1 Solutions to Monday’s Review Session

1.1 What is the domain of the functions \( f(x) = \frac{1}{x-3} \) and \( g(x) = \frac{\sqrt{x+2}}{x-3} \).

A lot of times when you are doing a domain problem you want to ask yourself where the function is undefined. For \( f(x) \), we could have a problem of dividing by zero. So, we must have that \( x \neq 3 \). Therefore, the domain for \( f(x) \) is simply:

\[
D : x \neq 3
\]

Now, for \( g(x) \) we have the added problem that we cannot take the square root of a negative number. Therefore we must make sure that everything underneath the radical is positive or greater than zero. Written in math symbols this means that:

\[
\begin{align*}
x + 2 & \geq 0 \\
\Rightarrow x & \geq -2.
\end{align*}
\]

Also, we have that \( x \neq 3 \) since we still don’t want to divide by zero. Therefore, the domain for \( g(x) \) is given by

\[
D : x \geq -2 \text{ and } x \neq 3.
\]

1.2 If \( f(x) = \sqrt{x} \), evaluate and completely simplify the difference quotient: \( \frac{f(x+h)-f(x)}{h} \).

The goal with any difference quotient problem is to eliminate the \( h \) in the denominator. What makes this problem difficult is the square root. To ”get rid” of the square root I am going to multiply the top and bottom of the fraction by the conjugate.

\[
\begin{align*}
\frac{f(x+h) - f(x)}{h} &= \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\
&= \frac{x+h - x}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \frac{1}{\sqrt{x+h} + \sqrt{x}}.
\end{align*}
\]
1.3 Find a possible formula for the following tables:

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>7</td>
<td>12</td>
<td>17</td>
<td>22</td>
<td>27</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>g(x)</td>
<td>3</td>
<td>5.1</td>
<td>8.67</td>
<td>14.739</td>
<td>25.89151</td>
</tr>
</tbody>
</table>

The first function is linear. To check if a table is a linear function check if it has a constant slope. The first table has a constant slope of 5. Also, the table has a y-intercept of 3. Therefore, the formula for the function is given by:

\[ f(x) = 5x + 3. \]

The second table is exponential. The way you check if a table is exponential is to take the quotient (i.e. divide) of consecutive \( y \) values and see if you get a constant. For this table we have that

\[
\frac{5.1}{3} = 1.7 \\
\frac{8.67}{5.1} = 1.7 \\
\frac{14.739}{8.67} = 1.7 \\
\frac{25.89151}{14.739} = 1.7.
\]

Therefore, this table is an exponential function.

Now, let’s see why this works. Suppose we have a function in the form \( g(x) = ab^x \) that fits the data points of some table. Now, let’s see what happens if we divide two consecutive \( y \)-values. For \( x = 2 \) and \( x = 1 \) we have that

\[ g(2)/g(1) = (ab^2)/(ab^1) = b \]

Now, let’s pick an arbitrary input \( n \). The next consecutive input is \( n + 1 \). If we then divide there corresponding outputs we have that

\[ g(n + 1)/g(n) = (ab^{n+1})/(ab^n) = b. \]

It follows that no matter what two consecutive points you pick their quotient will always be the constant \( b \).

So, if we return to our table above we have that \( b = 1.7 \) and \( a = 3 \), which is the \( y \)-intercept. So, the equation for \( g(x) \) is simply

\[ g(x) = 3(1.7)^x. \]

Also, recall that \( b = 1 + r \) where \( r \) is the decimal representation of the percent rate of change. Therefore, every time \( x \) increases by 1, \( g(x) \) increases by 70%.

Finally, let’s convert our exponential function into the form \( g(x) = ae^{kx} \). We already know that \( a = 3 \), so all we need to do is find \( k \). Well, since this is just another way to represent the same exponential function we must have that for all values of \( x \) the following must be true:

\[
\begin{align*}
3(1.7)^x &= 3e^{kx} \\
\Rightarrow \ln(1.7^x) &= \ln(e^{kx}) \\
\Rightarrow x \ln(1.7) &= kx \\
\Rightarrow \ln(1.7) &= k.
\end{align*}
\]

Remember that \( k \) is called the continuous growth rate.
1.4 What is the range and domain of the function $f(x) = 3 + 2 \ln(x - 4)$.

First, you cannot use your calculator to do this problem. One way to do this problem is to think about this as a transformation of the base graph $g(x) = \ln(x)$. The graph of $g(x) = \ln(x)$ is given below: Now, $\ln(x)$ has a range of $(-\infty, \infty)$ and a domain of $(0, \infty)$. Now, $f(x)$ just is this graph shifted right 4, vertically expanded by a factor of 2, and then vertically shifted up 3. This means that the range gets shifted to $(-\infty, \infty)$ (no change) and the domain is shifted to $(4, \infty)$.

Another way to do this problem is to think about the inverse of $f(x)$. The inverse of $f(x)$ is given by

$$f^{-1}(x) = e^{1/2(x-3)} + 4.$$ 

Now, the domain of $f^{-1}(x)$ is easily seen to be all real numbers since you can input any number. Now, the range of $f^{-1}(x)$ is easy to find since this is basically an exponential function that is shift up vertically 4 units. Therefore, the range is $(4, \infty)$. This means that $f(x)$ must have the opposite domain and range. In this case this means that the domain of $f(x)$ is $(4, \infty)$ and the range of $f(x)$ is $(-\infty, \infty)$.

Finally, the only asymptote of $f(x)$ is the line $x = 4$. 
1.5 If \((2, 3)\) lies on the graph of \(f(x)\). What point must lie on the graph of \(g(x) = -2 \cdot f(x - 4) + 3\).

This is a simple transformation problem. The graph of \(g(x)\) is related to the graph of \(f(x)\) by first shifting right 4, vertically expanding by 2, reflecting across the \(x\)-axis, and then shifting up 3. Therefore, \((2, 3)\) gets translated to the point \((6, -3)\).

Another way to think about this problem is to use our knowledge that \(f(2) = 3\). So if we evaluate \(g(6)\) we have that

\[
g(6) = -2 \cdot f(2) + 3 = -2 \cdot 3 + 3 = -3.
\]

This means that the point \((6, -3)\) lies on the graph of \(g(x)\).

1.6 Determine which of the following functions are odd, even, or neither.

- \(f(x) = \frac{x^2 + 1}{x^6 - 1}\)

\[
f(-x) = \frac{(-x)^2 + 1}{(-x)^6 - 1} = \frac{x^2 + 1}{x^6 - 1} = f(x).
\]

This function is even.

- \(f(x) = \frac{|x|}{x - 2}\)

\[
f(-1) = 1/(-3) = -1/3.
f(1) = 1/(-1) = -1.
\]

This function is neither even or odd.

- \(f(x) = \sin(x) \cos(x)\)

\[
f(-x) = \sin(-x) \cos(-x) = -\sin(x) \cos(x) = -f(x).
\]

This function is odd.