Millersville University Name Answer Key
Department of Mathematics
MATH 161, *Calculus I*, Test 1
September 19, 2006, 08:00-08:50AM

Please answer the following questions. Your answers will be evaluated on their correctness, completeness, and use of mathematical concepts we have covered. Please show all work and write out your work neatly. Answers without supporting work will receive no credit. The point values of the problems are listed in parentheses.

1. (7 points each) Using the properties of limits, evaluate the following limits if they exist. You may not use graphical or numerical evidence to support your answers; however, you may use such evidence to check your answers. If a limit fails to exist, please explain why in a sentence.

(a) $\lim\limits_{x \to 4} \frac{\sqrt{x+12}-4}{x-4}$ $= \lim\limits_{x \to 4} \frac{(\sqrt{x+12}-4)(\sqrt{x+12}+4)}{(x-4)(\sqrt{x+12}+4)}$

$= \lim\limits_{x \to 4} \frac{x+12-16}{(x-4)(\sqrt{x+12}+4)}$

$= \lim\limits_{x \to 4} \frac{1}{\sqrt{x+12}+4}$

$= 1/8$

(b) $\lim\limits_{x \to \infty} \frac{x^2+3x+1}{\sqrt{4x^4+7x^2+3}}$ $= \lim\limits_{x \to \infty} \frac{(x^2+3x+1)\,1/x^2}{\sqrt{4x^4+7x^2+3}\;\;1/x^2}$

$= \lim\limits_{x \to \infty} \frac{1+3/x+1/x^2}{\sqrt{(4x^4+7x^2+3)\,1/x^4}}$

$= \lim\limits_{x \to \infty} \frac{1+3/x+1/x^2}{\sqrt{4+7/x^2+3/4^4}}$

$= 1/2$

(c) $\lim_{x\to 2^+} \frac{x^2 - 3x - 4}{x^2 - 6x + 8}$ $= \lim_{x \to 2^+} \frac{(x-4)(x+1)}{(x-4)(x-2)}$

$= \lim_{x \to 2^+} \frac{x+1}{x-2}$

$= \infty$

(d) $\lim_{x\to 0} x^2 \cos\left(\frac{1}{x}\right)$

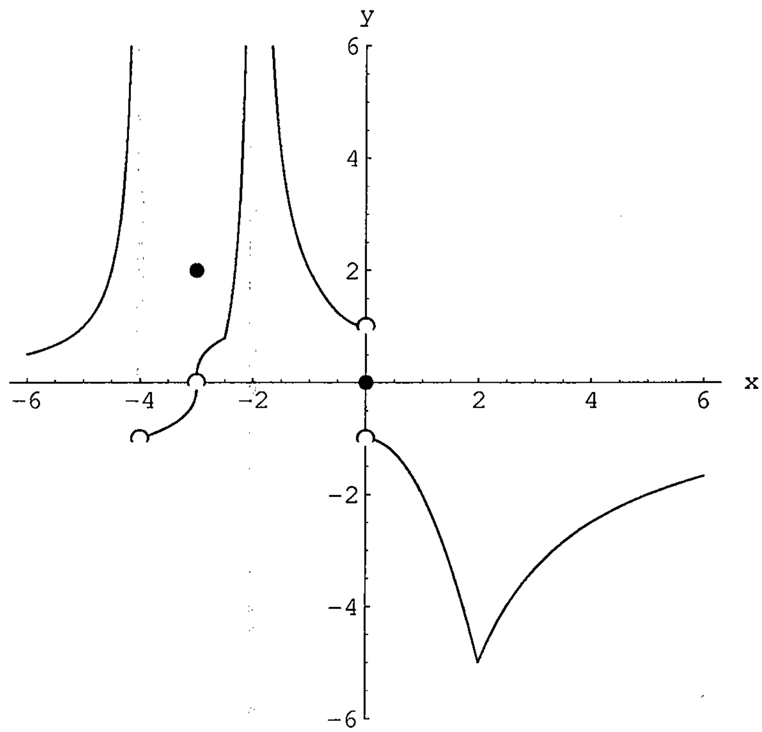
Since $-x^2 \le x^2 \cos(1/x) \le x^2$

and $\lim_{x \to 0} (-x^2) = 0 = \lim_{x \to 0} x^2$,

then by the Squeeze Theorem

$\lim_{x \to 0} x^2 \cos(1/x) = 0.$

2. Consider the function graphed below.



(a) (2 points each) Using the graph above, estimate the following limits.

i. $\lim_{x \to 0^-} f(x) =$ 1

ii. $\lim_{x \to 0^+} f(x) =$ -1

iii. $\lim_{x \to 0} f(x) =$ Does not exist

iv. $\lim_{x \to -2} f(x) =$ ∞

v. $\lim_{x \to 2^+} f(x) =$ -5

vi. $\lim_{x \to -4} f(x) =$ Does not exist

(b) (5 points) Using the graph above, identify the x values at which the function is discontinuous and identify the type of each discontinuity.

removable discontinuities : $x = -3$

jump discontinuities : $x = 0$

infinite discontinuities : $x = -4$, $x = -2$

3. (5 points) Consider the piecewise-defined function

$$f(x) = \begin{cases} \dfrac{1}{x^2 + cx - 3} & \text{if } x < -3 \\ \sqrt{x+7} & \text{if } x \geq -3 \end{cases}$$

Find a value for the constant c so that $f(x)$ is continuous at $x = -3$.

$$\lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} \frac{1}{x^2 + cx - 3} = \frac{1}{9 - 3c - 3} = \frac{1}{6 - 3c}$$

$$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} \sqrt{x+7} = 2.$$

If $f(x)$ is continuous at $x = -3$, then

$$\lim_{x \to -3^-} f(x) = \lim_{x \to -3^+} f(x)$$

$$\frac{1}{6-3c} = 2$$

$$1 = 12 - 6c$$

$$-11 = -6c$$

$$c = 11/6$$

4. (5 points) Using what we have learned in calculus (particularly about continuous functions) explain clearly why there is a value of x with $1 \leq x \leq 2$ for which

$$x - 4 \ln x = 0.$$

Let $f(x) = x - 4\ln x$. $f(x)$ is continuous on $[1, 2]$. $f(1) = 1 > 0$ while $f(2) \approx -0.77 < 0$. Thus according to the Intermediate Value Theorem there is a number c with $1 \leq c \leq 2$ such that $f(c) = 0$.

5. (6 points each) Using the definition of the derivative find the derivatives of the following functions. Results obtained by using "short-cut" derivative formulas will receive no credit (but may be used to check your answers).

(a) $f(x) = \dfrac{1}{x-2}$, for $x \neq 2$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left(\frac{1}{x+h-2} - \frac{1}{x-2}\right)}{h} \cdot \frac{(x+h-2)(x-2)}{(x+h-2)(x-2)}$$

$$= \lim_{h \to 0} \frac{x-2-(x+h-2)}{h(x+h-2)(x-2)}$$

$$= \lim_{h \to 0} \frac{-h}{h(x+h-2)(x-2)}$$

$$= \lim_{h \to 0} \frac{-1}{(x+h-2)(x-2)}$$

$$= \frac{-1}{(x-2)^2}$$

(b) $f(x) = \sqrt{x+3}$, for $x > -3$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

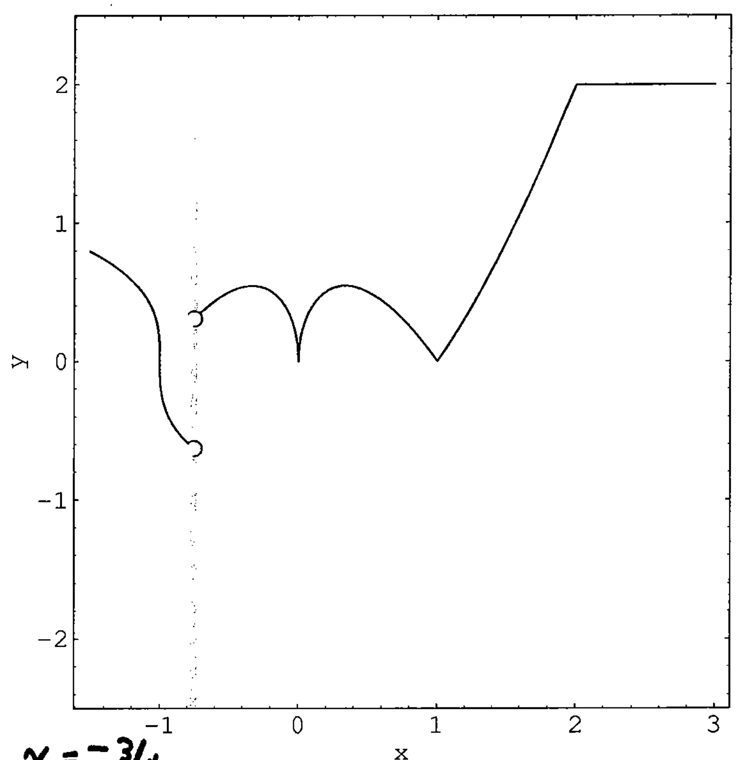
$$= \lim_{h \to 0} \frac{(\sqrt{x+h+3} - \sqrt{x+3})}{h} \cdot \frac{(\sqrt{x+h+3} + \sqrt{x+3})}{(\sqrt{x+h+3} + \sqrt{x+3})}$$

$$= \lim_{h \to 0} \frac{x+h+3-(x+3)}{h(\sqrt{x+h+3} + \sqrt{x+3})}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h+3} + \sqrt{x+3}}$$

$$= \frac{1}{2\sqrt{x+3}}$$

6. (6 points) Using the graph of $f(x)$ show in the figure below, find the x values at which the function will fail to have a derivative. Briefly explain why the function does not possess a derivative at those points.



discontinuity: $x = -3/4$

vertical tangent: $x = -1$

cusp: $x = 0$

corner: $x = 1$, $x = 2$

7. (5 points each) The position of a falling object is given by the function

$$s(t) = -16t^2 + 60t + 80.$$

(a) Find the average velocity of the object over the interval $1 \le t \le 2$.

$$v_{avg} = \frac{s(2) - s(1)}{2-1}$$

$$= \left(-16(2^2) + 60(2) + 80\right) - \left(-16(1^2) + 60(1) + 80\right)$$

$$= -64 + 120 + 80 + 16 - 60 - 80$$

$$= 12 \quad (\text{ft/sec})$$

(b) Find the instantaneous velocity of the object at $t = 2$.

$$v = \lim_{t \to 2} \frac{s(t) - s(2)}{t-2}$$

$$= \lim_{t \to 2} \frac{-16t^2 + 60t + 80 - 136}{t-2}$$

$$= \lim_{t \to 2} \frac{-16t^2 + 60t - 56}{t-2}$$

$$= \lim_{t \to 2} \frac{-4(4t^2 - 15t + 14)}{t-2} = \lim_{t \to 2} \frac{-4\cancel{(t-2)}(4t-7)}{\cancel{t-2}} = -4$$

(c) Is the object moving in the upward or downward direction at $t = 2$? You must justify your answer.

Downward since $v < 0$ at $t = 2$.

8. (6 points each) Given the function $f(x) = \dfrac{x^2 - 3x - 4}{x^2 - 6x + 8}$, using the appropriate limits

(a) find all the horizontal asymptotes of $f(x)$.

$$\lim_{x \to \infty} \frac{(x^2 - 3x - 4)(1/x^2)}{(x^2 - 6x + 8)(1/x^2)} = \lim_{x \to \infty} \frac{1 - 3/x - 4/x^2}{1 - 6/x + 8/x^2}$$

$$= 1$$

Horizontal asymptote at $y = 1$.

(Likewise, $\lim_{x \to -\infty} f(x) = 1$).

(b) find all the vertical asymptotes of $f(x)$.

There may be vertical asymptotes at $x = 2$ and $x = 4$ since the denominator becomes zero at this values.

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{(x-4)(x+1)}{(x-4)(x-2)} = \lim_{x \to 2^-} \frac{x+1}{x-2} = -\infty$$

Thus there is a vertical asymptote at $x = 2$.

$$\lim_{x \to 4} f(x) = \lim_{x \to 4} \frac{(x-4)(x+1)}{(x-4)(x-2)} = \lim_{x \to 4} \frac{x+1}{x-2} = 5/2.$$

Hence $f(x)$ has only a removable discontinuity at $x = 2$.