Lecture 7: Relaxation for 1-D problems.

Goals:

1. Define relaxation of problems.
2. Define w.l.s.c. envelope and discuss connection with convex envelope.
3. Prove relaxation theorems.

References:

1. Jost - Li Jost ~ Calculus of Variations Chapter 5
2. Dacorogna - Direct Methods in the Calculus of Variations. Chapter 9
3. Γ-convergence for Beginners - Braides - Chapter 1.

1. Review

a.) $\Omega \subset \mathbb{R}^n$

b.) $A = W_0^{1,q}(\Omega, \mathbb{R}) \longrightarrow$ 1-D case

c.) $I : A \to \mathbb{R}$ is defined by $I[f] = \int_\Omega L(x, f(x), Df)\,dx$.

d.) We will typically use the notation ~~L~~ $L(x, z, p)$ ~~to~~ when we are talking about properties of the Lagrangian.

Theorem ~ If $L(x, z, \cdot)$ is convex then $I[\cdot]$ is w.l.s.c.

Theorem - If L is coercive meaning
$$L(x, z, p) \geq -\gamma + \alpha |p|^q, \quad \text{for } 1 < q \leq \infty$$
and L is lower semicontinuous than L has a minimum.

In applications many problems fail to be w.l.s.c. (convex) how can we obtain useful information from the problem? There are three possibilities

1. Modify the problem in some way. (Singular Perturbation)
2. Add in new solutions (Completion)
3. Relax the problem.

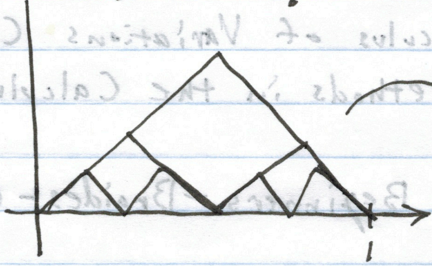
2. Relaxation

The key idea is that minimizing sequences capture a lot of information about the variational problem. We will assume coercivity and hence the convergence of a minimizing sequence.

example: (Bolza-Young)

$A = W_0^{1,4}$ and $I: A \to \mathbb{R}$ is defined by

$$I[f] = \int_0^1 (f'(x)^2-1)^2\,dx + \int_0^1 f(x)^2\,dx.$$



minimizing sequence f_n satisfies

$f_n \to 0$ and $I[f_n] \to 0$,

but $I[0] = 1$. We can use the minimizing sequences to define a new functional.

Definition - The relaxed functional $I^*[\cdot]$ is defined by:

$$I^*[f] = \inf_{f_n \to f}\left\{\liminf_{n\to\infty} I[f_n]\right\}$$

example:

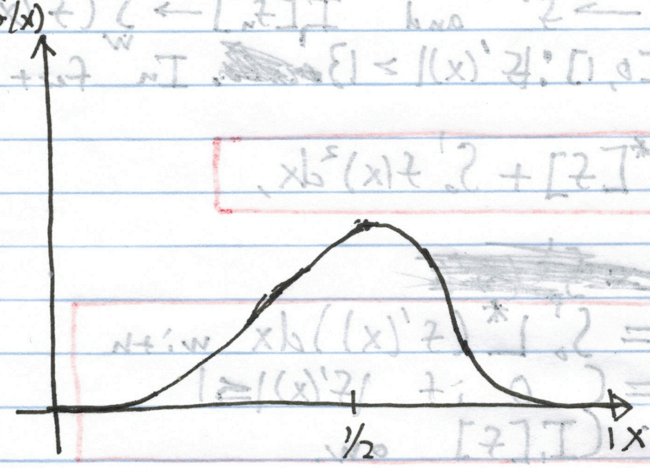
What is the relaxation of Bolza-Young? We have to think about charecterising ~~minimizing sequences~~ the energy of weakly convergent sequences in $W_0^{1,4}(\Omega)$. Let $f \in W_0^{1,4}(\Omega)$ and suppose $f_n \rightharpoonup f$. Then,

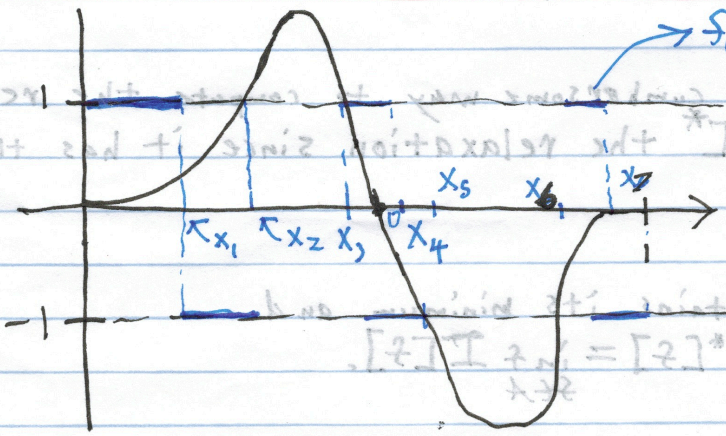
$$f_n' \xrightarrow{L^4} f' \quad \text{and} \quad f_n \xrightarrow{L^2} f.$$

So we really only care about the functional:

$$I_1[f] = \int_0^1 (f'(x)^2-1)^2\,dx.$$

For sets where $|f'(x)| \le 1$ we can construct sequences that oscillate in slope between -1 and 1 and ~~co~~ weakly converge to f





We are only interested where $|f'(x)| < 1$.

Recall, weak convergence is in some sense like convergence of averages. So, we select $|f_n'| = 1$ and switch signs to obtain the average and in the limit get weak convergence.

How do we select where x_1 should be? We want

$$\int_\Omega f_1'(x)\,dx = \int_\Omega f'(x)\,dx.$$

So,

$$x_1 - (x_2 - x_1) = \int_0^{x_2} f'(x)\,dx$$

$$\Rightarrow \boxed{x_1 = \frac{\int_0^{x_2} f'(x)\,dx + x_2}{2}} \qquad \boxed{x_7 = \frac{\int_{x_6}^{1} f'(x)\,dx + x_6 + 1}{2}}$$

Similarily

$$\boxed{x_4 = \frac{\int_{x_3}^{x_5} f'(x)\,dx + x_3 + x_5}{2}}$$

To construct f_2 do similar constructions on smaller sets.

So, we have $f_n' \to f'$ and $I_1[f_n] \to \int_W (f'(x)^2-1)^2\,dx$ where $W = \{x \in [0,1] : |f'(x)| > 1\}$. In fact it is true that:

$$\boxed{I^*[f] = I_1^*[f] + \int_0^1 f(x)^2\,dx,}$$

where
$$\boxed{\begin{aligned} &I_1^*[f] = \int_0^1 L^*(f'(x))\,dx \text{ with} \\ &L^*[f] = \begin{cases} 0 & \text{if } |f'(x)| \le 1 \\ I_1[f] & \text{o.w.} \end{cases}\end{aligned}}$$

This is a really cumbersome way to compute the relaxation! We call ~~this~~ I^* the relaxation since it has the following useful property:

Theorem: I^* obtains its minimum and
$$\min_{f\in A} I^*[f] = \inf_{f\in A} I[f].$$

proof: Let f_n be a minimizing sequence ~~that~~ and let f^* be its (weak) limit. Now,
$$I^*[f^*] = \inf_{g_n \to f^*} \left\{ \liminf I[g_n] \right\},$$
$$\Rightarrow I^*[f^*] \le \liminf_{n\to\infty} I[f_n] = \inf_{f\in A} I[f].$$

For contradiction suppose there exists $g \in A$ such that
$$I^*[g] < I^*[f^*].$$

Then $\exists$ a sequence $f_n \in A$ such that $f_n \to g$ and
$$\inf_{f\in A} I[f] \le \liminf_{n\to\infty} I[f_n] \le I^*[f^*] = \inf_{f\in A} I[f]$$

$$\Rightarrow I^*[f^*] = I^*[g]. \quad \blacksquare$$

3. Lower Semicontinuity

If a functional is coercive then l.s.c. is needed to guarantee convergence of minimizing sequences. Another method of relaxation is to find the "closest" l.s.c functional to the original problem.

Definition - Let $I: A \to \mathbb{R}$. Its (weak) l.s.c envelope scI is the greatest lower semicontinuous functional less than or equal to I:

$$scI = \sup\{\bar{I} \text{ l.s.c.} : \bar{I}[f] \leq I[f]\}.$$

Theorem - If I is coercive then scI assumes its minimum and

$$\min_A scI = \inf_A I.$$

Moreover, every minimizing sequence for I is a minimizing sequence for scI.

proof:

Let f_n be a minimizing sequence for I with weak limit f^*. Then,

$$sc\,I[f^*] \leq \liminf_{n\to\infty} sc\,I[f_n]$$

$$\leq \liminf_{n\to\infty} I[f_n] \quad \text{(limint inequality)}$$

$$= \inf_A I[f]$$

However the constant functional $J[g] = \inf_{f\in A} I[f]$ is l.s.c. and clearly $J[g] \leq I[g]$. Therefore,

$$J[g] = \inf_{f\in A} I[f] \leq sc\,I[g] \quad \text{(recovery sequence).}$$

Therefore, setting $g = f^*$ we have that $sc\,I[f^*] = \inf_{f\in A} I[f]$.

$$\Rightarrow sc\,I[f^*] = \inf_{f\in A} I[f] = J[g] \leq sc\,I[g].$$

different g. ▨

Theorem - If I is coercive then
$$sc\,I[f] = I^*[f].$$

proof: First we prove that $I^*[f]$ is l.s.c.. We have to check that for all $f_n \to f$ that:

$$\inf_{f_m \to f}\left\{\liminf I[f_m]\right\} \leq \liminf_{n\to\infty} \inf_{g_m \to f_n}\left\{\liminf I[g_m]\right\}$$

Otherwise, we could find some diagonal sequence $f_{n,m_n} \to f$ with

$$\inf_{f_m \to f}\left\{\liminf I[f_m]\right\} - \delta > I[f_{n,m_n}]$$

which is impossible. Therefore, I^* is w.l.s.c.

Also, $I^* \leq I$ and for every l.s.c. $\Phi \leq I$ we have for $f_n \to f$

$$\Phi[f] \leq \liminf_{n\to\infty} \Phi[f_n] \leq \liminf_{n\to\infty} I[f_n]$$

$$\Rightarrow \Phi[f] \leq I^*[f]$$

Therefore, I^* is the largest l.s.c. function below I. ∎

Corollary - $J = sc\,I$ if and only if the following conditions are satisfied

i.) Whenever $f_n \to f$
$$J[f] \leq \liminf_{n\to\infty} J[f_n] \quad \text{(liminf inequality)}$$

ii) For every f there exists a sequence $f_n \to f$ with
$$J[f] \geq \lim_{n\to\infty} J[f_n] \quad \text{(recovery sequence)}$$

example:

Define $I : L^p \to \mathbb{R}$ by

$$I[f] = \begin{cases} \int_\Omega |\nabla f|^p dx + \int_\Omega |f|^p dx, & \text{if } f \in \mathcal{E}'(\Omega) \\ \infty & \text{, o.w.} \end{cases}$$

Claim:

$$sc\, I = \begin{cases} \int_\Omega |\nabla f|^p dx + \int_\Omega |f|^p dx, & \text{if } f \in W^{1,p}(\Omega) \\ \infty & \text{, o.w.} \end{cases}$$

proof:

We need to show properties 1 and 2.

(i) (lim inf inequality).

Suppose $f_n \to f$. Select a subsequence f_{n_k} with the property
$$\lim_{k\to\infty} sc\, I[f_{n_k}] = \liminf_{n\to\infty} sc\, I[f_n].$$

We may also assume this is finite since $W^{1,p}(\Omega) \subset \mathcal{E}'$. Then, f_{n_k} is bounded in $W^{1,p}(\Omega)$ and hence has a weakly convergent subsequence which also converges strongly in $L^p(\Omega)$. This limit is of course f. Since norms are weakly lower semicontinuous we have that
$$sc\, I[f] \leq \lim_{k\to\infty} sc\, I[f_{n_k}] = \liminf_{n\to\infty} sc\, I[f_n] \leq \liminf_{n\to\infty} I[f_n].$$

(ii) (recovery sequence)

This follows from density.

4. Convexity

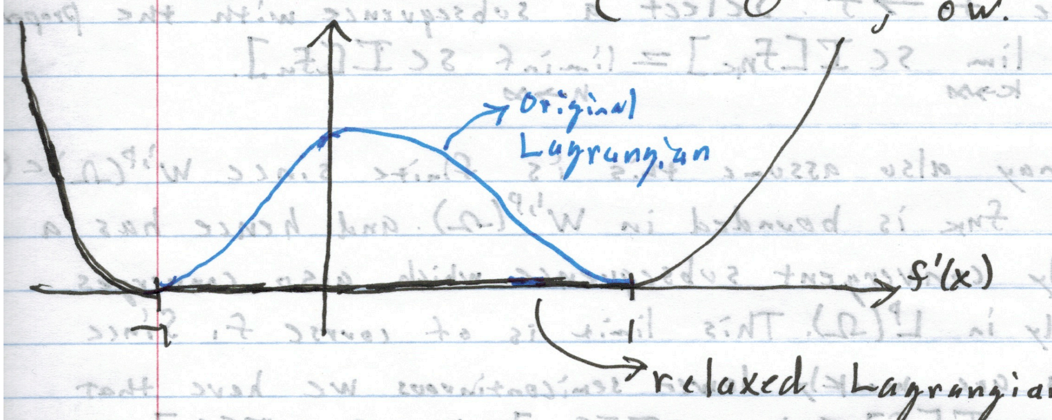
Theorem - If $L(p,z,x) = L(p)$ then I is w.l.s.c. if and only if L is convex.

Definition - The convex envelope of $L(p)$ is denoted by cnL and is the largest convex function below L.

$$cnL = \sup\{H : H \text{ is convex and } H \leq L\}.$$

Corollary: If $L(p,z,x) = L(p)$ then
$$sc\, I = I^* = cn\, I.$$

example:
$$I[f] = \int_0^1 (f'(x)^2 - 1)^2\,dx$$
is not convex. Its relaxation is
$$I^*[f] = \int_0^1 W(f'(x))\,dx,$$
where
$$W(f'(x)) = \begin{cases} (f'(x)^2-1)^2, & \text{if } f'(x) \geq 1 \\ 0, & \text{o.w.} \end{cases}$$



How to compute cn f → if possible draw a picture
Otherwise Legendre transform does the "trick"!

$$\boxed{\begin{aligned} L^{**}(p) &= \sup_{q \in \mathbb{R}^n} \{ \langle q, p \rangle - L^*(q) \} \\ L^*(q) &= \sup_{p \in \mathbb{R}^n} \{ \langle q, p \rangle - L(p) \}. \end{aligned}}$$

Theorem - If $L(p,z,x)$ is convex in p then I is w.l.s.c.

Corollary - $I_{cn}[f] = \int_\Omega cn_p\, L(p,z,x)\,dx \leq I^*[f]$.

To verify that $\int_\Omega cn_p\, L\,dx = I[f]$ we just show liminf and recovery sequence properties.

Remark - In fact for $L(p, z, x): \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$
$$sc\, I = I^* = cn\, I.$$

Summary. -

1. If a functional is not convex it is not clear a priori if a minimum exists.
2. Numerics need to be computed on the convexified problem.
3. Once a potential minimizer is computed for the relaxed problem ~~it is compared~~ it is compared with the value of the original functional. I.e.
   1. Construct minimizing sequence f_n such that $I^*[f_n] \to \min_A I^*[f]$. and let f^* be the (weak-limit) of f_n.
   2. Compute $I[f^*]$ if $I[f^*] = I^*[f]$ we have found our minimum!! (Also a theoretical tool).
   3. Note: Extra growth conditions
   $$L(p, z, x) \leq \alpha |p|^q + \gamma$$
   gives strong convergence which is also important for numerics.