J. Numer. Math., Vol. 14, No. 1, pp. 41–56 (2006)

Local analysis of discontinuous Galerkin methods applied to singularly perturbed problems

J. GUZMÁN*

Received March 1, 2005
Revised Version January 30, 2006

Abstract — We analyze existing discontinuous Galerkin methods on quasi-uniform meshes for singularly perturbed problems. We prove weighted L_2 error estimates. We use the weighted estimates to prove L_2 error estimates in regions where the solution is smooth. We also prove pointwise estimates in these regions.

Keywords: Discontinuous Galerkin, singularly perturbed problems

1. INTRODUCTION

We consider the problem

$$\begin{aligned} -\varepsilon\Delta u + u_x + cu &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

Here $\Omega \subset R^2$ is a domain with smooth boundary, f is smooth, $\varepsilon > 0$ is a small constant and $c(x) \geqslant 0$ with $c \in L_\infty(\Omega)$.

As we know the solution to this problem might develop layers. The standard continuous Galerkin discretization of this problem propagates error through out the domain. The streamline diffusion (SD) method developed by Brooks and Hughes [3] does much better in resolving layers. In fact, the numerical outflow boundary layers for the streamline diffusion method are contained an $O(\log(1/h)h)$ neighborhood of the outflow boundary.

In this paper we show that some discontinuous Galerkin (DG) methods will perform as well as the SD method in resolving layers. The DG method for the pure hyperbolic problem proposed by Reed *et al.* [10] which was further analyzed in [5,7], showed good results for the pure hyperbolic problem. The error in L_2 was shown to be $O(h^{k+1/2})$ assuming the solution is smooth and polynomials of degree k are used. In fact, Peterson [9] argued that this result is sharp. A strategy that has been

*Center for Applied Mathematics, Cornell University, 657 Rhodes Hall, Ithaca NY 14853
This work was supported by a Ford Foundation Fellowship and a Cornell-Sloan Fellowship.

used for singularly perturbed problems is to use [10] to discretize the convection–reaction part and use different discontinuous discretizations for the diffusion part (see [2,11]). The authors of the above papers proved global error estimates assuming the solution is smooth. If u_h denotes the DG approximation to (1.1), then their estimates take on the following form:

$$\|u - u_h\|_\Omega \leqslant C h^{k+1/2} \|u\|_{H^{k+1}(\Omega)}.$$

However, in general $\|u\|_{H^{k+1}(\Omega)}$ is large and depends on ε. Therefore, in this paper we prove error estimates in subdomains where the solution is smooth. That is, we consider a subdomain $\Omega_0 \subset \Omega$ where u is smooth and prove that

$$\|u - u_h\|_{\Omega_0} \leqslant C \log(1/h) h^{k+1/2}$$

where now C does not depend on ε. Moreover, we show that Ω_0 can be any domain as long as $\partial\Omega_0$ is $C\log(1/h)h$ distance away from the outflow boundary of Ω for a sufficiently large constant C. These estimates are exactly the estimates that were obtained in [4,6] for the SD method. However, the estimates in [4,6] assumed that $\inf_{x\in\Omega} c(x) > 0$. Here we show that we can remove this assumption. We also prove suboptimal max-norm error estimates.

The paper is organized as follows. In Section 2 we present the numerical methods. Then, in Section 3 we present some preliminary results. In Section 4 we give the L_2 error analysis. Finally, in Section 5 we prove max-norm estimates.

2. DG METHODS

Suppose we have a family of edge to edge triangulations $\{\mathscr{T}_h\}$ of Ω, $h = \sup_{T\in\mathscr{T}_h} h_T$ with $h_T = \operatorname{diam}(T)$; V_h will denote the finite dimensional space of functions that are polynomials of degree at most k on each element. We define $\mathscr{E}_h^\partial$ as the collection of boundary edge, $\mathscr{E}_h^0$ as the collection of interior edges corresponding to $\mathscr{T}_h$, and $\mathscr{E}_h = \mathscr{E}_h^0 \cup \mathscr{E}_h^\partial$. We further decompose our boundary edges as $\mathscr{E}_h^\partial = \mathscr{E}_h^- \cup \mathscr{E}_h^+$ where

$$\mathscr{E}_h^- = \{e \in \mathscr{E}_h^\partial : (n\cdot[1,0])|_e \leqslant 0\}$$

and

$$\mathscr{E}_h^+ \{e \in \mathscr{E}_h^\partial : (n\cdot[1,0])|_e > 0\}$$

where n is the outward unit normal to $\partial\Omega$. Notice that we have made the following assumption on the mesh: if $n\cdot[1,0] > 0$ for one point $(x,y)\in e$ for some $e\in\mathscr{E}_h^\partial$, then $n\cdot[1,0] > 0$ on all of e. We denote by S_e the union of elements that have e as an edge. On each edge, as in [1] we define the average and jump operators as follows: For $e\in\mathscr{E}_h^0$, q vector valued and φ scalar valued

$$\langle q\rangle = \tfrac{1}{2}(q_1 + q_2), \qquad [q] = q_1\cdot n_1 + q_2\cdot n_2$$
$$\langle \varphi\rangle = \tfrac{1}{2}(\varphi_1 + \varphi_2), \qquad [\varphi] = \varphi_1 n_1 + \varphi_2 n_2$$

where $S_e = T_1 \cup T_2$, $q_i = q|_{T_1}$, $\varphi_i = \varphi|_{T_i}$ and n_i is the outward normal to T_i, $i = 1,2$, for $e \in \mathscr{E}_h^\partial$

$$\langle q \rangle = q, \qquad [\varphi] = \varphi n.$$

The quantities $[q]$ and $\langle \varphi \rangle$ on boundary edges are not required so they are left undefined. Note that $[q]$ is a scalar and $[\varphi]$ a vector. If $e \in \mathscr{E}_h^0$, then, as in [5], $u^\pm(x,y) = \lim_{\delta \to 0} u(x \pm \delta, y)$ for $(x,y) \in e$.

Now we are ready to define our bilinear forms. First we write the bilinear form corresponding to the classical convection–reaction discretization:

$$\begin{aligned} B_1(u,v) \;&=\; \sum_{T \in \mathscr{T}_h} \int_T (u_x + cu)v \,\mathrm{d}x + \sum_{e \in \mathscr{E}_h^0} \int_e (u^+ - u^-)v^+ |n_x| \,\mathrm{d}s \\ &\quad + \sum_{e \in \mathscr{E}_h^-} \int_e uv|n_x| \,\mathrm{d}s. \end{aligned}$$

Using integration by parts we have the following

$$\begin{aligned} B_1(u,v) \;&=\; \sum_{T \in \mathscr{T}_h} \int_T u(-v_x + cv) \,\mathrm{d}x + \sum_{e \in \mathscr{E}_h^0} \int_e u^-(v^- - v^+)|n_x| \,\mathrm{d}s \\ &\quad + \sum_{e \in \mathscr{E}_h^+} \int_e uv|n_x| \,\mathrm{d}s. \end{aligned}$$

The bilinear form corresponding to the diffusion discretization is

$$\begin{aligned} B_2(u,v) \;&=\; \sum_{T \in \mathscr{T}_h} \varepsilon \int_T \nabla u \nabla v \,\mathrm{d}x - \sum_{e \in \mathscr{E}_h} \int_e \varepsilon(\langle \nabla_h u \rangle [v] + \gamma \langle \nabla_h v \rangle [u]) \,\mathrm{d}s \\ &\quad + \sum_{e \in \mathscr{E}_h} \eta \frac{\varepsilon}{h_e} \int_e [u][v] \,\mathrm{d}s. \end{aligned}$$

Here $\nabla_h \varphi$ is the piecewise defined function such that $\nabla_h \varphi = \nabla \varphi$ on each $T \in \mathscr{T}_h$. With a slight abuse of notation, from now on we let φ_x denote the piecewise defined function $\nabla_h \varphi \cdot [1,0]$ and $\varphi_y = \nabla_h \varphi \cdot [0,1]$. Also, n_x is the first coordinate of a unit normal to each edge (i.e. $n = [n_x, n_y]$ where n is a unit normal). If $\gamma = -1$ and $\eta > 0$, then we have the NIPG method which was considered in [11]. If $\gamma = 1$ and η sufficiently large, gives the IP method which was considered in [2].

The discontinuous approximation u_h is defined by

$$B(u_h, v) = (f, v) \quad \forall v \in V_h \tag{2.1}$$

where $B = B_1 + B_2$.

2.1. Approximation results

We first state trace and inverse inequalities. Let $T \in \mathscr{T}_h$ then we have

$$\|\psi\|_{\partial T} \leqslant C(h^{-1/2}\|\psi\|_T + h^{1/2}\|\nabla \psi\|_T) \tag{2.2}$$

where C is independent of T and ψ.

If $\psi \in V_h$, then

$$\|\nabla\psi\|_T \leqslant Ch^{-1}\|\psi\|_T \tag{2.3}$$

$$\|\psi\|_{\partial T} \leqslant Ch^{-1/2}\|\psi\|_T \tag{2.4}$$

and

$$\|\psi\|_{\partial T} \leqslant C(h^{1/2}\|\psi_x\|_T + \|\psi|n_x|^{1/2}\|_{\partial T}). \tag{2.5}$$

We used the following notation $\|\cdot\|_D = \|\cdot\|_{L_2(D)}$. The last inequality was used in [5].

Now we present a preliminary cut-off function. This function will differ from the one in [6] in order to handle the case that c is not bounded away from zero from below (e.g. no reaction term). One can construct a function with the following properties:

There exist positive constants c_1 and c_2 such that

$$c_1 \leqslant \varphi(t) \leqslant c_2, \quad t \leqslant 1$$

$$\varphi(t) = e^{-t}, \quad t \geqslant 0$$

$$\varphi(t) = 3 - \frac{1}{\log(|t|)+1}, \quad t \leqslant -1$$

$$\varphi'(t) < 0, \quad t \in (-\infty, \infty)$$

$$|\varphi^l(t)| \leqslant c_2|\varphi(t)|, \quad 1 \leqslant l \leqslant k+1, \quad t \in (-\infty, \infty)$$

$$|\varphi^l(t)| \leqslant c_2|\varphi'(t)|, \quad 2 \leqslant l \leqslant k+1, \quad t \in (-\infty, \infty)$$

and

$$|\varphi(t)| \leqslant c_2|t|(\log(|t|)+1)^2|\varphi'(t)|, \quad t \in (-\infty, \infty).$$

If we define $RO(D,v) = \max_{x\in D}|v(x)|/\min_{x\in D}|v(x)|$, then for any interval I of length 1

$$RO(I,\varphi) + RO(I,\varphi') \leqslant c_2.$$

We define our cut-off function

$$\omega(x,y) = \varphi\Big(\frac{x-A}{\rho}\Big)\varphi\Big(\frac{z_1-y}{\sigma}\Big)\varphi\Big(\frac{y-z_2}{\sigma}\Big).$$

Here $\rho = Kh\log(1/h)$ and $\sigma = Kh^{1/2}\log(1/h)$, where K is a sufficiently large constant that will be chosen later.

From the properties above it follows that $\omega_x < 0$ and that

$$|D_x^\alpha D_y^\beta \omega| \leqslant C\rho^{-\alpha}\sigma^{-\beta}|\omega|, \quad \alpha+\beta \leqslant k+1 \tag{2.6}$$

$$|D_x^\alpha D_y^\beta \omega| \leqslant C\rho^{-\alpha+1}\sigma^{-\beta}|\omega_x|, \quad \alpha \geqslant 1, \quad \alpha+\beta \leqslant k+1 \tag{2.7}$$

$$|\omega| \leqslant C(\log(1/h))^2|\omega_x| \tag{2.8}$$

$$RO(T,\omega) \text{ and } RO(T,\omega_x) \qquad \text{are bounded independently of } h \text{ on any element } T. \tag{2.9}$$

Property (2.8) makes it possible to handle the case of no reaction term. In fact, we can use this same cut-off function to prove error estimates for the SD method in the absence of a reaction term.

Now we can define a weighted norm

$$\begin{aligned} Q(v) \equiv \Big\{ & \varepsilon\|\omega\nabla_h v\|_\Omega^2 + \|(\omega|\omega_x|)^{1/2}v\|_\Omega^2 + \|\omega c^{1/2}v\|_\Omega^2 \\ & + \frac{1}{2}\sum_{e\in\mathscr{E}_h^0}\|\omega(v^+ - v^-)|n_x|^{1/2}\|_e^2 + \frac{1}{2}\sum_{e\in\mathscr{E}_h^\partial}\|\omega v|n_x|^{1/2}\|_e^2 \\ & + \sum_{e\in\mathscr{E}_h}\eta\frac{\varepsilon}{h_e}\|\omega[v]\|_e^2\Big\}^{1/2}. \end{aligned}$$

The following super-approximation result is similar to the super-approximation result found in [6].

Lemma 2.1. *There exists a constant C such that for $v \in V_h$ and $T \in \mathscr{T}_h$*

$$\begin{aligned} & h\|\omega^{-1}\nabla^2(\omega^2 v - P(\omega^2 v))\|_T + \|\omega^{-1}\nabla(\omega^2 v - P(\omega^2 v))\|_T \\ & \quad + \frac{1}{h}\|\omega^{-1}(\omega^2 v - P(\omega^2 v))\|_T \leqslant Ch^{-1/2}K^{-1/2}\|(\omega|\omega_x|)^{1/2}v\|_T \end{aligned} \tag{2.10}$$

where P denotes the L_2 projection operator into V_h.

Proof. By approximation properties of P we know that

$$\begin{aligned} & h\|\nabla^2(\omega^2 v - P(\omega^2 v))\|_T + \|\nabla(\omega^2 v - P(\omega^2 v))\|_T \\ & \quad + \frac{1}{h}\|\omega^2 v - P(\omega^2 v)\|_T \leqslant Ch^k \sum_{|\alpha|+|\beta|+|\gamma|=k+1}\|D^\alpha\omega D^\beta\omega D^\gamma v\|_T. \end{aligned}$$

Note that $D^\gamma v = 0$ if $|\gamma| = k+1$ since v is in our subspace. Therefore, we assume that $|\gamma| \leqslant k$. First suppose that $\alpha_2 + \beta_2 \neq 0$, where $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$. In this case, by using (2.6) and (2.3), we have

$$\|D^\alpha\omega D^\beta\omega D^\gamma v\|_T \leqslant C\rho^{-\alpha_1-\beta_1}\sigma^{-\alpha_2-\beta_2}h^{-|\gamma|}\|\omega^2 v\|_T.$$

Using the definition of ρ and σ, we have that

$$\rho^{-\alpha_1-\beta_1}\sigma^{-\alpha_2-\beta_2}h^{|\gamma|} = K^{-|\alpha|-|\beta|}\log(1/h)^{-|\alpha|-|\beta|}h^{-k-1+(\alpha_2+\beta_2)/2}.$$

Since $|\alpha|+|\beta|=k+1-|\gamma|\geqslant 1$, $\alpha_2+\beta_2\neq 0$ and by using (2.8), we have

$$h^k\|D^\alpha\omega D^\beta\omega D^\gamma v\|_{L_2(T)}\leqslant CK^{-1}h^{-1/2}\|\omega|\omega\omega_x|^{1/2}v\|_T.$$

On the other hand, suppose $\alpha_2+\beta_2=0$. By (2.6) and (2.7) we have

$$\|D^\alpha\omega D^\beta\omega D^\gamma v\|_T\leqslant C\rho^{-\alpha_1-\beta_1+1/2}h^{-|\gamma|}\|\omega|\omega\omega_x|^{1/2}v\|_T.$$

Again using the definition of ρ, we have

$$h^k\|D^\alpha\omega D^\beta\omega D^\gamma v\|_T\leqslant K^{-1/2}h^{-1/2}\|\omega|\omega\omega_x|^{1/2}v\|_T.$$

The result now follows by multiplying through by ω^{-1} and using (2.9). $\square$

3. MAIN RESULT

We can now prove our main result.

Theorem 3.1. *Let $\psi=P(u)-u_h$ where u_h solves* (2.1) *for either the* NIPG *or* IP *methods. Let K be sufficiently large. If $\varepsilon\leqslant h$, then there exists a constant C such that*

$$\begin{aligned}Q^2(\psi)\ &\leqslant\ C\big(h^{-1}\|\omega(u-P(u))\|_\Omega^2+h\|\omega\nabla_h(u-P(u))\|_\Omega^2\\&\qquad+h^3\|\omega\nabla_h^2(u-P(u))\|_\Omega^2\big).\end{aligned}$$

Proof. It can easily be shown that

$$\begin{aligned}Q^2(\psi)\ &=\ B(\psi,\omega^2\psi)-2\varepsilon\sum_{T\in\mathscr{T}_h}\int_T\omega\psi\nabla\omega\nabla\psi\,\mathrm{d}x\\&\quad+(1+\gamma)\varepsilon\sum_e\int_{e\in\mathscr{E}_h}\langle\omega\nabla_h\psi\rangle[\omega\psi]\,\mathrm{d}s\\&\quad+2\gamma\varepsilon\sum_e\int_{e\in\mathscr{E}_h}\langle\psi\nabla\omega\rangle[\omega\psi]\,\mathrm{d}s.\end{aligned}$$

By (2.6) and (2.8), we have

$$\left|\varepsilon\sum_{T\in\mathscr{T}_h}\int_T\omega\psi\nabla\omega\nabla\psi\,\mathrm{d}x\right|\leqslant CK^{-1/2}(\varepsilon\|\omega\nabla_h\psi\|_\Omega^2+\|(\omega|\omega_x|)^{1/2}\psi\|_\Omega^2).$$

Using (2.9), (2.4), and the arithmetic–geometric mean inequality we obtain

$$\begin{aligned}(1+\gamma)\varepsilon\sum_{e\in\mathscr{E}_h}\int_e\langle\omega\nabla_h\psi\rangle[\omega\psi]\,\mathrm{d}s\ &\leqslant\ (1+\gamma)\frac{1}{8}\varepsilon\|\omega\nabla_h\psi\|_\Omega^2\\&\quad+(1+\gamma)C^*\sum_{e\in\mathscr{E}_h}\frac{\varepsilon}{h_e}\|\omega[\psi]\|_e^2.\end{aligned}\tag{3.1}$$

Here C^* depends on the constant arising from inverse estimates.

Again, by (2.6), (2.9), (2.8) and (2.4), we see that

$$\left|\varepsilon \sum_{e\in\mathscr{E}_h} \int_e \langle \psi \nabla\omega\rangle [\omega\psi]\,\mathrm{d}s\right| \leqslant CK^{-1/2}\Big(\sum_{e\in\mathscr{E}_h} \frac{\varepsilon}{h_e}\|[\omega\psi]\|_e^2 + \|(\omega|\omega_x|)^{1/2}\psi\|_\Omega^2\Big).$$

If we have $\gamma = -1$ (NIPG), then both sides of (3.1) will be zero. In this case, by making K large enough we have that

$$Q^2(\psi) \leqslant CB(\psi,\omega^2\psi). \tag{3.2}$$

On the other hand if, $\gamma = 1$ (IP) then as long as $\eta > 2C^*$ and K large enough we again have (3.2).

Using the orthogonality property, we see that

$$B(\psi,\omega^2\psi) = B(\psi,E) + B(u-P(u),P(\omega^2\psi)) \tag{3.3}$$

where $E = \omega^2\psi - P(\omega^2\psi)$. We first bound $B(\psi,E) = B_1(\psi,E) + B_2(\psi,E)$.

Since $\psi_x \in V_h$,

$$\begin{aligned} B_1(\psi,E) &= \sum_{e\in\mathscr{E}_h^0} \int_e (\psi^+ - \psi^-)E^+|n_x|\,\mathrm{d}s + \sum_{e\in\mathscr{E}_h^-} \int_e \psi E|n_x|\,\mathrm{d}s \\ &\quad + \int_\Omega c\psi E\,\mathrm{d}x. \end{aligned}$$

By using the Cauchy–Schwarz inequality, (2.2), (2.10), and the boundedness of c, we get

$$\begin{aligned} B_1(\psi,E) &\leqslant C\sum_{e\in\mathscr{E}_h^0} \|\omega(\psi^+ - \psi^-)|n_x|^{1/2}\|_e K^{-1/2}\|(\omega|\omega_x|)^{1/2}\psi\|_{S_e} \\ &\quad + Ch^{1/2}K^{-1/2}\|\omega c^{1/2}\psi\|_{L_2(\Omega)}\|(\omega|\omega_x|)^{1/2}\psi\|_{L_2(\Omega)}. \end{aligned}$$

Therefore,

$$B_1(\psi,E) \leqslant CK^{-1/2}Q^2(\psi). \tag{3.4}$$

Using (2.2) and (2.10), we see that

$$|B_2(\psi,E)| \leqslant CK^{-1/2}Q^2(\psi). \tag{3.5}$$

We only bound one of the terms of $|B_2(\psi,E)|$ to illustrate this.

$$\begin{aligned} \sum_{e\in\mathscr{E}_h} \varepsilon \int_e \langle \nabla_h E\rangle [\psi]\,\mathrm{d}s &\leqslant C\varepsilon \sum_{e\in\mathscr{E}_h} (h^{-1/2}\|\omega^{-1}\nabla_h E\|_{S_e} + h^{1/2}\|\omega^{-1}\nabla_h^2 E\|_{S_e})\|[\omega\psi]\|_e \\ &\leqslant C\sum_{e\in\mathscr{E}_h} K^{-1/2}\|(\omega|\omega_x|)^{1/2}\psi\|_{S_e}\Big(\frac{\varepsilon}{h_e}\Big)^{1/2}\|[\omega\psi]\|_e \\ &\leqslant CK^{-1/2}Q^2(\psi). \end{aligned}$$

Now we bound $B(u-P(u),P(\omega^2\psi))$. Since $P(\omega^2\psi)\in V_h$, we have that

$$\begin{aligned}B_1(u-P(u),P(\omega^2\psi)) &= \sum_{e\in\mathscr{E}_h^0}\int_e (u-P(u))^-(P(\omega^2\omega)^- - P(\omega^2\psi)^+)|n_x|\,\mathrm{d}s\\ &\quad+\sum_{e\in\mathscr{E}_h^+}\int_e (u-P(u))P(\omega^2\psi)|n_x|\,\mathrm{d}s\\ &\quad+\int_\Omega c(u-P(u))P(\omega^2\psi)\,\mathrm{d}x.\end{aligned}$$

Applying (2.4), the triangle inequality, (2.10) and the arithmetic–geometric mean inequality, we have

$$\begin{aligned}&\sum_{e\in\mathscr{E}_h^0}\int_e (u-P(u))^-(P(\omega^2\psi)^- - P(\omega^2\psi)^+)|n_x|\,\mathrm{d}s\\ &\quad\leqslant \sum_{e\in\mathscr{E}_h^0}\big\{(h^{-1/2}\|\omega(u-P(u))\|_{S_e}+h^{1/2}\|\omega\nabla_h(u-P(u))\|_{S_e})\\ &\qquad\qquad\times(\|\omega^{-1}((\omega^2\psi)^+-(\omega^2\psi)^-)|n_x|^{1/2}\|_e+\|\omega^{-1}(E^+-E^-)|n_x|^{1/2}\|_e)\big\}\\ &\quad\leqslant C(h^{-1}\|\omega(u-P(u))\|_\Omega^2+h\|\omega\nabla_h(u-P(u))\|_\Omega^2)\\ &\qquad+\delta\sum_{e\in\mathscr{E}_h^0}\|\omega(\psi^- -\psi^+)|n_x|^{1/2}\|_e^2+K^{-1}Q^2(\psi)\end{aligned}$$

where $\delta>0$ will be chosen later. After similar arguments for the last terms in $B_1(u-P(u),P(\omega^2\psi))$, we conclude that

$$\begin{aligned}B_1(u-P(u),P(\omega^2\psi)) &\leqslant C(h^{-1}\|\omega(u-P(u))\|_\Omega^2+h\|\omega\nabla_h(u-P(u))\|_\Omega^2)\\ &\quad+\delta Q^2(\psi)+K^{-1}Q^2(\psi).\end{aligned}\tag{3.6}$$

We now bound the terms of $B_2(u-P(u),P(\omega^2\psi))$. By the Cauchy–Schwarz inequality, the triangle inequality, (2.6) and (2.10), we see that

$$\begin{aligned}&\sum_{T\in\mathscr{T}_h}\varepsilon\int_T\nabla(u-P(u))\nabla P(\omega^2\psi)\,\mathrm{d}x\\ &\quad\leqslant\sum_{T\in\mathscr{T}_h}\varepsilon\|\omega\nabla(u-P(u))\|_T(\|\omega^{-1}\nabla E\|_T+\|\omega^{-1}\nabla(\omega^2\psi)\|_T)\\ &\quad\leqslant C\varepsilon\|\omega\nabla_h(u-P(u))\|_\Omega^2+\delta(\varepsilon\|\omega\nabla_h\psi\|_\Omega^2+\|(\omega|\omega_x|)^{1/2}\psi\|_{L_2(T)})\\ &\qquad+K^{-1}Q^2(\psi).\end{aligned}$$

Again, applying the Cauchy–Schwarz, the triangle inequality, (2.2), (2.10) and the arithmetic–geometric mean inequality, we get that

$$\begin{aligned}
&\sum_{e\in\mathscr{E}_h} \varepsilon \int_e \langle \nabla_h(u-P(u))\rangle [P(\omega^2\psi)]\,\mathrm{d}s \\
&\quad \leqslant \frac{\varepsilon}{h}\sum_{e\in\mathscr{E}_h}\big(h^{1/2}\|\omega\nabla_h(u-P(u))\|_{S_e}+h^{3/2}\|\omega\nabla_h^2(u-P(u))\|_{S_e}\big) \\
&\qquad\qquad \times(\|[\omega\psi]\|_e+\|\omega^{-1}[E]\|_e) \\
&\quad \leqslant C(h\|\omega\nabla_h(u-P(u))\|_\Omega^2+h^3\|\omega\nabla_h^2(u-P(u))\|_\Omega^2) \\
&\qquad + K^{-1}Q^2(\psi)+\delta\sum_{e\in\mathscr{E}_h}\frac{\varepsilon}{h_e}\|[\omega\psi]\|_e^2.
\end{aligned}$$

By bounding the last two terms of $B_2(u-P(u),P(\omega^2\psi))$ in a similar fashion, we arrive at

$$\begin{aligned}
B_2(u-P(u),P(\omega^2\psi)) \leqslant\ & C\big(h^{-1}\|\omega(u-P(u))\|_\Omega^2+\|\omega\nabla(u-P(u))\|_\Omega^2 \\
& +h^3\|\omega\nabla^2(u-P(u))\|_\Omega^2\big)+4\delta Q^2(\psi)+4K^{-1}Q^2(\psi). \quad (3.7)
\end{aligned}$$

Finally, taking K large enough in (3.4), (3.5), (3.6), (3.7), and choosing δ sufficiently small in (3.6) and (3.7) we arrive at our result. □

Remark 3.1. By using Lemma 4.1, we can improve Theorem 3.1 so that $Q^2(\psi)$ also contains the term $\sum_{T\in\mathscr{T}_h} h\|\omega\psi_x\|_T^2$.

Now we can state a error estimate away from the layers.

Corollary 3.1. Let K, ρ and σ be as in Theorem 3.1. Let

$$\Omega_0 = \{(x,y)\in\Omega : x\leqslant A,\ z_1\leqslant y\leqslant z_2\}$$

and

$$\Omega_s^+ = \big\{(x,y)\in\Omega : x\leqslant A+s\log(1/h)\rho,\ z_1-s\log(1/h)\sigma\leqslant y\leqslant z_2+s\log(1/h)\sigma\big\}.$$

Let h_0 and m be such that $h_0^m\leqslant\varepsilon$. If $\|u\|_{H^2(\Omega)}\leqslant C\varepsilon^{-2}$ and $\|u\|_{H^k(\Omega^+_{k+1+2m})}<C$, then

$$\|u-u_h\|_{\Omega_0}\leqslant C\log(1/h)h^{k+1/2},\qquad h\leqslant h_0.$$

Proof. By the triangle inequality and the properties of the L_2-projection operator, it is enough to establish

$$\|P(u)-u_h\|_{\Omega_0}\leqslant C\log(1/h)h^{k+1/2},\qquad h\leqslant h_0.$$

It follows form the properties of ω, that

$$\|P(u)-u_h\|_{\Omega_0} \leqslant C\|\omega(P(u)-u_h)\|_{\Omega} \leqslant C\log(1/h)\|(\omega|\omega_x|)^{1/2}(P(u)-u_h)\|_{\Omega}.$$

Therefore, from Theorem 3.1 and properties of ω, we have

$$\begin{aligned}\|P(u)-u_h\|_{\Omega_0} \leqslant\ & C\log(1/h)(h^{-1/2}\|u-P(u)\|_{\Omega_s^+} \\ & + h^{1/2}\|\nabla_h(u-P(u))\|_{\Omega_s^+} + h^{3/2}\|\nabla_h^2(u-P(u))\|_{\Omega_s^+}) \\ & + C\log(1/h)h^s(h^{-1/2}\|u-P(u)\|_{\Omega} \\ & + h^{1/2}\|\nabla_h(u-P(u))\|_{\Omega} + h^{3/2}\|\nabla_h^2(u-P(u))\|_{\Omega}).\end{aligned}$$

From approximation properties, we have

$$\begin{aligned}\big(h^{-1/2}\|u-P(u)\|_{\Omega_s^+} + h^{1/2}\|\nabla_h(u-P(u))\|_{\Omega_s^+} \\ + h^{3/2}\|\nabla_h^2(u-P(u))\|_{\Omega_s^+}\big) \leqslant h^{k+1/2}\|u\|_{H^k(\Omega_s^+)}.\end{aligned}$$

Using the triangle inequality and inverse estimates, we see that

$$\begin{aligned}\big(h^{-1/2}\|u-P(u)\|_{\Omega} + h^{1/2}\|\nabla_h(u-P(u))\|_{\Omega} \\ + h^{3/2}\|\nabla_h^2(u-P(u))\|_{\Omega}\big) \leqslant Ch^{-1/2}\|u\|_{H^2(\Omega)}.\end{aligned}$$

The result now follows by letting $s = k+1+2m$. □

Remark 3.2. In the case that $\inf_{x\in\Omega} c(x) > 0$, we can show that $\|u-u_h\|_{\Omega_0} \leqslant Ch^{k+1/2}$.

In the next section we will need a weighted stability estimate. By following the ideas of Theorem 3.1 we can prove the following theorem.

Theorem 3.2. *Let u_h solve* (2.1) *for either the* NIPG *or* IP *methods. Let K be sufficiently large. If $\varepsilon \leqslant h$, then there exists a constant C such that*

$$Q(u_h) \leqslant C\|\omega f\|_{\Omega}.$$

Here C is independent of h, u_h and f.

4. APPROXIMATE GREEN'S FUNCTION BOUNDS AND L_∞ ESTIMATES

In this section we prove suboptimal L_∞ bounds. In order to do so, we need bounds on the approximate Green's function. In this direction, for $(x_0,y_0) \in \Omega$ define the rectangle containing (x_0,y_0)

$$\Omega_0 = \big\{(x,y) \in \Omega : x \leqslant x_0 + C_1\log(1/h)\rho,\ |y-y_0| \leqslant C_1\log(1/h)\sigma\big\}.$$

Here C_1 is a sufficiently large constant which we specify below. The approximate Green's function $G \in V_h$ with reversed wind direction satisfies

$$B(v, G) = v(x_0, y_0) \quad \forall v \in V_h.$$

Using Theorem 3.2 (with the wind direction reversed) and applying the techniques used in [6] we can prove the following estimate.

Corollary 4.1. There exists a constant C_1 (in the definition of Ω_0) independent of h such that

$$\|G\|_{L_\infty(\widetilde{\Omega\setminus\Omega_0})} + \|\nabla_h G\|_{L_\infty(\widetilde{\Omega\setminus\Omega_0})} \leqslant Ch^{k+2}.$$

We used the notation

$$\widetilde{D} = \bigcup_{\substack{T \in \mathscr{T}_h \\ T \cap D \neq \varnothing}} T.$$

In order to prove pointwise estimates, we need a global bound on G. This requires an extra stability estimate. The following result was proving for the IP method (reversed wind direction and $c(x) \equiv 0$) in Lemma A.1 in [2]. The proof for the NIPG method is similar.

Lemma 4.1. *There exist positive constants C_2 and C_3 such that for every $v \in V_h$*

$$\begin{aligned}
&h\|v_x\|_\Omega^2 + \varepsilon\|v_y\|_\Omega^2 + \|c^{1/2}v\|_\Omega^2 \\
&\quad + \frac{1}{2}\sum_{e \in \mathscr{E}_h^0} \|(v^+ - v^-)|n_x|^{1/2}\|_e^2 + \frac{1}{2}\sum_{e \in \mathscr{E}_h^\partial} \|v|n_x|^{1/2}\|_e^2 \\
&\quad + \sum_{e \in \mathscr{E}_h} \eta\frac{\varepsilon}{h_e}\|\omega[v]\|_e^2 \leqslant C_2 B(C_3 v - h v_x, v).
\end{aligned}$$

We will also need the following lemma.

Lemma 4.2. *Suppose $v \in V_h$ and suppose that $T_1, T_2 \in \mathscr{T}_h$ share a common edge e. Then,*

$$\|v^1 - v^2\|_e \leqslant Ch^{1/2}(\|v_x^1\|_{T_1} + \|v_x^2\|_{T_2}) + C\|(v^1 - v^2)|n_x|^{1/2}\|_{\partial T_1}$$

where $v^1 = v|_{T_1}$ and $v^2 = v|_{T_2}$.

Proof. We extend v^1 and v^2 to all of R^2 in the natural way. By (2.5), we have

$$\begin{aligned}
\|v^1 - v^2\|_e &\leqslant Ch^{1/2}\|v_x^1 - v_x^2\|_{T_1} + C\|(v^1 - v^2)|n_x|^{1/2}\|_{\partial T_1} \\
&\leqslant Ch^{1/2}(\|v_x^1\|_{T_1} + \|v_x^2\|_{T_1}) + C\|(v^1 - v^2)|n_x|^{1/2}\|_{\partial T_1}.
\end{aligned}$$

Since v^2 lies in a finite dimensional space and T_1 and T_2 belong to a shape regular mesh and share a common edge we have

$$\|v_x^2\|_{T_1} \leqslant C\|v_x^2\|_{T_2}.$$

This completes the proof. $\square$

Our proof of global estimates for G is very similar to the proof given by Niijima [8] for the streamline diffusion method.

Theorem 4.1. *There exists a constant C independent of h such that*

$$\begin{aligned}&\|c^{1/2}G\|_\Omega^2 + h\|G_x\|_\Omega^2 + \varepsilon\|G_y\|_\Omega^2 \\ &\quad + \sum_{e\in\mathscr{E}_h^0} \|(G^+ - G^-)|n_x|^{1/2}\|_e^2 + \sum_{e\in\mathscr{E}_h^\partial} \|G|n_x|^{1/2}\|_e^2 \\ &\quad + \sum_{e\in\mathscr{E}_h} \eta\frac{\varepsilon}{h_e}\|\omega[G]\|_e^2 \leqslant C\log(1/h)^2 h^{-1}.\end{aligned}$$

Proof. By Lemma 4.1 we have

$$\begin{aligned}&h\|G_x\|_\Omega^2 + \varepsilon\|G_y\|_\Omega^2 + \|c^{1/2}G\|_\Omega^2 \\ &\quad + \frac{1}{2}\sum_{e\in\mathscr{E}_h^0} \|(G^+ - G^-)|n_x|^{1/2}\|_e^2 + \frac{1}{2}\sum_{e\in\mathscr{E}_h^\partial} \|G|n_x|^{1/2}\|_e^2 \\ &\quad + \sum_{e\in\mathscr{E}_h} \eta\frac{\varepsilon}{h_e}\|\omega[G]\|_e^2 \leqslant C_2 B(C_3G - hG_x, G) = C_2(C_3G(x_0,y_0) - hG_x(x_0,y_0)).\end{aligned}$$

First, by an inverse estimate and the arithmetic–geometric mean inequality, we have

$$hG_x(x_0,y_0) \leqslant \|G_x\|_\Omega \leqslant \delta h\|G_x\|_\Omega^2 + Ch^{-1}$$

where $\delta > 0$ will be chosen later.

Let $(x_m,y_0) \in \Omega\backslash\Omega_0$ such that $|x_m - x_0| \leqslant C\log(1/h)\rho$. If we now draw the line from (x_0,y_0) to (x_m,y_0), then this line will intersect the elements $T_0,T_1,\ldots,T_m$ at the points $(x_1,y_0),(x_2,y_0),\ldots,(x_m,y_0)$, respectively. By adding and subtracting the right and left hand limits of G at the points (x_i,y_0) and applying the Fundamental Theorem of calculus, we have

$$-G(x_0,y_0) = \sum_{i=0}^{m-1}\int_{x_i}^{x_{i+1}} G_x(s,y_0)\,\mathrm{d}s + \sum_{i=1}^{m-1}(G^+(x_i,y_0) - G^-(x_i,y_0)) + G^-(x_m,y_0). \tag{4.1}$$

By an inverse estimate, we know that

$$\int_{x_i}^{x_{i+1}} G_x(s,y_0)\,\mathrm{d}s \leqslant (x_{i+1} - x_i)\|G_x\|_{L_\infty(T_i)} \leqslant h^{-1}\|G_x\|_{L_1(T_i)}.$$

Furthermore, since $\mathrm{meas}(T_0 \cup T_1 \cup \cdots \cup T_m) \leqslant Ch(\log(1/h)\rho)$, we have that

$$\sum_{i=0}^{m-1} \int_{x_i}^{x_{i+1}} G_x(s,y_0)\,\mathrm{d}s \leqslant C\log(1/h)\left(\sum_{i=1}^{m-1} \|G_x\|_{T_i}^2\right)^{1/2}.$$

Applying the arithmetic–geometric mean inequality, we have

$$\sum_{i=0}^{m-1} \int_{x_i}^{x_{i+1}} G_x(s,y_0)\,\mathrm{d}s \leqslant \delta h \sum_{T\in\mathscr{T}_h} \|G_x\|_T^2 + C\log(1/h)^2 h^{-1}.$$

Using inverse estimates on the edges, we have

$$G^+(x_i,y_0) - G^-(x_i,y_0) \leqslant Ch^{-1/2}\|(G^+ - G^-)\|_{e_i}$$

where $e_i \subset \partial T_i$ is an edge containing (x_i,y_0). By Lemma 4.2, we have

$$G^+(x_i,y_0) - G^-(x_i,y_0) \leqslant Ch^{-1/2}\|(G^+ - G^-)|n_x|^{1/2}\|_{\partial T_i} + C(\|G_x\|_{T_i} + \|G_x\|_{T_{i+1}}).$$

Therefore,

$$\sum_{i=1}^{m-1}(G^+(x_i,y_0) - G^-(x_i,y_0)) \leqslant Ch^{-1/2}\sum_{i=1}^{m-1} \|(G^+ - G^-)|n_x|^{1/2}\|_{\partial T_i} + C\sum_{i=1}^{m-1} \|G_x\|_{T_i}.$$

Applying the arithmetic–geometric mean inequality we have

$$\begin{aligned}\sum_{i=1}^{m-1}(G^+(x_i,y_0) - G^-(x_i,y_0)) \;\leqslant\;& \delta \sum_{i=1}^{m-1} \|(G^+ - G^-)|n_x|^{1/2}\|_{\partial T_i}^2 + \delta h \sum_{i=1}^{m-1} \|G_x\|_{T_i}^2 \\ &+ C\log(1/h)^2 h^{-1}.\end{aligned}$$

Here we used that $m \leqslant C\log(1/h)$.

By Corollary 4.1, we have

$$G^-(x_m,y_0) \leqslant Ch^{k+2}.$$

By choosing δ sufficiently small we arrive at our result. □

Now we prove pointwise estimates.

Theorem 4.2. *Assume that*

$$\|u\|_{C^{k+1}(\widetilde{\Omega_0})} + \|\nabla u\|_{L_1(\Omega)} + \varepsilon\|\nabla^2 u\|_{L_1(\Omega)} + \|u\|_{\Omega} \leqslant C.$$

If $\varepsilon \leqslant h$, then

$$|(u-u_h)(x_0,y_0)| \leqslant Ch^{k+1/4}\log(1/h)^2.$$

Proof. By the definition of G and the orthogonality property of $u-u_h$, we have

$$(u_h - P(u))(x_0,y_0) = B(u_h - P(u), G) = B(u - P(u), G).$$

One can show using the Cauchy–Schwarz inequality, inverse estimates, and properties of the L_2-projection, that

$$\begin{aligned} B_{\Omega\setminus\Omega_0}(u-P(u),G) \;&\leqslant\; C\Big(\|\nabla u\|_{L_1(\Omega)} + \varepsilon\|\nabla^2 u\|_{L_1(\Omega)} + \frac{1}{h}\|u\|_\Omega\Big) \\ &\quad \times \Big(\|G\|_{L_\infty(\widetilde{\Omega\setminus\Omega_0})} + \|\nabla_h G\|_{L_\infty(\widetilde{\Omega\setminus\Omega_0})}\Big). \end{aligned}$$

Here $B_D(w,v)$ are the terms of $B(w,v)$ with integration restricted to D. Therefore, using our hypothesis and Corollary 4.1, we have

$$B_{\Omega\setminus\Omega_0}(u-P(u),G) \leqslant Ch^{k+1}.$$

Now we bound $B_{\Omega_0}(u-P(u),G)$:

$$\begin{aligned} &B_{\Omega_0}(u-P(u),G) \\ &\quad = \sum_{T\in\mathscr{T}_h, T\cap\Omega_0\neq\varnothing} \int_T \big\{\varepsilon\nabla(u-P(u))\nabla(G) + (u-P(u))(-G_x + cG)\big\}\,\mathrm{d}x \\ &\qquad - \sum_{e\in\mathscr{E}_h^0, e\cap\Omega_0\neq\varnothing} \int_e \Big\{\varepsilon(\langle\nabla_h(u-P(u))\rangle[G] + \gamma\langle\nabla_h G\rangle[u-P(u)] \\ &\qquad\qquad + \frac{\eta}{h_e}[u-P(u)][G]\Big\}\,\mathrm{d}s \\ &\qquad + \sum_{e\in\mathscr{E}_h^0, e\cap\Omega_0\neq\varnothing} \int_e (u-P(u))^-(G^- - G^+)|n_x|\,\mathrm{d}s. \end{aligned}$$

By Hölders inequality, approximation properties of P, and the fact that $\mathrm{meas}(\Omega_0) \leqslant C(\log(1/h))h^{1/2}$ we have

$$\sum_{T\in\mathscr{T}_h, T\cap\Omega_0\neq\varnothing} \int_T \varepsilon\nabla(u-P(u))\nabla(G)\,\mathrm{d}x \leqslant C\varepsilon h^k\|u\|_{C^{k+1}(\widetilde{\Omega_0})}\log(1/h)h^{1/4}\|\nabla_h G\|_{\Omega_1}.$$

If we apply Theorem 4.1 and our hypothesis, we get that

$$\sum_{T\in\mathscr{T}_h, T\cap\Omega_0\neq\varnothing} \int_T \varepsilon\nabla(u-P(u))\nabla(G)\,\mathrm{d}x \leqslant C(\varepsilon/h)^{1/2}\log(1/h)^2 h^{k+1/4}.$$

Since $G_x \in V_h$ we have

$$\sum_{T\in\mathscr{T}_h, T\cap\Omega_0\neq\varnothing} \int_T (u-P(u))G_x\,\mathrm{d}x = 0.$$

It can easily be shown that

$$\sum_{T\in\mathscr{T}_h, T\cap\Omega_0\neq\varnothing}\int_T (u-P(u))cG\,\mathrm{d}x \leqslant Ch^{k+5/4}\|u\|_{C^{k+1}(\widetilde{\Omega_0})}\|c^{1/2}G\|_{\Omega_0} \leqslant C\log(1/h)h^{k+3/4}.$$

By applying Hölder's inequality, approximation properties of P, the fact that $\eta > 0$, and Theorem 4.1, we obtain

$$\begin{aligned}
&-\sum_{e\in\mathscr{E}_h^0, e\cap\Omega_0\neq\varnothing}\int_e \varepsilon(\langle\nabla_h(u-P(u))\rangle[G])\,\mathrm{d}s\\
&\qquad\leqslant \varepsilon h^k\|u\|_{C^{k+1}(\Omega_1)}\sum_{e\in\mathscr{E}_h^0, e\cap\Omega_0\neq\varnothing}\|[G]\|_{L_1(e)}\\
&\qquad\leqslant C\varepsilon^{1/2}h^{k+1}\Bigg(\sum_{e\in\mathscr{E}_h^0, e\cap\Omega_0\neq\varnothing}\frac{\eta}{h}\|[G]\|_e^2\Bigg)^{1/2}\Bigg(\sum_{e\in\mathscr{E}_h^0, e\cap\Omega_0\neq\varnothing}1\Bigg)^{1/2}\\
&\qquad\leqslant C\varepsilon^{1/2}\log(1/h)^2h^{k+1/4}.
\end{aligned}$$

In the last inequality we used that

$$\Bigg(\sum_{e\in\mathscr{E}_h^0, e\cap\Omega_0\neq\varnothing}1\Bigg)^{1/2}\leqslant\log\left(\frac{1}{h}\right)h^{-3/4},$$

since there are at most $C(\log(1/h))^2h^{-3/2}$ triangles in Ω_0.

Similarly, we obtain

$$-\sum_{e\cap\Omega_0\neq\varnothing}\int_e(\varepsilon/h)[u-P(u)][G]\,\mathrm{d}s\leqslant C\varepsilon^{1/2}\log(1/h)^2h^{k+1/4}.$$

Using Hölders, approximation properties of P, inverse estimates and Theorem 4.1, we see that

$$\begin{aligned}
&-\sum_{e\in\mathscr{E}_h^0, e\cap\Omega_0\neq\varnothing}\int_e\varepsilon\gamma\langle\nabla_hG\rangle[u-P(u)]\,\mathrm{d}s\\
&\qquad\leqslant C\varepsilon\log(1/h)h^{k+1/4}\|u\|_{C^{k+1}(\widetilde{\Omega_0})}\|\nabla_hG\|_{\Omega_0})\leqslant\log(1/h)^2h^{k+1/4}.
\end{aligned}$$

By Hölder's inequality, approximation properties of P and Theorem 4.1, we have that

$$\begin{aligned}
&\sum_{e\in\mathscr{E}_h^0, e\cap\Omega_0\neq\varnothing}\int_e(u-P(u))^-(G^--G^+)|n_x|\,\mathrm{d}s\\
&\qquad\leqslant C\log(1/h)h^{k+3/4}\|u\|_{C^{k+1}(\widetilde{\Omega_0})}\Bigg(\sum_{e\in\mathscr{E}_h^0, e\cap\Omega_0\neq\varnothing}\|(G^+-G^-)|n_x|^{1/2}\|_e^2\Bigg)^{1/2}\\
&\qquad\leqslant C\log(1/h)^2h^{k+1/4}.
\end{aligned}$$

Our result now follows since $\varepsilon \leqslant h$. $\square$

Remark 4.1. In the piecewise linear case, if we add artificial crosswind diffusion, then we can improve the pointwise estimates from $\log(1/h)^2 h^{5/4}$ to $\log(1/h)^2 \times h^{11/8}$. This modification was done to the streamline diffusion method in [6,8]. However, this estimate will still be suboptimal. Optimal max-norm estimates ($O(h^{3/2})$) for these DG methods and for the streamline diffusion method assuming general quasi-uniform meshes is still an open problem.

Acknowledgments. The author would like to thank Lars Wahlbin and Alfred Schatz for many useful conversations. The author also thanks the referee for many useful comments that lead to a better presentation of the results.

REFERENCES

1. D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini, Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.* (2002) **39**, 1749 – 1779.
2. J. Gopalakrishnan and G. Kanschat, A multilevel discontinuous Galerkin method. *Numer. Math.* (2003) **95**, 527 – 550.
3. T. J. R. Hughes and A. N. Brooks, A multidimensional upwind scheme with no crosswind diffusion. In: *Analytical and Numerical Approaches to Asymptotic Problems in Analysis* (Eds. O. Axelsson, L. S. Frank, and A. van der Sluis), North-Holland, Amsterdam, 1981, pp. 99 – 116.
4. C. Johnson, U. Navert, and J. Pitkäranta, Finite element methods for linear hyperbolic problems. *Comp. Meth. Appl. Mech. Engrg.* (1984) **45**, 285 – 312.
5. C. Johnson and J. Pitkäranta, An analysis of the discontinuous Galerkin method for a scalar hyperbolic equation. *Math. Comp.* (1986) **46**, 1 – 26.
6. C. Johnson, A. H. Schatz, and L. B. Wahlbin, Crosswind smear and pointwise error estimates in streamline diffusion finite element methods. *Math. Comp.* (1987) **49**, 25 – 38.
7. P. Lesiant and P.-A. Raviart, On a finite element method for solving the neutron transport equation. In: *Mathematical Aspects for Finite Elements in Partial Differential Equations* (Ed. C. de Boor), Academic Press, New York, 1974.
8. K. Niijima, Pointwise error estimates for the streamline diffusion finite element scheme. *Numer. Math.* (1990) **56**, 707 – 719.
9. T. E. Peterson, A note on the convergence of the discontinuous Galerkin method for a scalar hyperbolic equation. *SIAM J. Numer. Anal.* (1991) **28**, 133 – 140.
10. W. H. Reed, T. R. Hill, F. W. Brinkley, and K. D. Lathrop, Trident: A two dimensional multigroup, triangular mesh, explicit neutron transport code. *LA-6735-MS, Los Alamos Scientific Laboratory*, Los Alamos, 1977.
11. E. Süli, Ch. Scwab, and P. Houston, hp-DGFEM for partial differential equations with nonnegative characteristic form. In: *Discontinuous Galerkin Finite Element Methods. Lecture Notes in Computational Science and Engineering* (Eds. B. Cockburn, G. Karniadakis, and C.-W. Shu), Springer-Verlag, 2000, Vol. 11, pp. 221 – 230.