{aligned}
(\mathcal{A}\,\underline{\mathbf{Z}}_1(m), \underline{\mathbf{Z}}_1(\mu)) =& -(\mathbf{R}_1(\mu), \nabla\cdot\underline{\mathbf{Z}}_1(m)) - (\underline{\mathbf{L}}_1(\mu), \underline{\mathbf{Z}}_1(m)) && \text{by (3.6b)}\\
=& -(\mathbf{R}_1(\mu), \mathbf{S}_1(m)) && \text{by (3.6c), (3.6f)}\\
=& -(\mathbf{Q}_1(\mu), \mathbf{S}_1(m)) - \hat{t}^2(\mathbf{S}_1(\mu), \mathbf{S}_1(m)) && \text{by (3.6d), (9.69)}\\
=&(\mathsf{u}_1(\mu), \nabla\cdot(\mathbf{S}_1(m))) - \hat{t}^2(\mathbf{S}_1(\mu), \mathbf{S}_1(m))\\
& -\langle \mu, \mathbf{S}_1(m)\cdot\boldsymbol{n}\rangle && \text{by (3.6a)}\\
=& -\hat{t}^2(\mathbf{S}_1(\mu), \mathbf{S}_1(m)) - \langle \mu, \mathbf{S}_1(m)\cdot\boldsymbol{n}\rangle. && \text{by (3.6e)}
\end{aligned}$$

This proves (9.68c).

Next we prove (9.68e).

$$\begin{aligned}
(\mathcal{A}\,\underline{\mathbf{Z}}_2(\boldsymbol{\mu}), \underline{\mathbf{Z}}_1(\mu)) =& -(\mathbf{R}_1(\mu), \nabla\cdot\underline{\mathbf{Z}}_2(\boldsymbol{\mu})) - (\underline{\mathbf{L}}_1(\mu), \underline{\mathbf{Z}}_2(\boldsymbol{\mu})) && \text{by (3.6b)}\\
=& -(\mathbf{R}_1(\mu), \mathbf{S}_2(\boldsymbol{\mu})) && \text{by (3.7c), (3.7f)}\\
=& -\hat{t}^2(\mathbf{S}_1(\mu), \mathbf{S}_2(\boldsymbol{\mu}))\\
& +(\mathsf{u}_1(\mu), \nabla\cdot\mathbf{S}_2(\boldsymbol{\mu})) - \langle \mu, \mathbf{S}_2(\boldsymbol{\mu})\cdot\boldsymbol{n}\rangle && \text{by (3.6a), (3.6d), (9.69)}\\
=& -\hat{t}^2(\mathbf{S}_1(\mu), \mathbf{S}_2(\boldsymbol{\mu})) - \langle \mu, \mathbf{S}_2(\boldsymbol{\mu})\cdot\boldsymbol{n}\rangle && \text{by (3.7d).}
\end{aligned}$$

This proves (9.68e). This completes the proof of the theorem.

□

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