Blending the Finite Element Method with the Penalty Immersed Boundary Method to Investigate Hemodynamics in the Aorta

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Introduction

The purpose of this eight-week research project was to gain a greater understanding of the mathematics of fluid flow interacting with an elastic immersed boundary. We focused on one of penalty Immersed Boundary applications that analyzes the interaction of blood flow with the aortic wall (as studied in the research paper titled *Blood Flow in a Compliant Vessel by the Immersed Boundary Method*) and sought to understand their numerical methods. Once we understood the mathematics involved, we approximated solutions to the necessary equations using the Finite Element Method and wrote various codes using MATLAB to simulate the interactions of fluid flow with an immersed boundary.

Background Information

The term "Immersed Boundary" was coined in 1972 by Charles S.Peskin in his famous mathematical work regarding the simulation of cardiac mechanics and associated blood flow. Known as Immersed Boundary method, it became more pervasive but has remained mainly applicable to problems involving a moving elastic massless boundary immersed in fluid. The massless assumption makes the numerical task easier by allowing the use of a constant boundary density in the Navier-Stokes equation and therefore the Fast Fourier method can be used to solve it. In the case of a massive boundary, like the arterial wall of the aorta, it will require a more complicated numerical scheme to deal with a wall density, which is now variable and therefore makes the Fast Fourier irrelevant. Also, this makes it necessary to take into account the effect of gravity on the boundary mass. In order to steer away from these complications and stay much closer to the IB method, a modified approach called the penalty Immersed Boundary Method (pIB) was developed, which gives mass to the immersed boundary without spreading it to the fluid and therefore keeping the density constant.

Mathematical Model for the penalty Immersed Boundary

$$\rho_0\left(\frac{\partial U}{\partial t} + U.\nabla U\right) = -\nabla p + \mu \nabla^2 U + f \qquad (1)$$

$$\nabla * U = 0 \qquad (2)$$

$$f(x,t) = \int F(r,s,t)\,\delta\left(x - X(r,s,t)\right) drds \qquad (3)$$

$$\frac{\partial X}{\partial t}(r,s,t) = U\left(X(r,s,t),t\right) = \int U(x,t)\,\delta\left(x - X(r,s,t)\right) drds \qquad (4)$$

$$F = F_E + F_K \qquad (5)$$

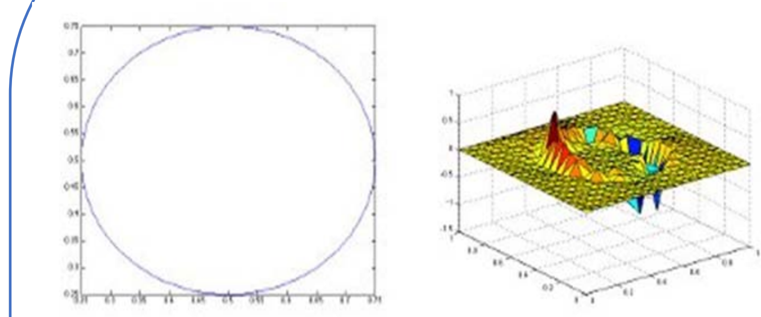
$$F_E = -\frac{\partial E}{\partial X} \qquad (6)$$

$$F_K(r,s,t) = K\left(Y(r,s,t) - X(r,s,t)\right) \qquad (7)$$

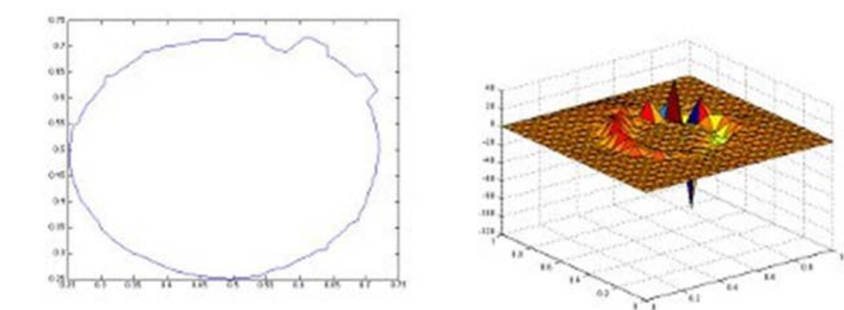
$$M(r,s)\frac{\partial^2 Y}{\partial t^2} = -F_K(r,s,t) - M(r,s)\,g\,e_3 \qquad (8)$$

- 1 and 2 are the Navier- Stokes equations, with function *U* as the fluid velocity;
- 3 and 4 include the two dimensional Dirac Delta function, which regulates the local interaction between forces and velocities along the immersed boundary;
- 5, 6, 7 and 8 are the massless immersed boundary (*X*) and the massive boundary (*Y*) written in Lagrangian form.

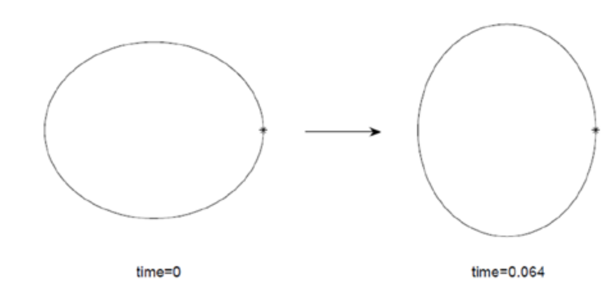
Preliminary Results



Immersed Boundary and Fluid Velocity at T = 0



Immersed Boundary and Fluid Velocity at T = 10)^(-6)

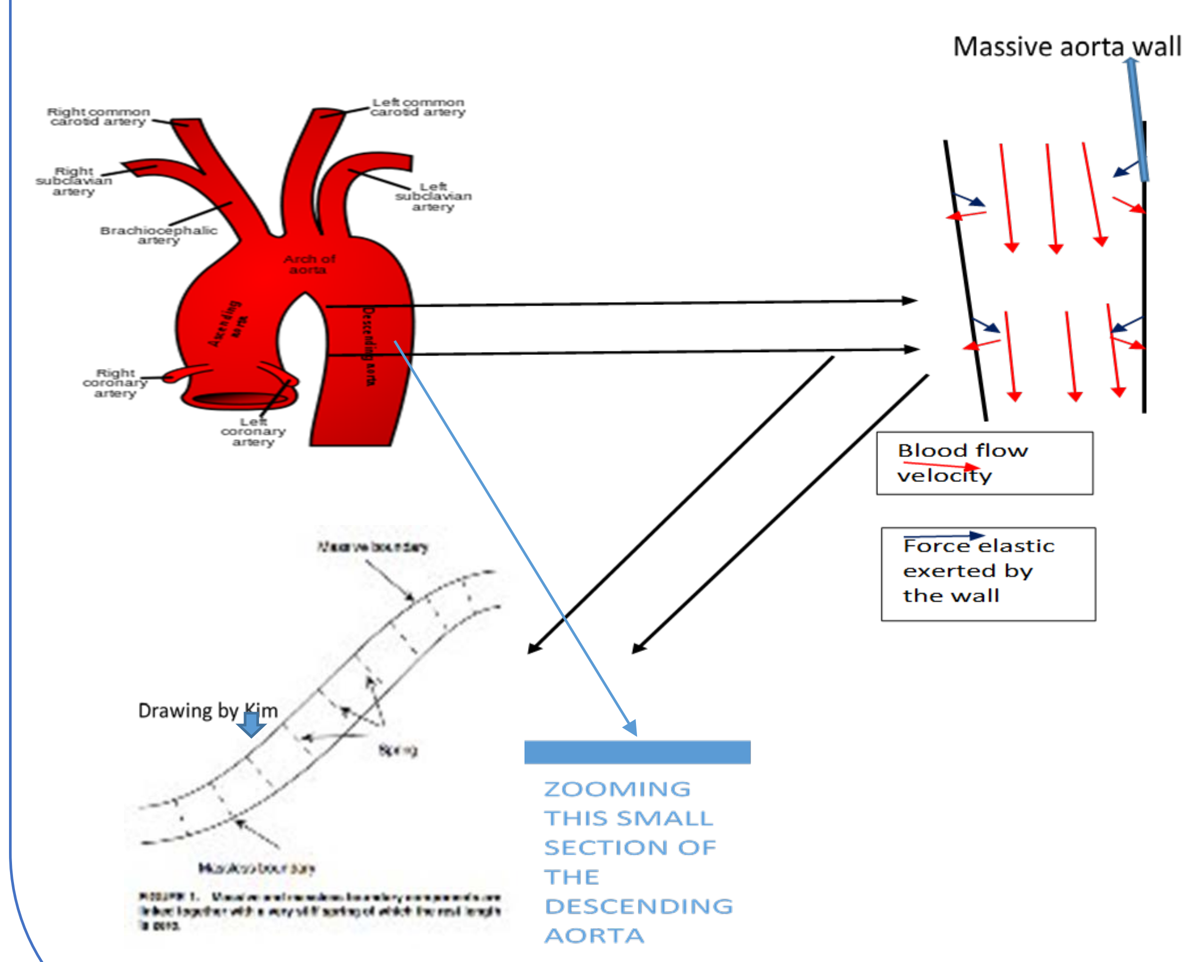


Reference simulation performed by Peskin used to test the accuracy of our code

Fig. 2. Motion of a massive elastic boundary (closed curve).

PIB Concept and its Advantages

The pIB model uses an innovative way to assign mass to the immersed boundary or the arterial wall in our case without having to overhaul the numerical scheme used to treat the case of a massless boundary. To do so, a twin boundary system (a massive (Y) and a massless (X)) is conceived. Y is made of massive boundary points that are not coupled to each other but mapped one to one to each massless boundary (X) marker through some stiff collection of springs. X plays the same role as the massless immersed boundary of the original IB method. Y moves according to Newton Laws of motion (F=ma) and the only forces that are acting on are the gravitational force and the forces of the spring oscillated by the movement of the immersed boundary during its interaction with the fluid velocity. This concept insures that the mass is given to the immersed boundary X without spreading the mass to the fluid and therefore keeping the density of the Navier-stokes equation constant.



Methods

We begin using the Finite Element Method by turning the original Navier-Stokes equations into a weak formulation:

$$1)\ \frac{1}{t_i - t_{i-1}}\left(\int_\Omega \boldsymbol{U}(\boldsymbol{x},t_i) * \mathbf{V}(\boldsymbol{x}) + \upsilon\int_\Omega \nabla\boldsymbol{U}(\boldsymbol{x},t_i) : \nabla\mathbf{V}(\boldsymbol{x}) - \int_\Omega \boldsymbol{P}(\boldsymbol{x},t_i)\, div\mathbf{V} + \int_\Omega (\boldsymbol{U} * \nabla\boldsymbol{U}) * \mathbf{V}(\boldsymbol{x})\right) = \int_\Omega \boldsymbol{f} * \mathbf{V}(\boldsymbol{x}) + \frac{1}{t_i - t_{i-1}}\int_\Omega \boldsymbol{U}(\boldsymbol{x},t_{i-1}) * \mathbf{V}(\boldsymbol{x})$$

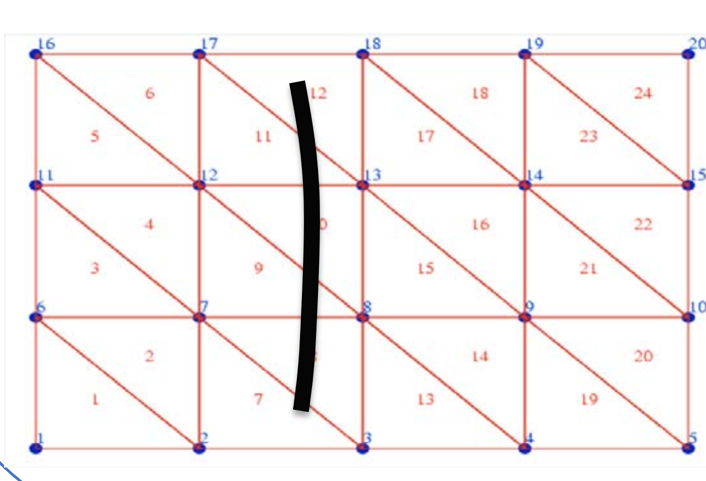
$$2)\ \int_\Omega div\boldsymbol{U}(\boldsymbol{x},t_i)\,\mathbf{q}(\boldsymbol{x}) = 0$$

Next, we discretize this formulation into a linear system of finite elements which we then use Matlab to solve.

$$\begin{pmatrix} A + \frac{1}{k}M & -C \\ C^T & 0 \end{pmatrix} * \begin{bmatrix} u_1^{j+1} \\ \vdots \\ u_{n+m}^{j+1} \\ p_1^{j+1} \\ \vdots \\ p_{n+m}^{j+1} \end{bmatrix} = \begin{bmatrix} \vec{F} \\ \vec{0} \end{bmatrix} + \begin{bmatrix} \frac{1}{k}M\overline{U}^j \\ \vec{0} \end{bmatrix}$$

A = stiffness Matrix
M= Mass Matrix
U= Velocity Vector
P= Fluid Pressure Vector
F= Force density vector of the Immersed Boundary on the Fluid
C= Pressure Matrix

In particular, we create a triangular mesh on our domain and use this linear system to approximate the solution to the Navier-Stokes on each triangle



Note that in this case, the immersed boundary only intersects triangles 7, 8, 9, 10, 11, and 12. Thus, we only update the force on the right-hand side of the linear system on these specific triangles. Finally, we run the code and see how the immersed boundary moves as it interacts with the fluid over time

Discussion

We ran our code with a sample reference test given in one of Charles Peskin's papers. As seen from above, our results did not match those of the reference test. After only 10^(-6) seconds, our simulation of the immersed boundary starts losing stability whereas in the reference test, it remains stable. We're still in the debugging process and we hope to find out whether it is a bug in our code or a stability problem (in which case we would have to carefully refer back to the literature).

Conclusion:

We spent the first 2-3 weeks learning about the finite element method and familiarizing ourselves with Matlab.
Thus, after 8 weeks, we were only able to reach the debugging process of our code. And because our results did not match the results of the reference test, we have yet to simulate the aortic walls with our code.
As for future plans, we hope to finish debugging our code and finally simulate hemodynamics in the aorta.

Acknowledgments:

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References

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