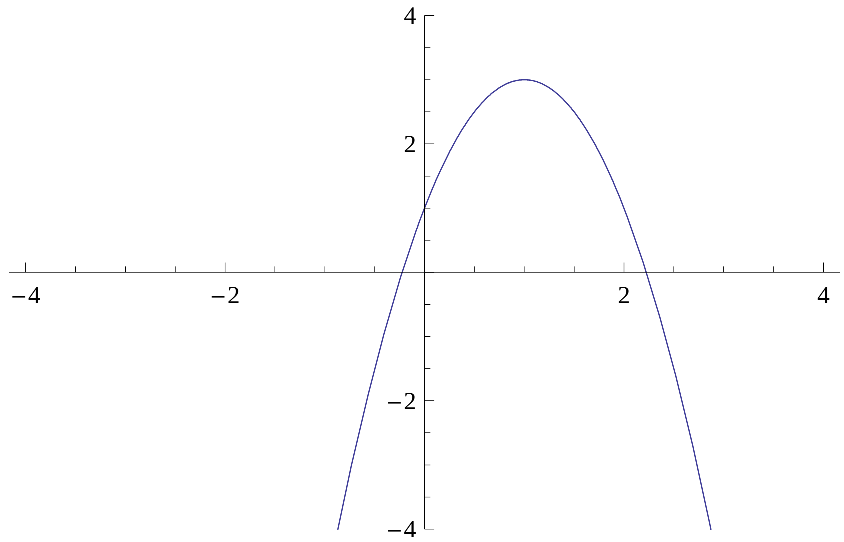
1 Solutions to Tuesdays's Review Session

1.1 What is the maximum value of the function $f(x) = -2x^2 + 4x + 1$.

If you want to find the maximum value of a function a good place to start is to graph the function to at least get an idea of what the function looks like and where the maximum value is. The graph of $f(x)$ is given below. So, we can see from the graph that the highest point on the graph is given



by the vertex. This means all we need to do is complete the square to get the vertex.

$$\begin{aligned}
-2x^2 + 4x + 1 &= -2(x^2 - 2x) + 1 \\
&= -2(x^2 - 2x + 1 - 1) + 1 \\
&= -2\left[(x-1)^2 - 1\right] + 1 \\
&= -2(x-2)^2 + 2 + 1 \\
&= -2(x-2)^2 + 3
\end{aligned}$$

Therefore, the vertex is given by the coordinates $(2, 3)$. Now the maximum of a function is the highest value the function reaches. So the maximum is 3.

1.2 Solve the equation $\ln(3x+2)+\ln(x-3)-3=2$.

With any log equation our first goal should be to eliminate the logarithms. To do this we have to make sure we use the properties correctly.

$$\ln(3x+2)+\ln(x-3)-3=2$$
$$\Rightarrow \ln((3x+2)(x-3))=5$$
$$\Rightarrow (3x+2)(x-3)=e^5$$

Now, this is a quadratic equation so our goal is to get everything on one side of the equation and try to factor or use the quadratic formula.

$$\Rightarrow 3x^2-7x-6=e^5$$
$$\Rightarrow 3x^2-7x-6-e^5=0$$
$$\Rightarrow x=\frac{7\pm\sqrt{49-4\cdot 3\cdot(-6-e^5)}}{6}$$
$$\Rightarrow x=\frac{7\pm\sqrt{121+e^5}}{6}$$

Finally, these are supposed to be solutions to the original log problem. But you will see that the logarithms above are undefined for the negative root. Therefore, the final solutions is given by:

$$x=\frac{7+\sqrt{121+e^5}}{6}$$

1.3 Find possible formulas for the tables below

x	2	5	8	11	14	17
f(x)	10	5	0	5	10	5

x	1	2	3	4
g(x)	.18	.108	.0648	.03888

The first table seems to be a periodic function so we should try to model it using a periodic function in the form $f(x)=k+A\sin(B(x-h))$ or $f(x)=k+A\cos(B(x-h))$. From the table we can figure out the following facts:

- The period is 12.
- The amplitude is 5.
- The midline is $y=5$.

Now, if we choose to use a $\sin(x)$ function to model the graph we have to use a shift of 11 to the right since a sine function starts at the midline and increases (think about the graph of $\sin(x)$). Putting this information together we get the function:

$$f(x)=5+5\sin\left(\frac{\pi}{6}(x-11)\right)$$

Now, if we instead use a base graph of $\cos(x)$ then the horizontal shift is 2 to the right since a cosine graph starts at its maximum value. Putting everything together we get the function:

$$f(x)=5+5\sin\left(\frac{\pi}{6}(x-2)\right).$$

For the second table it is clear that the function is not linear or periodic. So lets see if it is an exponential function. Dividing the outputs we have that:

$$\begin{aligned} .108/.18 &= .6 \\ .0648/.108 &= .6 \\ .3888/.0648 &= .6 \end{aligned}$$

Therefore, we can model the table by an exponential function in the form $g(x) = ab^x$. We know that $b = .6$ so now we need to just find a. Well, we know that $g(1) = .18$. Therefore, we know that:

$$.18 = a(.6)^1$$

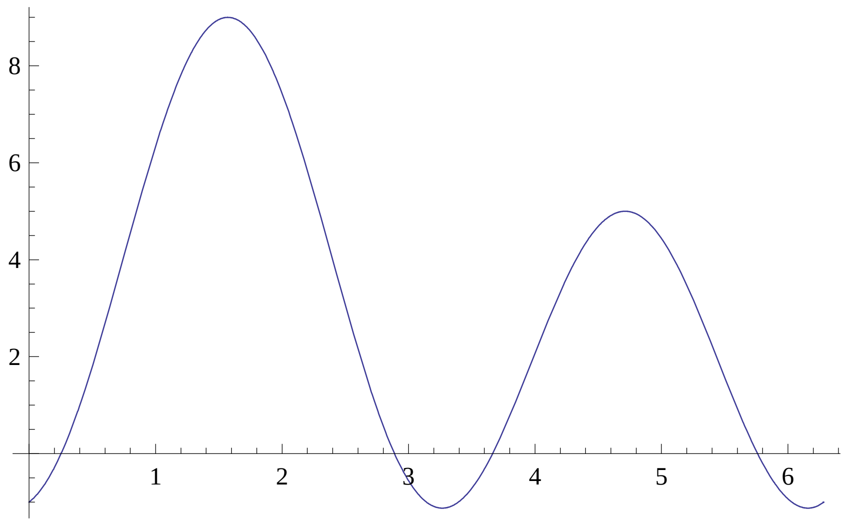
$$\Rightarrow .18/.6 = a$$

$$\Rightarrow a = .3.$$

Therefore, $g(x) = .3(.6)^x$.

1.4 Find all solutions $0 < \theta < 2\pi$, satisfying $8\sin^2(\theta) + 2\sin(\theta) - 1 = 0$.

One problem with solving trig equations is that they can have a ton of solutions. So it would be nice to know how many solutions there are before we try to solve the problem. One way you can figure out how many solutions there are is to graph the function $f(\theta) = 8\sin^2(\theta) + 2\sin(\theta) - 1$ on the interval $0 < \theta < 2\pi$ and count the number of times it crosses the horizontal axis. We can see



from the graph that there are four zeros. Now, to actually solve this problem you have to recognize that the equation is factorable and can be factored into the form

$$(4\sin(\theta) - 1)(2\sin(\theta) + 1) = 0$$

$$\Rightarrow 4\sin(\theta) - 1 = 0 \text{ and } 2\sin(\theta) + 1 = 0$$

$$\Rightarrow \sin(\theta) = \frac{1}{4} \text{ and } \sin(\theta) = -\frac{1}{2}$$

$$\Rightarrow \theta = \sin^{-1}\left(\frac{1}{4}\right) \text{ and } \theta = \sin^{-1}\left(-\frac{1}{2}\right)$$

$$\Rightarrow \theta = .25268 \text{ and } \theta = -\frac{\pi}{6}$$

Now, $-\frac{\pi}{6}$ is not in the interval $0 < \theta < 2\pi$ so we should add 2π to it. Therefore, we have the two solutions

$$\theta = .25268 \text{ and } \theta = \frac{11\pi}{6}.$$

To get another solution we need to think about another angle ϕ such that $\sin(\phi) = \frac{1}{4}$. Well, we know that the sine function is positive is in the second quadrant. So, using .25268 as a reference angle we have that another solution is $\pi - .25268$. If you have trouble with these type of problems, draw pictures!! Finally, we need to get one more solution. We want to find an angle ψ such that $\sin(\psi) = -1\frac{1}{2}$. Since sine is negative in the third quadrant we can use the reference angle $\frac{\pi}{6}$ to get that the fourth solution is $pi + \frac{\pi}{6}$. Therefore, the four solutions to this problem are given by:

$$.25268, \frac{11\pi}{6}, \pi - .25268 \text{ and } \frac{7\pi}{6}.$$

1.5 Prove the following identities

- $\cos^4(\theta) - \sin^4(\theta) = \cos(2\theta)$.
- $\sec(2\theta) = \frac{\sec^2(\theta)}{2 - \sec^2(\theta)}$.

$$\begin{aligned}
\cos^4(\theta) - \sin^4(\theta) &= (\cos^2(\theta) - \sin^2(\theta))(\cos^2(\theta) + \sin^2(\theta)) \\
&= (\cos^2(\theta) - \sin^2(\theta)) \cdot 1 \\
&= \cos(2\theta).
\end{aligned}$$

$$\begin{aligned}
\frac{\sec^2(\theta)}{2 - \sec^2(\theta)} &= \frac{\frac{1}{\cos^2(\theta)}}{2 - \frac{1}{\cos^2(\theta)}} \\
&= \frac{\frac{1}{\cos^2(\theta)}}{\frac{2\cos^2(\theta) - 1}{\cos^2(\theta)}} \\
&= \frac{1}{2\cos^2(\theta) - 1} \\
&= \frac{1}{\cos(2\theta)} \\
&= \sec(2\theta).
\end{aligned}$$

1.6 Suppose a population grows according to the formula $P(t) = 35e^{.7t}$.

- After how many years will the population reach 500 people?
- By what percentage does the population grow each year?

The first problem is a simple exponential equation. We need to solve the equation $500 = 35e^{.7t}$.

$$\frac{500}{35} = e^{.7t}$$

$$\Rightarrow \ln(\frac{100}{7}) = .7t$$

$$\Rightarrow t = \frac{\ln(\frac{100}{7})}{.7}$$

$$\Rightarrow t = 3.79 \text{ years.}$$

Now, to find the percentage that this population grows at we need to convert the exponential into the form ab^t. We know that $a = 35$ so we need to just find b. Setting the two equations equal to each other we have that:

$$35b^t = 35e^{.7t} = 35(e^{.7})^t$$

$$\Rightarrow b = e^{.7}$$

$$\Rightarrow b = 2.0138.$$

Therefore, the percentage the population grows in one year is 101%.