# Lecture 4: Convexity (1-D case)

## Goals:

1. For 1-D problems, lower semicontinuity $\Leftrightarrow$ convex
2. Existence
3. Uniform convexity $\Rightarrow$ uniqueness
4. Improvement to strong convergence.

## Notation:

- $\Omega \subset \mathbb{R}^n$ bounded set with smooth boundary
- $L: \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$ is the Lagrangian.
- We will write

$$L(p, z, x) = L(p_1, \dots, p_n, z, x_1, \dots, x_n)$$

The variables are proxies for

$$p = \nabla f$$
$$z = f$$

- $I[f] = \int_\Omega L(\nabla f, f, x)\,dx, \quad f = g$ on $\partial\Omega$.

## 1. Euler-Lagrange Equations

Lets do a formal calculation to derive the E-L equations (see lecture 2).

$$\cancel{\delta I = \int_\Omega \left(\frac{\partial L}{\partial p}\delta \nabla f + \frac{\partial L}{\partial z}\delta f\right)dx}$$

$$\delta I = \int_\Omega \left(\frac{\partial L}{\partial p}\cdot \nabla \delta f + \frac{\partial L}{\partial z}\delta f\right)dx$$

$$= \int_\Omega \left(-\nabla\cdot\frac{\partial L}{\partial p} + \frac{\partial L}{\partial z}\right)\delta f\,dx.$$

If $f \in C^2(\Omega)$ minimizes $I$ then

$$\boxed{\left.\left(\frac{\partial L}{\partial z} - \sum_{i=1}^n \frac{\partial}{\partial x_i}\frac{\partial L}{\partial p_i}\right)\right|_f = 0}$$

## Special Case:

1. If $\Omega \subset \mathbb{R}$ and $L$ is independant of $z$ then

$$\boxed{\frac{\partial L}{\partial p} = c}$$

2. $\Omega \subset \mathbb{R}$. The second form of the E-L equations

$$\boxed{\left.\left(\frac{d}{dx}\left[L - f'(x)\frac{\partial L}{\partial p}\right] = \frac{\partial L}{\partial x}\right)\right|_f.}$$

~~proof~~:

Differentiating we have that

$$\frac{d}{dx}\left(L - f'(x)\frac{\partial L}{\partial p}\right) = \frac{\partial L}{\partial x} + f'(x)\frac{\partial L}{\partial z} + \cancel{f''(x)\frac{\partial L}{\partial p}}$$
$$- \cancel{f''(x)\frac{\partial L}{\partial p}} - f'(x)\frac{d}{dx}\left(\frac{\partial L}{\partial p}\right)$$
$$= \frac{\partial L}{\partial x} + f'(x)\underbrace{\left(\frac{\partial L}{\partial z} - \frac{d}{dx}\frac{\partial L}{\partial p}\right)}_{=0 \text{ E-L.}}$$

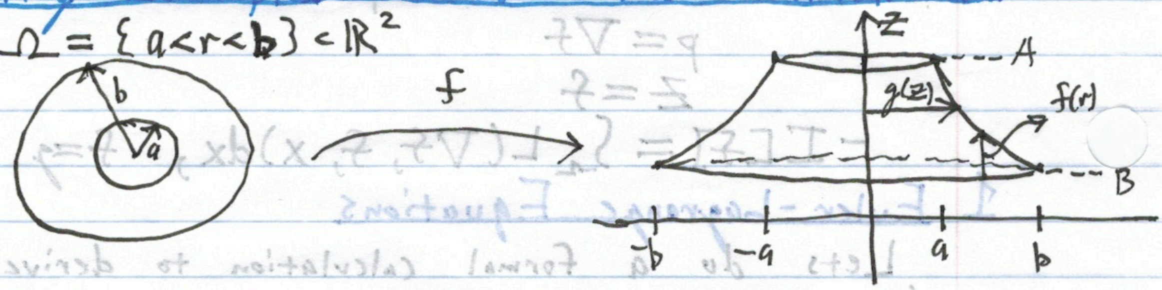
■

3. If $L$ is independant of $x$ then

$$\boxed{L - f'(x)\frac{\partial L}{\partial p} = C} \quad \text{or} \quad \boxed{f'(x) = 0}$$

## 2 Interesting Example - (Minimal Surfaces of Revolution)

Let $\Omega = \{a<r<b\} \subset \mathbb{R}^2$




Let $I : \mathcal{A} \to \mathbb{R}$ be defined by

$$* \quad I[f] = \int_\Omega \sqrt{1+|\nabla f|^2}\, dx\,dy = 2\pi \int_a^b \sqrt{1+f'(r)^2}\, r\, dr$$

where $\mathcal{A} = \{f \in W^{1,1}(\Omega) : f(a) = A,\ f(b) = B\}$.

Let $g$ be the radial distance from the $z$-axis. We can also study

$$** \quad I^*[g] = 2\pi \int_A^B g(z)\sqrt{1+g'(z)^2}\, dz$$

A physical upper bound on $I$ is the following:

$$\boxed{\inf_{f \in \mathcal{A}} I[f] \le \pi(a^2+b^2)}$$

(Area of two disjoint discs)

If we look at the 2nd version of the E-L equations for $I^*$ we have:

$$-g(z)\sqrt{1+g'(z)^2} + \frac{g(z)\,g'(z)^2}{\sqrt{1+g'(z)^2}} = C$$

$$\Rightarrow -g(z)(1+g'(z)^2) + g(z)g'(z)^2 = C(1+g'(z)^2)^{1/2}$$

$$\boxed{g(z)^2 = C^2(1+g'(z)^2)}$$

The solution to this equation is then

$$\boxed{g(z) = c_1^{-1}\cosh(c_1 z + c_2)}$$

$c_1$ and $c_2$ are determined by boundary conditions $g(B)=b$ and $g(A)=a$.

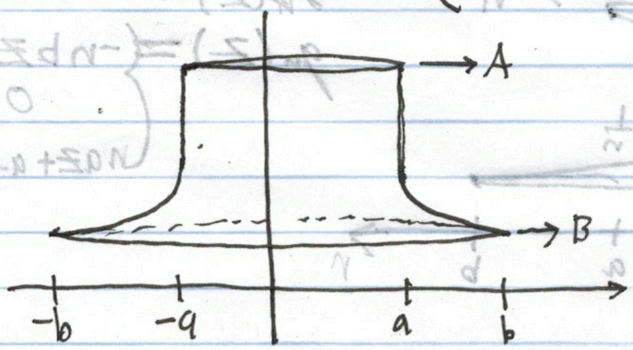
From boundary conditions we have that:

$$A - B = c_1^{-1}(\cosh(c_1 a + c_2) - \cosh(c_1 b + c_2)) > 0$$

This cannot be satisfied if $b-a$ is too large.

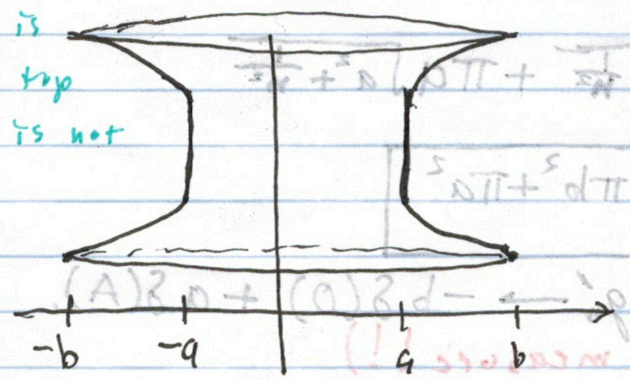
However, we can construct a weak solution

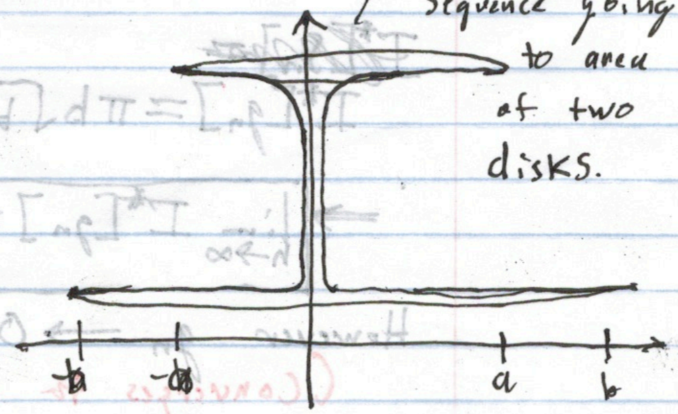
$$g(z) = \begin{cases} c_1^{-1}\cosh(c_1 z + c_2), & B<z<z^* \\ a, & z^*<z<A \end{cases}$$




However if $A-B$ is large enough then this cannot beat area of two disks.

(picture is incorrect, top radius is not a).



Ignore.

## 3. Existence

If $I: A \to \mathbb{R}$, with $A = \{f \in W^{1,p}(\Omega) : f = g \text{ on } \partial\Omega\}$ ~~is convex~~ defined by

$$I[f] = \int_\Omega W(\nabla f) + f \cdot h \, dx$$

satisfies

1. $W$ is convex
2. $W$ has "$p^{th}$ power growth"

$$c_1(|\nabla f|^p - 1) \le W(\nabla f) \le c_2(|\nabla f|^p + 1)$$

then if $p > 1$, $\exists f^* \in W^{1,p}(\Omega)$ such that

$$I[f^*] = \liminf_{f \in A} I[f].$$

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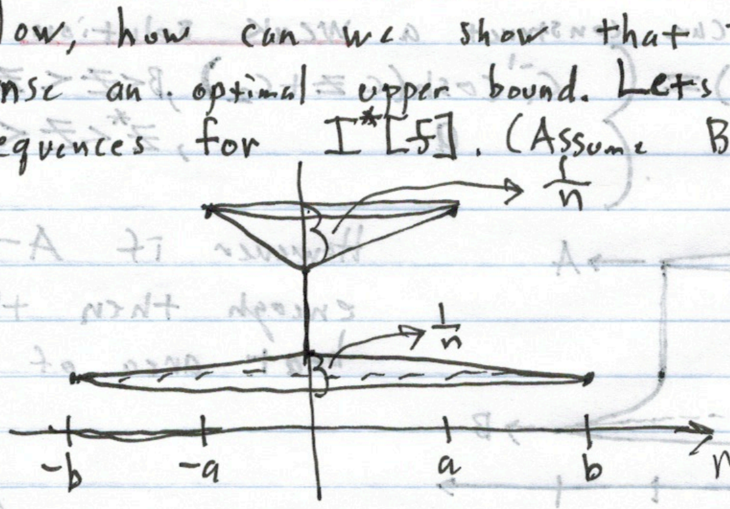
Lets find a lower bound:

$$I[f] = 2\pi \int_a^b r\sqrt{1 + f'(r)^2}\, dr \ge 2\pi a \int_a^b -f'(r)\, dr = 2\pi a(A - B).$$

If $A$ is large enough then

$$I[f] \ge \pi(b^2 + a^2)$$ (In class we found a lower bound for $I^*$)

Now, how can we show that $\pi(b^2+a^2)$ is in some sense an optimal upper bound. Lets look at minimizing sequences for $I^*[f]$. (Assume $B = 0$).



$$g_n(z) = \begin{cases} -nbz + b, & z < \frac{1}{n} \\ 0, & \frac{1}{n} < z < A - \frac{1}{n} \\ naz + a - naA, & A - \frac{1}{n} < z < A \end{cases}$$

$$I^*[g_n] = \pi b\sqrt{b^2 + \frac{1}{n^2}} + \pi a\sqrt{a^2 + \frac{1}{n^2}}$$

$$\Rightarrow \boxed{\lim_{n\to\infty} I^*[g_n] = \pi b^2 + \pi a^2}$$

However $g_n \to 0$, $g_n' \to -b\delta(0) + a\delta(A)$.

(Converges to a measure!!)

## 3. Existence

What structure on $L$ might gives us existence??
Assume the following problem:

$$I[f] = \cancel{\int_\Omega L(p,z,x)\,dx} = \int_\Omega L(\nabla f, f, x)\,dx$$

with $A = \{f \in W^{1,q}(\Omega) : f = g \text{ on } \partial\Omega\}$. We assume the following coercivity condition for $1 < q \leq \infty$:

$$L(p,z,x) \geq \alpha |p|^q - \beta.$$

Let $\{f_n\}$ be a minimizing sequence for $I$. Coercivity gives $\exists$ a subsequence $\{f_{n_k}\}$ and $f^*$ such that

$$\nabla f_{n_k} \xrightarrow{L^q} \nabla f^* \quad \text{and} \quad f_{n_k} \xrightarrow{L^q} f^*.$$

To show $f^*$ is a minimizer we must prove lower semicontinuity

$$I[f^*] \leq \liminf_{k\to\infty} I[f_{n_k}]$$

If $f$ is a smooth minimizer then if we define

$$i(t) = I[u+tv], \quad v \in C_c^\infty(\Omega) \quad (\text{actually need } v \in C_c^{0,1})$$

then

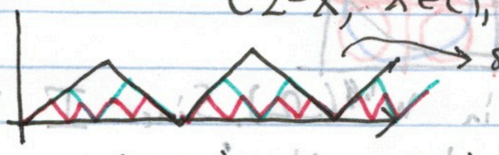
$$i''(0) = \int_\Omega \left( v_{x_i} L_{p_i p_j} v_{x_j} + 2 v_{x_i} v L_{p_i z} + v^2 L_{zz} \right) d\Omega \geq 0.$$

Set

$$v(x) = \varepsilon \eta(x) \rho\left(\frac{x\cdot\xi}{\varepsilon}\right), \quad \xi \in \mathbb{R}^n, \quad \eta \in C_c^\infty(\Omega)$$

and $\rho$ is the 2-periodic function defined by

$$\rho(x) = \begin{cases} x, & x \in (0,1) \\ 2-x, & x \in (1,2) \end{cases}$$




Substituting this in we obtain

$$i''(0) = \int_\Omega \Big( \eta(x)^2 \xi_i L_{p_i p_j} \xi_j \cdot \rho'^2 + 2\varepsilon \eta_{x_i} \rho L_{p_i p_j} \xi_j \cdot \rho'$$
$$+ \varepsilon^2 \eta_{x_i} \rho L_{p_i p_j} \eta_{x_j} \rho + 2\varepsilon^2 \eta_{x_i} \eta \rho^2 L_{p_i z} + 2\varepsilon \eta^2 \xi_i \rho' \rho L_{p_i z}$$
$$+ \varepsilon^2 \eta^2 \rho^2 \Big) dx$$

Taking limit $\varepsilon \to 0$ we have the Legendre Hadamard necessary condition:

$$\boxed{\xi_i L_{p_i p_j} \xi_j \geq 0.}$$ $\longrightarrow$ This means the function is convex in its highest derivative.

For now, lets just consider

$$I[f] = \int_\Omega L(\nabla f)dx$$

Theorem - $I$ is lower semicontinuous with respect to weak convergence in $W^{1,q}(\Omega)$ if and only if $L$ is convex.

~~proof~~:

($\Rightarrow$)

Fix $p \in \mathbb{R}^n$, we want to show $\nabla^2 L|_p \geq 0$. Construct a sequence as follows: (For simplicity we assume $\Omega$ is a cube).

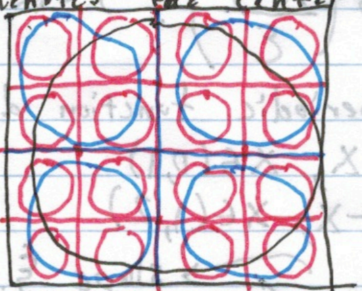
a.) Fix any $v \in C_c^\infty(\Omega)$.

b.) For $k \in \mathbb{N}$, subdivide $Q$ into disjoint subcubes $\{Q_\ell\}_{\ell=1}^{2^{kn}}$ of side length $\frac{1}{2^k}$.

c.) Define

$$u_k = \frac{1}{2^k} v(2^k(x - x_\ell)) + \vec{p}\cdot\vec{x}, \quad x \in Q_\ell$$

$x_\ell$ denotes the center of $Q_\ell$ and $u = p \cdot x$.



Clearly $u_k \rightharpoonup u$ in $W^{1,q}(\Omega)$. Since $I$ is weakly lower semicontinuous we have that:

$$|\Omega| L(p) = I[u] \leq \liminf_{k\to\infty} I[u_k]$$

However, we know that

~~$I[u_k] = \int |v(2^k(x-x_\ell)) + \vec{p}\cdot\vec{x}, \ x \in Q_\ell$~~

~~$I[u_k] = 2^{kn}\int_{Q_\ell} L(p + Dv(2^k(x - x^\ell)))dx$~~

$$I[u_k] = 2^{kn}\int_{Q_1} L(p + Dv(2^k(x - x^\ell)))dx = \int_Q L(p + Dv)dx.$$

Therefore, Mistake this should be $I[v] \le I[v+v]$

$$I[v] \le I[\nabla v + \nabla v].$$

Consequently, $v = p \cdot x$ is a minimum subject to its own boundary conditions. Therefore, $L$ is convex.

($\Leftarrow$).

Suppose $u_n \to v$ in $W^{1,q}(\Omega)$ and assume $L$ is the max of finitely many affine functions

$$L(p) = \max_{j=1,\dots,m} (b^j \cdot p + c^j)$$

Write should be $\nabla v$ should be $\nabla v$

$$E_j = \{x \in \Omega : L(\nabla f) = b^j \cdot \nabla f + c^j\}.$$

Then $\Omega = \bigcup_{j=1}^{m} E_j$, and we may as well assume the sets to be disjoint. From weak convergence we have

$$L[v] = \int_\Omega L(\nabla v)\,dx = \sum_{j=1}^{m} \int_{E_j} (b^j \cdot Dv + c^j)\,dx$$

$$= \lim_{k\to\infty} \sum_{j=1}^{m} \int_{E_j} (b^j \cdot Du_k + c^j)\,dx$$

$$\le \liminf_{k\to\infty} \sum_{j=1}^{m} \int_{E_j} L(\nabla u_k)\,dx$$

$$= \liminf_{k\to\infty} L[u_k].$$

The inequality is a result of the max formula. In the general case, we can write $F(p) = \lim_{m\to\infty} F^m(p)$ for $F^m(p) = \max_{j=1,\dots,m} (b^j \cdot p + c^j)$ and apply monotone convergence theorem.

■

**Remark:** This works since affine functions are weakly continuous and a convex function is the supremum of affine functions. Convexity gives us the natural nonlinearity that is compatible with the linear structure of weak convergence.

**Theorem** - Assume $L$ is smooth, and in addition the mapping
$$p \longmapsto L(p, z, x)$$
is convex for each $z \in \mathbb{R}$, $x \in \Omega$. Then, $I[\cdot]$ is weakly lower semicontinuous in $W^{1,q}(\Omega)$.

*proof*: The idea of the proof is like we did before, except we need to use if $f_n \xrightarrow{W^{1,q}} f$ then $f_n \xrightarrow{L^q} f$. We then have to use Egoroff's theorem to deduce uniform convergence
$$\int_{G_\varepsilon} L(\nabla f, f, x)\,dx \leq \liminf_{k\to\infty} I[f_k].$$
Where $G_\varepsilon$ is a set satisfying $|G_\varepsilon - \Omega| \longrightarrow 0$. Then apply monotone convergence theorem. ∎

**Theorem (Existence)**: Assume $L$ satisfies the coercivity condition
$$L(p, z, x) \geq \alpha |p|^q - \beta$$
and is convex in $p$. Suppose also $\mathcal{A} \neq \emptyset$. Then there exists $f^* \in \mathcal{A}$ such that
$$I[f^*] = \min_{f\in\mathcal{A}} I[f].$$

*proof*: Let $\{f_n\}$ be a minimizing sequence and w.l.o.g. assume $\beta = 0$. Therefore,
$$I[f] \geq \alpha \|\nabla f\|^q_{L^q(\Omega)}.$$
Therefore,
$$\sup_k \|\nabla f_k\|_{L^q(\Omega)} < \infty.$$
Now, let $w \in \mathcal{A} \Rightarrow f_k - w \in W_0^{1,q}(\Omega)$. Therefore, by Poincare's inequality we have that
$$\|f_k\|_{L^q(\Omega)} \leq \|f_k - w\|_{L^q(\Omega)} + \|w\|_{L^q(\Omega)} \leq C\|\nabla f_k - \nabla w\|_{L^q(\Omega)} + C$$
$$\leq C.$$
Therefore, $\|\nabla f_k\|_{W^{1,q}(\Omega)}$ is bounded.

Therefore, there exists a subsequence $f_{n_k}$ such that $f_{n_k} \xrightarrow{W^{1,q}} f^*$. From lower semi-continuity we have that

$$I[f^*] \leq \liminf_{k\to\infty} I[f_{n_k}].$$

$\blacksquare$

**Definition** - $L(p,x)$ is $\boxed{\text{uniformly convex}}$ if there exists $\theta > 0$ such that

$$L_{p_i p_j} \xi_i \xi_j \geq \theta |\xi|^2.$$

**Theorem** - Suppose $L(p,x)$ is uniformly convex. Then a minimizer $f \in A$ of $I[\cdot]$ is unique.

*proof*: $u, \tilde{u} \in A$

Suppose ~~$f, \tilde{f} \in A$~~ both minimize $I$ over $A$. Then $v = \frac{u+\tilde{u}}{2} \in A$.

We claim $I[v] \leq \frac{I[u]+I[\tilde{u}]}{2}$.

From uniform convexity we have that

$$L(p,x) \geq L(q,x) + D_p L(q,x)\cdot(p-q) + \frac{\theta}{2}|p-q|^2.$$

Set $q = (Du + D\tilde{u})/2$, $p = Du$ and integrate

$$I[u] \geq I[v] + \int_\Omega D_p L\left(\frac{Du+D\tilde{u}}{2}, x\right)\cdot\left(\frac{Du - D\tilde{u}}{2}\right) + \frac{\theta}{8}|Du - D\tilde{u}|^2 dx$$

$\boxed{\text{Similarily}}$

$$I[\tilde{u}] \geq I[v] + \int_\Omega D_p L\left(\frac{Du+D\tilde{u}}{2}, x\right)\cdot\left(\frac{D\tilde{u} - Du}{2}\right) + \frac{\theta}{8}|Du - D\tilde{u}|^2 du.$$

$$\Rightarrow I[v] + \frac{\theta}{8}\int_\Omega |Du - D\tilde{u}|^2 dx \leq \frac{I[u]+I[\tilde{u}]}{2}$$

$$\Rightarrow I[v] \leq \frac{I[u]+I[\tilde{u}]}{2}$$

Since $u$ and $\tilde{u}$ are minimizers we have that

$$I[v] = I[u]$$

$$\Rightarrow \int_\Omega |Du - D\tilde{u}|^2 dx = 0.$$

$\Rightarrow u = \tilde{u}$ in $W^{1,p}(\Omega)$ from boundary conditions.

$\blacksquare$

## 4. Improvement to Strong Convergence

**Theorem** – If $A = \{f \in W^{1,q}(\Omega) : f = g \text{ on } \partial\Omega\}$ ~~any~~ and $q = 2$ then if $L(p)$ is strictly convex meaning

$$\xi^T \cdot D^2 L\, \xi \geq \theta |p|^2, \quad \gamma > 0$$

then a minimizing sequence converges strongly to $f \in W^{1,2}(\Omega)$.
if $\boxed{|L(p)| \leq C(1+|p|^2)}$ (growth condition)

~~proof~~:

From strict convexity we have that $\forall q, p \in \mathbb{R}^n$

$$F(q) \geq F(p) + DF(p)\cdot(q-p) + \frac{\gamma}{2}|q-p|^2.$$

Set $p = Df$ and $q = Df_k$ and integrate over $\Omega$ to find

$$I[f_k] \geq I[f] + \int_\Omega DL(Df)\cdot(Df_k - Df)\,dx + \frac{\gamma}{2}\int_\Omega |Df_k - Df|^2 dx$$

stake, F should be L.

Now, $|DF(p)| \leq C(1+|p|)$ from convexity and the growth condition. Since $Df_k \rightharpoonup Df$ in $L^2$ it follows that since $DF \in L^2$ ~~the~~ and $I[f_k] \to I[f]$ that

$$Df_k \to Df.$$

■

## 5. Euler-Lagrange Equations Returned

**Definition** – We say $f \in A$ is a weak solution of the E-L equations provided

$$\boxed{\int_\Omega [L_{p_i}(Df, f, x) v_{x_i} + L_z(Df, f, x) v]\,dx = 0}$$

for all $v \in W_0^{1,q}(\Omega)$. Here we assume that

$$A = \{f \in W^{1,q}(\Omega) : f = g \text{ on } \partial\Omega\}.$$

**Theorem** – Assume $L$ satisfies the following growth conditions.

1. $|L(p, z, x)| \leq C(|p|^q + |z|^q + 1)$
2. $|D_p(p, z, x)| \leq C(|p|^{q-1} + |z|^{q-1} + 1)$
3. $|D_z(p, z, x)| \leq C(|p|^{q-1} + |z|^{q-1} + 1)$.

and $f^* \in A$ satisfies $I[f^*] = \min_{f \in A} I[f]$ then $f^*$ ~~satisfies~~ is a weak solution of the E-L equations.

*proof*:
Let $\tau \neq 0$ and write the difference quotient

$$\frac{i(\tau)-i(0)}{\tau} = \int_\Omega \frac{L(\nabla f + \tau \nabla v, f+\tau v, x) - L(\nabla f, f, x)}{\tau}\,dx = \int_\Omega L^\tau(x)\,dx,$$

where for $v \in W_0^{1,q}(\Omega)$, $i(\tau) = I[f+\tau v]$. Clearly,

$$L^\tau(x) \to L_{p_i}(\nabla f, f, x) v_{x_i} + L_z(\nabla f, f, x)\,dx,$$

pointwise. We would like to use D.C.T. to put the limit inside. We have a candidate for the limit so lets rewrite:

$$L^\tau(x) = \frac{1}{\tau}\int_0^\tau \frac{d}{ds} L(\nabla f + s\nabla v, f+sv, x)\,ds.$$

$$= \frac{1}{\tau}\int_0^\tau \left[L_{p_i}(\nabla f + s\nabla v) v_{x_i} + L_z(\nabla f + s\nabla v, f+sv, x) v\right] ds$$

We now use Young's inequality and our growth conditions:

$$\cancel{|L_{p_i}(\nabla f + s\nabla v) v_{x_i}| \le C(|\nabla f + s\nabla v|^q + \frac{|v_{x_i}|^q}{q} + 1}$$

$$\cancel{|L_z(}$$

$$\begin{aligned}
|L_{p_i}(\nabla f + s\nabla v, f+sv, x) v_{x_i}| &\le C(|\nabla f + s\nabla v|^{q-1} + |f+sv|^{q-1} + 1)|v_{x_i}| \\
&\le C(|\nabla f|^{q-1} + |f|^{q-1} + |\nabla v|^{q-1} + |v|^{q-1} + 1)|v_{x_i}| \\
&\le C(|\nabla f|^q + |f|^q + |\nabla v|^q + |v|^q + 1) \\
&\quad + \frac{|v_{x_i}|^q}{q}
\end{aligned}$$

Similarily,

$$|L_z(\nabla f + s\nabla v, f+sv, x)||v| \le C(|\nabla f|^q + |f|^q + |\nabla v|^q + |v|^q + 1) + \frac{|v|^q}{q}.$$

This gives us our bounding function. Passing to the limit gives us the result. ∎

Recall: Inequality $ab \le \frac{a^p}{p} + \frac{b^q}{q}$, where $\frac{1}{p} + \frac{1}{q} = 1$

4.11.

**Theorem** - If the joint mapping $(p, z) \mapsto L(p, z, x)$ is convex for each $x$, then each weak solution is a minimizer.

~~pnf~~:

From convexity $(p, z) \mapsto L(p, z, x)$ we have

$$L(\nabla f^*, f^*, x) + D_p L(\nabla f^*, f^*, x) \cdot (\nabla h - \nabla f^*) + D_z(\nabla f^*, f^*, x) \cdot (h - f^*) \leq L(\nabla h, h, x),$$

for all $h \in \mathcal{A}$. Consequently, $h - f \in W_0^{1,q}(\Omega)$ and integrating we have that

$$I[f^*] \leq I[h],$$

where the second term on the left vanishes ~~since~~ since $f^*$ solves the weak E.L. equations.

$\blacksquare$

## 6. Summary:

1. Coercivity:
$$L(p, z, x) \geq \alpha_1 |p|^q - \alpha_2$$
2. If the mapping $p \mapsto L(p, z, x)$ is convex then $L$ is w.l.s.c.
3. coercivity + w.l.s.c. $\Longrightarrow$ existence
4. If $L(p, z, x) = L(p, x)$ and the mapping $p \mapsto L(p, x)$ is uniformly convex ~~then~~ and coercive the minimizer is unique.
5. If $|L(p)| \leq C(1 + |p|^2)$ ~~we~~ this growth condition guarantees strong convergence of minimizing sequence.
6. Growth conditions also imply weak solutions of E-L. equations are minimizers.
7. Growth conditions + convexity in $p$ and $z$ guarantee solutions of (weak) E-L. equations are minimizers (not just local minimizers).