1 Solutions to First Day Problems

#1. Consider the function

$$f(x) = \begin{cases} \frac{\sin^2(3ax)}{x^2}, \ x \neq 0 \\ k, \ x = 0. \end{cases}$$

Find the value of k that makes the function continuous.

Solution: Remember for a function to be continuous it should have no holes, breaks, or jumps. To be more mathematical precise we say that function is continuous at some point b if

$$\lim_{x \to b} f(x) = f(b).$$

So, to make sure this function is continuous we really only need to check the limit as $x \to 0$. To do a limit algebraically we do the following:

- Do the dumb thing and just plug the value in. YOU NEED TO BE CAREFUL HERE. If there is a jump the limit will not exist but the function value will.
- If doing the dumb thing does not work and you get something like $0/0$ or ∞/∞ you need to do L'Hospital's rule.
- If L'Hospital's rule does not work you can try factoring and hopefully the bad things will cancel.
- As a last ditch effort you could make a table of values near the point you are trying to take the limit of.

So lets take our limit. Doing the dumb thing we get $0/0$. So we need to use L'Hospitals rule.

$$\lim_{x \to 0} \frac{\sin^2(3ax)}{x^2} = \lim_{x \to 0} \frac{6\sin(3ax)\cos(3ax)}{2x} = \frac{0}{0}.$$

This means we have to use L'Hospital's rule again.

$$\lim_{x \to 0} \frac{6\sin(3ax)\cos(3ax)}{2x} = \lim_{x \to 0} (9\cos(3ax) - 9\sin(3ax)) = 9.$$

Don't forget to answer the question. The value of k that makes the function continuous is $k = 9$.

#2. Find the following limits

- $\lim_{x\to\infty} \tanh(7x)$
  Solution:
  $$\lim_{x\to\infty} \tanh(7x) = \lim_{x\to\infty} \frac{e^{7x} - e^{-7x}}{e^{7x} + e^{-7x}} = \lim_{x\to\infty} \frac{e^{7x}}{e^{7x}} \frac{1 - e^{-14x}}{1 + e^{-14x}} = \lim_{x\to\infty} \frac{1 - e^{-14x}}{1 + e^{-14x}} = 1.$$

- c
  Solution: Let $f(x) = 15^{2x}$. Then, from the definition of the derivative we know that
  $$\lim_{h\to 0} \frac{15^{6+2h} - 15^6}{h} = f'(3).$$
  So, lets just find $f'(3)$ using techniques from chapter 3.
  $$f'(x) = 2\ln(15)15^{2x} \Rightarrow f'(3) = 2\ln(15)15^6.$$

- $\lim_{x\to 0} \frac{1 - \cosh(3x)}{x}$
  Solution:
  $$\lim_{x\to 0} \frac{1 - \cosh(3x)}{x} = \lim_{x\to 0}(-3\sinh(x)) = 0.$$

- $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin(x)}\right)$
  Solution: If we do the dumb thing we get ∞-∞, which is an indeterminate form. This means we need to do more work. The only thing we can really do is add fractions.
  $$\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin(x)}\right) = \lim_{x\to 0} \frac{\sin(x) - x}{x\sin(x)}$$
  Using L'Hospital's rule we now have that
  $$\lim_{x\to 0} \frac{\sin(x) - x}{x\sin(x)} = \lim_{x\to 0} \frac{\cos(x) - 1}{\sin(x) + x\cos(x)} = \frac{0}{0}$$
  Using L'Hospital's rule again we have
  $$\lim_{x\to 0} \frac{\cos(x) - 1}{\sin(x) + x\cos(x)} = \lim_{x\to 0} \frac{-\sin(x)}{\cos(x) + \cos(x) - x\sin(x)} = 0.$$

#3. Find the following derivatives, assuming a and k are constants

- $\frac{d}{d\theta}\sqrt{a^2 - \sin^2(\theta)}$
Solution: We are going to need to use the chain rule a couple of times.

$$\frac{d}{d\theta}\sqrt{a^2 - \sin^2(\theta)} = \frac{1}{2}\left(a^2 - \sin^2(\theta)\right)^{-1/2}(-2)\sin(\theta)\cos(\theta).$$

- $\frac{d}{dt}\frac{\ln(kt)+t}{\ln(kt)-t}$
Solution: We need to use the quotient rule and chain rule.

$$\frac{d}{dt}\frac{\ln(kt)+t}{\ln(kt)-t} = \frac{[\ln(kt)-t][1/t+1] - [\ln(kt)+t][1/t-1]}{[\ln(kt)-t]^2} = \frac{-2}{[\ln(kt)-t]^2}$$

#4. Suppose $g'(3) = 4$ and $g(3) = 6$. Let $h(t) = 7t^2 g(t^{1/3})$. What is $h'(27)$?
Solution: Just differentiate and plug in 3.

$$h'(t) = 14tg(t^{1/3}) + 7t^2\left(\frac{1}{3}\right)t^{-2/3}g'(t^{1/3}) \Rightarrow h'(27) = 14 \cdot 27g(3) + 7 \cdot (27)^2/3 \cdot (27)^{-2/3}g'(3).$$

Fill in the values for $g(3)$ and $g'(3)$.

#5. Let $f(x)$ be defined by

$$f(x) = \begin{cases} x^2 + x + 3, \; x \leq 1 \\ ax^2 + c, \; x > 1 \end{cases}$$

What values of a and c make $f(x)$ continuous and differentiable?

Solution: To make the function continuous we need to make sure each piece of the function meets at $x = 1$. We also need to make sure that the derivatives of each piece meet at $x = 1$. Differentiating each piece we have that

$$f'(x) = \begin{cases} 2x + 1, \; x \leq 1 \\ 2ax, \; x > 1 \end{cases}$$

So at $x = 1$ we need $1 + 1 = a + c$ and $2 \cdot 1 + 1 = 2a \cdot 1$. Therefore $a = \frac{3}{2}$ and $c = \frac{1}{2}$.
#6. Suppose

$$\frac{4f^2P}{1-f^2} = k,$$

where k is a constant. Find $\frac{df}{dP}$.

Solution: Implicitly differentiating we have that

$$\begin{aligned}
& \frac{d}{dP}\frac{4f^2P}{1-f^2} = \frac{d}{dP}k \\
\Rightarrow\ & \frac{(1-f^2)\frac{d}{dP}(4f^2P) - 4f^2P\frac{d}{dP}(1-f^2)}{(1-f^2)^2} = 0 \\
\Rightarrow\ & \frac{(1-f^2)(8f\frac{df}{dP}P + 4f^2) - 4f^2(-2)fP\frac{df}{dP}}{(1-f^2)^2} = 0 \\
\Rightarrow\ & \frac{8f\frac{df}{dP}P + 4f^2 - 8f^3P\frac{df}{dP} - 4f^4 + 8f^3P\frac{df}{dP}}{(1-f^2)^2} = 0 \\
\Rightarrow\ & \frac{8f\frac{df}{dP}P + 4f^2 - 4f^4}{(1-f^2)^2} = 0 \\
\Rightarrow\ & 2f\frac{df}{dP}P + f^2 - f^4 = 0 \\
\Rightarrow\ & \frac{df}{dP} = \frac{f^3 - f}{2P}
\end{aligned}$$

2 Solutions to Second Day Problems

#1. On what intervals is the function $f(x) = x^4 - 4x^3$ both decreasing and concave up?

Solution: To find where a function is increasing/decreasing look first where the function switches from being increasing to decreasing or vice versa. This is equivalent to find the critical points and testing whether they are local maximums or minimums. Differentiating we have that

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3).$$

The critical points are therefore $x = 0$ and $x = 3$. Now we just need to check the value of the derivative in between the critical points. $f'(4) = 64$, $f'(2) = -16$, and $f'(-1) = -16$. Therefore the function is decreasing for $x < 0$ and $0 < x < 3$ and is increasing for $x > 3$. To check the concavity of the function we need to find the second derivative. Differentiating again we have

$$f''(x) = 12x^2 - 24x = 12x(x-2).$$

The possible inflection points are $x = 0$ and $x = 2$. Now $f''(-1) = 36$, $f''(1) = -12$, and $f''(3) = 36$. Therefore the function is concave down for $0 < x < 2$ and is concave up for $x < 0$ and $x > 2$. It follows that the function is decreasing and concave up on the interval $2 < x < 3$.

#2. For what values of a are $f(x) = a^x$ and $g(x) = 1 + x$ tangent at $x = 0$.

Solution: Two functions are tangent at a point if they have the exact same tangent line at that point. Both functions share the point $(1, 0)$ in common so we only need to check if their tangent lines have the same slope. Differentiating with respect to x we have that $f'(x) = \ln(a)a^x$ and $g'(x) = 1$. Therefore we need $\ln(a) = 1$. Solving this equation we get $a = e$.

#3. Use the local linearization of $f(x) = \sqrt{1+x}$, near $x = 3$ to approximate the value of $\sqrt{4.2}$.

Solution: First local linearization is nothing more than fancy words to mean find the equation of the tangent line to $f(x)$ at $x = 3$. To find the equation of a line we need two things, the slope and a point. Now, $f(3) = 4$ and differentiating $f'(x) = \frac{1}{2}(1+x)^{-1/2} \Rightarrow f'(3) = 1/4$. Therefore the local linearization near $x = 3$ is given by

$$f(x) \approx \frac{1}{4}(x-3) + 4.$$

Plugging in $x = 3.2$ we have that

$$\sqrt{4.2} = f(3.2) \approx \frac{1}{4}(.2) + 4 = \frac{2}{40} + 4 = 81/40.$$

#4. Find constants a and b so that the function $f(x) = axe^{bx}$ has a local max at $(1/3, 1)$.

Solution: We need the following conditions to be satisfied:

$$f(1/3) = 1 \text{ and } f'(1/3) = 0$$

Therefore,

$$\begin{aligned} \frac{a}{3}e^{b/3} &= 1 \text{ and } ae^{b/3} + \frac{ab}{3}e^{b/3} = 0 \\ \frac{a}{3}e^{b/3} &= 1 \text{ and } ae^{b/3}\left(1 + \frac{b}{3}\right) = 0 \\ \frac{a}{3}e^{b/3} &= 1 \text{ and } b = -3 \\ a &= 3e \text{ and } b = -3 \end{aligned}$$

#5. A rectangle is formed by having one side of the rectangle on the x-axis and two vertices on the curve $y = \dfrac{1}{1+x^2}$. Find the dimensions of the rectangle that maximize its area.

Solution: For any optimization problem you should write down your goals first. Our goal is to maximize the area. Then you try to find a formula for this quantity. Sometimes this formula will involve more than one variable and cannot be differentiated. In this case you need another equation to eliminate a variable. Anyway for our problem the area in terms of x is given by:

$$A = 2x\frac{1}{1+x^2}.$$

Differentiating we have that

$$\frac{d}{dt}A = \frac{(1+x^2)2 - 2x(2x)}{(1+x^2)^2} = \frac{1-2x^2}{(1+x^2)^2}.$$

It is clear that the critical point we are interested in is $x = 1$. We need to test this point to see if it is a maximum or minimum. For $x = 2$ we have that $\frac{dA}{dt} = -3/25$ and for $x = 0$ we have that $\frac{dA}{dt} = 1$. Therefore, $x = 1$ maximizes the area. We should still answer the question. The dimensions of the rectangle are $2 \times 1/2$.

#6. If

$$\frac{dP}{dt} = kP(L-P), \;\; k, L > 0.$$

- For what values of P is the population increasing?
  Solution: In order for the population to be increasing we need $\frac{dP}{dt} > 0$. This means $P < L$.

- Find

$$\frac{d^2P}{dt^2}.$$

  Solution: Differentiating we have that

$$\begin{aligned}
\frac{d^2P}{dt^2} &= \frac{d}{dt}\left(kPL - kP^2\right) \\
\Rightarrow \frac{d^2P}{dt^2} &= k\left(L\frac{dP}{dt} - 2P\frac{dP}{dt}\right). \\
\Rightarrow \frac{d^2P}{dt^2} &= k(L-2P)\frac{dP}{dt}. \\
\Rightarrow \frac{d^2P}{dt^2} &= k(L-2P)kP(L-P). \\
\Rightarrow \frac{d^2P}{dt^2} &= k^2P(L-2P)(L-P).
\end{aligned}$$

3 Solutions to Third Day Problems

#1. Oil is leaking out of a ruptured tank at a rate given by $r(t) = 50e^{-.02t}$ in thousands of liters per minute. How many liters leaked out during the first hour?

Solution: First, we want to find the water that leaked out in one hour. This quantity has units of volume. The function $r(t)$ is the rate of change of the water as a function of time in minutes. I.e, $r(t) = \frac{dw}{dt}$, where w is the amount of water. Therefore,

$$\int_0^1 r(t)\,dt = \int_0^1 \frac{dw}{dt}\,dt = w(1) - w(0).$$

That is the integral gives us how much water flowed in the first hour. So,

$$\int_0^1 50e^{-.02t}\,dt = -\frac{1}{.02}e^{-.02t}\Big|_0^1 = -\frac{1}{.02}\left(e^{-.02} - 1\right).$$

#2. Let $F(x) = \displaystyle\int_{\pi/2}^{x} \frac{\sin(t)}{t}\,dt$. Find the global maximums and minimums of this function on $[\pi/2, 3\pi/2]$.

Solution: Remember to find global max/mins you need to check the critical points and the end-points of the interval. First, $F'(x) = \dfrac{\sin(x)}{x}$. This function has critical points when the function is zero **OR** the derivative is undefined and the function itself is defined. The only critical point on this interval is $x = \pi$. The values of the function at the endpoints are given by

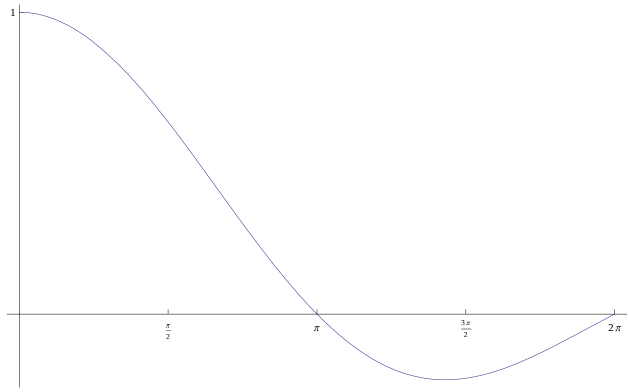
$$F(\pi/2) = \int_{\pi/2}^{\pi/2} \frac{\sin(t)}{t}\,dt = 0,$$

$$F(3\pi/2) = \int_{\pi/2}^{3\pi/2} \frac{\sin(t)}{t}\, dt.$$

Also, the value at the critical point is

$$F(\pi) = \int_{\pi/2}^{\pi} \frac{\sin(t)}{t}\, dt.$$

Now, these integrals cannot be evaluated directly. Instead we will look at the graph of the function and look at the area underneath the curve to determine which quantities are bigger.



It is clear from the picture that $F(0)$ is the global min while $F(\pi)$ is the global max.