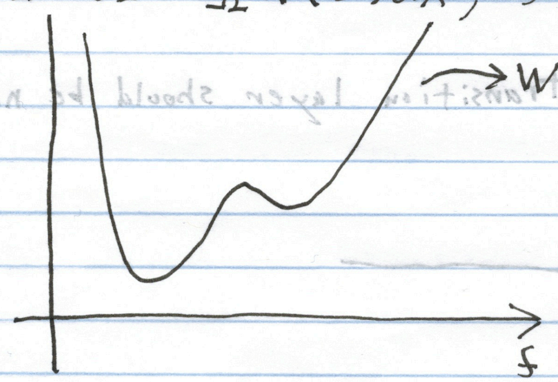
Lecture 8: Γ-convergence

Goals:

- Γ-convergence for 1-D problems
- Gradiant theory of phase transitions
- Dimension reduction
- Connection with relaxation.

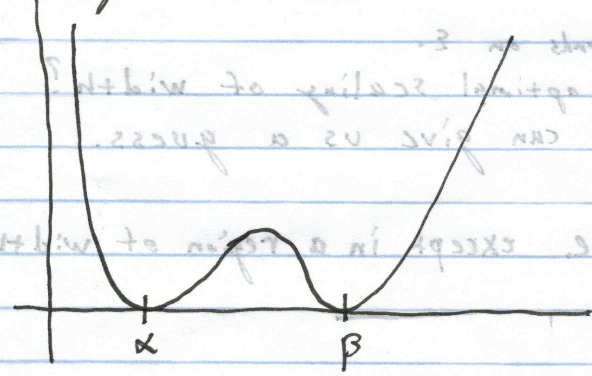
1. Example:

$$I[f] = \int_\Omega W(f)\,dx, \quad A = \{ f \in L^1(\Omega) : \int_\Omega f\,dx = C \}.$$



The two minimum of f are "phases" and W is the free energy.

W.l.o.g. we can assume W is of the following form



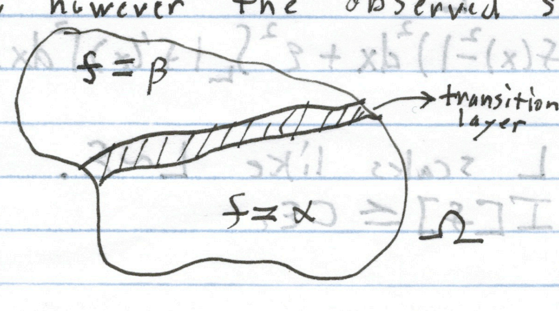
by make the affine change of coordinates

$$W(f) + c_1 f + c_2.$$

since

$$\int (c_1 f + c_2) = c_1 C + c_2.$$

The minimizers are clearly any functions which equal α, β a.e. and satisfy the ~~density cons~~ mass constraint. Practically however the observed state looks like:



We can add in a penalty for transition layers

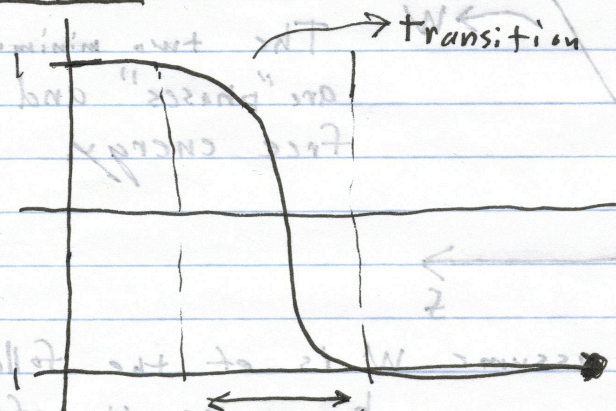
$$I[f] = \int_\Omega \left(W(f) + \varepsilon^2 |\nabla f|^2 \right) dx,$$

with ε a spall parameter. Lets consider the "simple" example:

$$I[f] = \int_0^1 \left(f(x)^2 - 1 \right)^2 + \varepsilon^2 \int_0^1 |f'(x)|^2 dx.$$

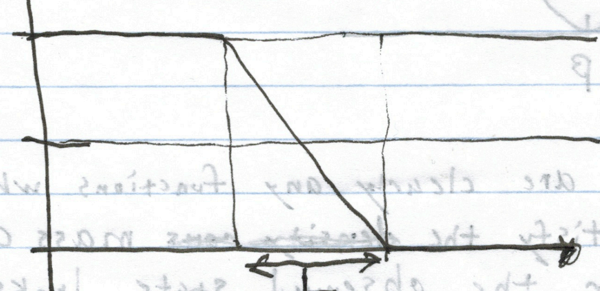
The question we are interested in is what is the behaviour of the minimizers as $\varepsilon \to 0$?

<u>Intuition:</u>



1. How can we guess optimal scaling of width?
   An upper bound can give us a guess.
   Let

   $f_L = 1$ a.e. except in a region of width L



$$I[f_L] = \int_L \left(f(x)^2 - 1 \right)^2 dx + \varepsilon^2 \int_L |f'(x)|^2 dx \leq 4L + \frac{\varepsilon^2}{L}$$

The optimal L scales like $L \sim \varepsilon$.

$$\Rightarrow \min I[f] \leq C\varepsilon.$$

2. We would guess that the width scales like ε. We can get better information by finding a lower bound.

$$\int_0^1 (f(x)^2-1)^2 dx + \varepsilon^2 f'(x)^2 dx \geq 2\varepsilon \int_0^1 |(f(x)^2-1)| \cdot |f'(x)| dx.$$
$$= 2\varepsilon \int_0^1 \left|\frac{d}{dx} g(f(x))\right| dx$$

where

$$g'(t) = |t^2-1| \Rightarrow g(t) = \int_{-1}^{t} |1-s^2| ds.$$

For $-1<t<1$ we have that $g(t) = t - \frac{1}{3}t^3 \Big|_{-1}^{t}$

Consequently,

$$2\varepsilon \int_0^1 \left|\frac{d}{dx} g(f(x))\right| dx = 2(g(1)-g(-1))\varepsilon = 8/3 . \varepsilon.$$

So we have that

$$\int_0^1 (f(x)^2-1)^2 dx + \varepsilon^2 f'(x)^2 dx \geq \frac{8}{3}\varepsilon.$$

Note we can get the optimal form of the transition layer by setting

$$|(f(x)^2-1)| = \varepsilon |f'(x)| \Rightarrow f_\varepsilon(x) = \tanh\left(\frac{x-x_0}{\varepsilon}\right)$$

for a transition layer centered at x_0.

From this construction we see that 1-transition layer is ~~preffered~~ prefered.

Can we make this intuitive construction rigorous in some way? What we would really like is a limit process for functionals

* The inequality above is known as the Modica-Mortola inequality.

* We have shown that the optimal solution has two phases with a transition layer of width $L \approx \varepsilon$. We will make this precise in a bit. For now, lets look at the analogous problem in 2-d.

$$f(x)^4 - 2f(x)^2 + 1 \le 2$$

$$0 \le (a-b)^2 \le a^2+b^2$$ [redacted]

$$\Rightarrow a^2+b^2 \ge 2|a|\cdot|b|.$$

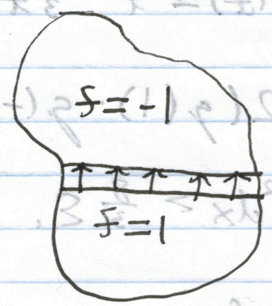
2. Example:

[redacted]

$$I[f] = \int_\Omega (f(x)^2-1)^2\,dx + \varepsilon^2\int_\Omega |\nabla f|^2\,dx,$$

with again the constraint $\int_\Omega f(x)\,dx = C$. By taking an ansatz of a linear transition layer we obtain the upper bound

$$\inf_{f\in A} I[f] \le C\varepsilon.$$



A lower bound for the energy of a transition layer can also be found!

$$\int_\Omega (f(x)^2-1)^2\,dx + \varepsilon^2\int_\Omega |\nabla f|^2\,dx \ge 2\varepsilon\int_\Omega |\nabla f|\cdot|f(x)^2-1|\,dx$$

$$= 2\varepsilon\int_\Omega |\varphi'(f)\,\nabla f|\,dx$$

$$= 2\varepsilon\int_\Omega |\nabla\varphi(f)|\,dx,$$

where $\varphi' = |t^2-1| \Rightarrow \varphi(t) = \int_{-1}^{t} |s^2-1|\,ds.$

Integrating across the "jump" we have that:

$$\int_\Omega (f(x)^2-1)^2\,dx + \varepsilon^2\int_\Omega |\nabla f|^2\,dx \ge 2[\varphi(1)-\varphi(-1)]\cdot \text{Length}(T.L.)$$

(T.L. = transition layer).

$$\Rightarrow \int_\Omega (f(x)^2-1)^2\,dx + \varepsilon^2\int_\Omega |\nabla f|^2\,dx \ge \frac{8}{3}\varepsilon\,\mathrm{Per}_\Omega\{x : f(x)=1\}.$$

* The previous calculation is very formal!

* Remark: optimal transition layer is then

$$f_\varepsilon(x) = \tanh\left(\frac{d(x, T.L.)}{\varepsilon}\right)$$

3. Our first Γ-limit

We would like to say in some sence that

$$\text{"}\lim_{\varepsilon\to 0}\text{"}\left(I_\varepsilon[f] = \int_\Omega (f(x)^2-1)^2\,dx + \varepsilon^2 \int_\Omega |\nabla f|^2\,dx \right) \text{"="}\ \frac{8}{3}\,\varepsilon\,\mathrm{Per}_\Omega\{f(x)=1\ a.e.\} + \text{h.o.t.}$$

Of course this doesen't even make sense formally since the r.h.s. contains ε.

$\Gamma\text{-}\lim_{\varepsilon\to 0}$ ⟵

$$\boxed{\text{"}\lim_{\varepsilon\to 0}\text{"}\ \frac{1}{\varepsilon} I_\varepsilon[f] = \begin{cases} \frac{8}{3}\,\mathrm{Per}_\Omega\{f(x)=1\ a.e.\}, & \text{if } f(x)=\pm 1\ a.e. \\ \infty, & o.w. \end{cases}}$$

Let $\mathcal{I}_\varepsilon = \frac{1}{\varepsilon} I$ and

$$\boxed{I_0 = \begin{cases} \frac{8}{3}\,\mathrm{Per}_\Omega\{f(x)=1\ a.e.\} & \text{if } f(x)=\pm 1\ a.e. \\ \infty & o.W. \end{cases}}$$

<u>Observations</u>:

1. If $f_\varepsilon \to f_0$ in L^1 as $\varepsilon\to 0$, then
$$I_0[f_0] \le \liminf_{\varepsilon\to 0} I_\varepsilon[f_\varepsilon].$$ (liminf inequality)

~~proof:~~

~~a) If $I_0[f_0]=\infty$ let $f_\varepsilon \to f_0$ in L^1. Then either $f_\varepsilon \to f_0$ in L^4 in which case taking limit $\varepsilon\to 0$ yields $I_\varepsilon[f_\varepsilon]\to\infty$ or $f_\varepsilon \not\to f_0$ in L^4 meaning $\|f_0\|_{L^4}=\infty$ which yields the same result upon taking limits.~~

is is embarassing gibberish. ⟵

2. There exists $f_\varepsilon \to f_0$ in L^1 as $\varepsilon\to 0$ such that
$$I_\varepsilon[f_\varepsilon] \to I_0[f_0]$$ (recovery sequence)

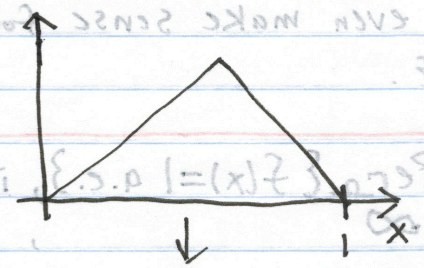
3. If f_ε satisfies $I_\varepsilon[f_\varepsilon]$ is uniformly bounded then f_ε is compact in $L^1(\Omega)$. This is why it is important to use L^1 topology.

* We will make these statements precise in a bit.
* Items 1 and 2 comprise the definition of a Γ-lim.

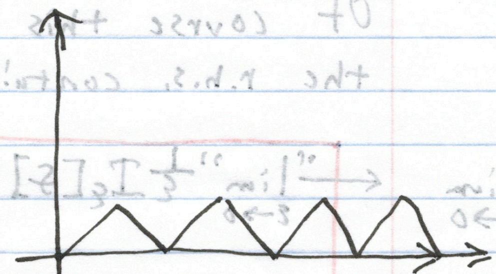
Consequences:

1. $I_\varepsilon[f] = \int_0^1 [\varepsilon f''(x)^2 + \frac{1}{\varepsilon}(f'(x)^2-1)^2 + \lambda f(x)^2]\,dx$

$$\Gamma\text{-}\lim_{\varepsilon\to 0} I_\varepsilon[f] = \begin{cases} \int_0^1 \lambda f(x)^2 + \frac{8}{3}\#\text{teeth}, & \text{if } |f'(x)|=1 \text{ a.e.} \\ \infty, & \text{o.w.} \end{cases}$$



optimum for small λ



optimum for large λ

Remark: Γ-convergence is stable under ~~higher~~ lower order perturbations. The relevant topology of the Γ-lim is $W^{1,1}$ the lower order term $\lambda f(x)^2$ is compact in this topology for a minimizing sequence so it does not affect the Γ-lim.

$$\Gamma\text{-}\lim_{\varepsilon\to 0} J_\varepsilon + J = J_0 + J.$$

2. Claim: Γ-convergence $\Rightarrow$ Convergence of minimizers, and convergence of minimizing value.

~~proof~~:

a.) If $\Gamma\text{-}\lim_{\varepsilon\to 0} I_\varepsilon = I_0$ then

$$\limsup_{\varepsilon\to 0} (\min I_\varepsilon) \le \min I_0 \leftarrow \text{(claim!)}$$

This follows from the recovery sequence

$$\limsup_{\varepsilon\to 0} (\min I_\varepsilon[X_\varepsilon]) \le \limsup_{\varepsilon\to 0} I_\varepsilon[X_\varepsilon] = \min I_0$$

b.) By lim-inf inequality we have

$$\min I_0 \le \liminf_{\varepsilon\to 0} \min I_\varepsilon$$

4. Γ-lim

Definition - A sequence $I_\varepsilon : X \to \mathbb{R}$ Γ-converges in X to $I_0 : X \to \mathbb{R}$ if $\forall f \in X$:

$$\boxed{\begin{array}{ll} 1.\ I_0[f] \le \liminf_{\varepsilon\to 0} I_\varepsilon[f_\varepsilon], \text{ if } f_\varepsilon \to f & \text{(liminf inequality)} \\ 2.\ \text{There exists } f_\varepsilon \to f \text{ such that} & \\ \quad I_0[f] \ge \limsup_{\varepsilon\to 0} I_\varepsilon[f_\varepsilon] & \text{(recovery sequence)} \end{array}}$$

We then write $\Gamma\text{-}\lim_{\varepsilon\to 0} I_\varepsilon = I_0$.

Comment: Property 2 can be equivalently written as there exists $f_\varepsilon \to f$ such that
$$I_0[f] = \lim_{\varepsilon\to 0} I_\varepsilon[f_\varepsilon].$$

Comment: The previous definition consits of upper and lower estimates. Moreover, it gives us l.s.c. for free (we will see this later). To obtain a decent notion for existence of a minimum we will also need a notion of coerceivity.

Definition: A functional $I : X \to \mathbb{R}$ is coercive if for all $M \in \mathbb{R}$ the set $\{I \le M\}$ is precompact in X with its equipped topology. A functional is mildly coercive if $\exists K \subset X$ compact such that $\inf_X I[f] = \inf_K I[f]$. A sequence is mildly equi-coercive if $\exists K \subset X$ compact such that $\inf_X I[f] = \inf_K I_\varepsilon$ for all ε.

Note! We don't need equi-coercivity here.

Theorem - If $\Gamma\text{-}\lim_{\varepsilon\to 0} I_\varepsilon = I_0$ then
$$\inf_X I_0 \le \lim_{\varepsilon\to 0} \inf_X I_\varepsilon.$$

proof.

We will assume that the minimum is obtained since it changes the arguments little.

a.) Let f_ε be a recovery sequence for a minimizer f^* of I_0. Then,

$$\limsup_{\varepsilon \to 0} \min_{X} I_\varepsilon[f] \leq \limsup_{\varepsilon \to 0} I_\varepsilon[f_\varepsilon] = \min_{X} I_0[f^*]$$

b.) By the liminf inequality we have that

$$\min_{X} I_0[f] \leq \liminf_{\varepsilon \to 0} \min_{X} I_\varepsilon[f].$$

Items a) and b) $\Rightarrow \min_{X} I_0 = \lim_{\varepsilon \to 0} \min_{X} I_\varepsilon.$

■

Theorem - If $\Gamma\text{-}\lim_{\varepsilon \to 0} I_\varepsilon = I_0$ and I_ε is mildly equi-coercive then if f_ε minimizes I_ε then there exists a subsequence f_{ε_K} and $f^* \in X$ such that $f_{\varepsilon_K} \to f^*$ and $\min_{X} I_0[f] = I^*[f^*]$.

~~proof~~: From equi-coercivity we get compactness so that there exists f^* and a subsequence such that $f_{n_K} \to f^*$. Therefore,

$$I_0[f^*] \leq \liminf_{K \to \infty} I_\varepsilon[f_{\varepsilon_K}] = \lim_{\varepsilon \to 0} \min_{X} I_\varepsilon[f] = \min_{X} I_0[f].$$

$\Rightarrow f^*$ is a minimum.

■

Theorem - If $\Gamma\text{-}\lim_{\varepsilon \to 0} I_\varepsilon = I_0$ then I_0 is l.s.c.

~~proof~~:

Otherwise $\exists$ some $f \in X$ and a sequence $f_\varepsilon \to f$ such that

$$I_0[f] \geq \lim_{\varepsilon \to 0} I_0[f_\varepsilon]$$

By Γ-convergence $\exists$ a sequence $f_{\varepsilon,\delta}$ such that

$$\lim_{\delta \to 0} f_{\varepsilon,\delta} = f_\varepsilon$$

and

$$\lim_{\delta \to 0} I_\delta[f_{\varepsilon,\delta}] = I_0[f_\varepsilon]$$

Let $\eta = \frac{1}{4}\left(I_0[f] - \lim_{\varepsilon \to 0} I_0[f_\varepsilon]\right) > 0$. For every $\varepsilon > 0$ we can find $\delta(\varepsilon)$ with

$$I_{\delta(\varepsilon)}[f_{\varepsilon,\delta(\varepsilon)}] - I_0[f_\varepsilon] < \eta \quad *$$

Choose ε small enough that

$$I_0[f] - I_0[f_\varepsilon] > 3\eta \quad ** \text{ (Follows from def.)}$$

and

$$I_0[f] - I_{\delta(\varepsilon)}[f_{\delta(\varepsilon)}] < \eta \quad *** \text{ (liminf inequality)}$$

$$\Rightarrow \eta < I_{\delta(\varepsilon)}[f_{\varepsilon,\delta(\varepsilon)}] - I_{\delta(\varepsilon)}[f_{\varepsilon,\delta(\varepsilon)}] = 0$$

which is a contradiction.

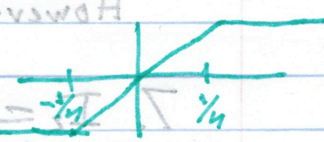
■

examples:

1. $\mathbb{X}=[0,1]$, $f_n(x)=nx^2$

$$\Gamma\text{-}\lim_{n\to\infty} f_n(x) = \begin{cases} 0, & x=0 \\ \infty, & x\neq 0. \end{cases}$$

2. $\mathbb{X}=\mathbb{R}$, $f_n(x)=\begin{cases} 1, & x\geq 1/n \\ nx, & -\frac{1}{n}\leq x\leq \frac{1}{n} \\ -1, & x\leq -\frac{1}{n} \end{cases}$



$$\Gamma\text{-}\lim_{n\to\infty} f_n(x) = \begin{cases} 1, & x>0 \\ -1, & x\leq 0 \end{cases}$$

* Pointwise and Γ-lim are not the same.

3. example:

$\mathbb{X}=[0,1]$, $f_n(x)=n(x-\frac{1}{n})^2 = nx^2 - 2 + \frac{1}{n}$

$$\Gamma\text{-}\lim_{n\to\infty} f_n(x) = \begin{cases} 0, & x=0 \\ \infty, & x\neq 0. \end{cases}$$

* Lower order perturbations do not contribute to the Γ-lim. Take the sequence $x_n = 1/n$.

4. example:

$f_n(x) = \frac{1}{nx} + x$, $\mathbb{X}=[0,1]$

* Note if $g_n(x)=x$ then $\Gamma\text{-}\lim_{n\to\infty} g_n = x$.

If there exists x_n such that $f_n(x_n) \to \Gamma\text{-}\lim_{n\to\infty} g_n(x) = x$ then

$$\Gamma\text{-}\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} f_n(x_n) \geq \inf_{x_n\to x} \liminf_{n\to\infty} f_n(x_n) \geq \Gamma\text{-}\lim_{n\to\infty} f_n(x) \geq \Gamma\text{-}\lim_{n\to\infty} g_n(x).$$

$$\Rightarrow \Gamma\text{-}\lim_{n\to\infty} g_n = \Gamma\text{-}\lim_{n\to\infty} f_n$$

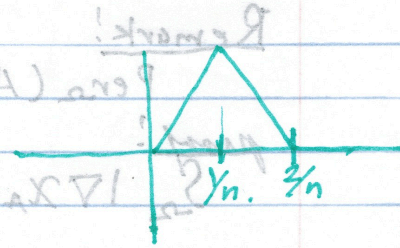
Take, $x_n = \begin{cases} x, & x\neq 0 \\ 1/\sqrt{n}, & x=0 \end{cases}$ $\longrightarrow$ This comes from balancing the two terms.

5. example:

Does $\Gamma\text{-}\lim_{\varepsilon\to 0}(-I_\varepsilon) = -\Gamma\text{-}\lim_{\varepsilon\to 0} I_\varepsilon$?

No, let

$$f_n(x) = \begin{cases} nx, & 0<x<1/n \\ 2-nx, & \frac{1}{n}<x<2/n \\ 0, & \text{o.w.} \end{cases}$$



6. example:

$f_n(x) = \sin(nx)$

$\Gamma\text{-}\lim_{n\to\infty} f_n(x) = -1$

However, f_n does not converge pointwise a.e.

7. $I_\varepsilon = I$ for all ε.

$\Gamma\text{-}\lim_{\varepsilon\to 0} I_\varepsilon = scI = I^*$.

* we have shown this before and it was one of the characterizations of the relaxation.

5. Modica-Mortola

Claim: If

$$I_\varepsilon = \frac{1}{\varepsilon}\int_\Omega (f(x)^2-1)^2\,dx + \varepsilon\int_\Omega |\nabla f|^2\,dx, \quad \int_\Omega f^2(x)\,dx = c$$

Then,

$$\Gamma\text{-}\lim_{\varepsilon\to 0} I_\varepsilon = \begin{cases} \frac{8}{3}\,Per_\Omega\{f(x)=1\}, & \text{if } |f(x)|=1 \text{ a.e.} \\ \infty, & \text{o.w.} \end{cases}$$

w.r.t. L^1 topology. Moreover, I_ε is equicoercive in L^1.

Definition - Define the space of functions of bounded variation $BV(\Omega)$ by

$$BV(\Omega) = \{f \in L^1(\Omega) : \int_\Omega |\nabla f| < \infty\}$$

where we are using the notation.

$$\boxed{\int_\Omega |\nabla f| = \sup_{\substack{g\in C_c^\infty(\Omega,\mathbb{R}^n)\\ |g|\le 1 \text{ p.w.}}} \int_\Omega f\cdot\nabla g\,dx.}$$

Remark: $f \in B.V.(\Omega) \iff f\in L^1$ and ∇f is a vector valued measure on Ω with finite total variation $\int|\nabla f|$.

Remark:

$$Per_\Omega(A) = \int_\Omega |\nabla \chi_A|$$

proof:

$$\int_\Omega |\nabla\chi_A| = \sup_{\substack{g\in C_c^\infty\\ |g|\le 1}} \int_A \nabla\cdot g = -\sup_{\substack{g\in C_c^\infty\\ |g|\le 1}} \int_{\partial A} g\,ds = Per_\Omega(A).$$

properties of B.V.

1. If $f_\varepsilon \to f$ in $L^1(\Omega)$ then

$$\int_\Omega |\nabla f| \le \liminf_{\varepsilon\to 0} \int_\Omega |\nabla f_\varepsilon|$$

~~proof~~:

Let $g \in C_0^\infty$ with $|g| \le 1$. Then,

$$\int_\Omega f\, \nabla \cdot g\, dx = \lim_{n\to\infty} \int_\Omega f_n \nabla \cdot g \le \liminf_{n\to\infty} \int_\Omega |\nabla f_n|$$

Taking sup over both sides gives us the result. ■

2. If $f_n \to f$ in $L^1(\Omega)$ and $\int |\nabla f_n| < M$ then $f \in B.V.(\Omega)$

~~proof~~:

If $g \in C_0^\infty$ then for $i = 1, \dots, n$

$$\lim_{n\to\infty} \int_\Omega g\,(D_i f_n) = -\lim_{n\to\infty} \int_\Omega f_n D_i g = -\int_\Omega f\, D_i g$$

$$\Rightarrow \left| \int_\Omega f\, D_i g \right| \le \sup|g| \liminf_{n\to\infty} \int_\Omega |\nabla f_n| < \infty.$$ ■

3. Bounded sets in B.V. are compact in L^1.

~~proof~~:

By mollification there exists $h_n \in C^\infty$ with $\|f_n - h_n\|_{L^1} < \frac{1}{n}$, $\int |\nabla h_n| \le M+1$

$\Rightarrow h_n$ is bounded in $W^{1,1}(\Omega)$ so h_n is compact in L^1.

So, $h_n \xrightarrow{L^1} f$ and $f_n \xrightarrow{L^1} f$. By property 2 it follows that $f \in BV$. ■

~~proof~~ of Γ-convergence

1. (liminf inequality)

First we show that it sufficies to consider f such that $|f(x)| = 1$ a.e. since if $f_\varepsilon \to f$ in $L^1(\Omega)$ then

$$\liminf_{\varepsilon\to 0} \int \left(\varepsilon |\nabla f_\varepsilon|^2 + \tfrac{1}{\varepsilon}(f_\varepsilon^2 - 1)^2\right) dx \ge \liminf_{\varepsilon\to 0} \tfrac{1}{\varepsilon} \int (f_\varepsilon^2 - 1)^2 dx$$

$$\ge \liminf_{\varepsilon\to 0} \int \tfrac{1}{\varepsilon} (f_\varepsilon^2 - 1)^2 dx$$

$$= \infty \text{ if } f \not\equiv 1 \text{ a.e.}$$

Now we show that it is sufficient to consider f_ε such that

$$-1 \le f_\varepsilon \le 1$$

since otherwise we could replace f_ε by its truncation

$$f_\varepsilon^* = \begin{cases} 1, & f_\varepsilon \ge 1 \\ f_\varepsilon, & -1 \le f_\varepsilon \le 1 \\ -1, & f_\varepsilon < 1 \end{cases}$$

without changing the L^1 limit.

Now suppose $-1 < f_\varepsilon < 1$ and $f_\varepsilon \to f$ in L^1 with $|f| = 1$ a.e.
Then,

$$I_\varepsilon[f_\varepsilon] \ge 2 \int_\Omega |\nabla \varphi(f_\varepsilon)| \, dx,$$

where

$$\varphi(t) = \int_{-1}^{t} |1 - t^2| \, dt \quad \text{(lower bound)}$$

By D.C.T. $\varphi(f_\varepsilon) \to \varphi(f)$. By l.s.c. of B.V. norm we have

$$\liminf_{\varepsilon \to 0} I_\varepsilon[f_\varepsilon] \ge 2 \int_\Omega |\nabla \varphi(f)| = \tfrac{8}{3} \operatorname{Per}\{|f| = 1\}.$$

2. (Recovery Sequence)

$$\int \left(\varepsilon |\nabla f_\varepsilon|^2 + \tfrac{1}{\varepsilon}(f_\varepsilon^2 - 1)^2 \right) dx \ge 2 \int |\nabla \varphi(f_\varepsilon)| \, dx$$

equality is obtained when

$$f_\varepsilon = \pm \tanh\left(\frac{\operatorname{dist}(X, TL)}{\varepsilon} \right)$$

~~This is essentially the~~

This is essentially the recovery sequence. ∎