# Lecture 3: Tools From Analysis

**Goals:**

1. Definition of weak convergence
2. Sobolev spaces
3. Properties of weak convergence
4. Direct method applied to Dirichlet's principle.

**Key points:**

1. compactness in weak topologies
2. Sobolev embedding theorems.

**Sources:**

1. Evans - PDEs
2. Evans - weak convergence methods for nonlinear PDEs.

## 1 Modes of Convergence

**Notation:**

$\Omega \to$ bounded domain in $\mathbb{R}^n$, generally we assume simply connected with smooth boundary.

$\partial\Omega \to$ boundary of $\Omega$.

**Definition** - Let $1 \le p \le \infty$. The space $L^p(\Omega)$ is the space of Lebesgue measurable functions $f : \Omega \to \mathbb{R}^n$ such that

$$\boxed{\begin{cases} \int_\Omega |f|^p dx < \infty, & p \ne \infty \\ \operatorname{ess\,sup}_\Omega |f| < \infty, & p = \infty \end{cases}}$$

The $L^p$-norm of $f$ is defined by

$$\boxed{\|f\|_{L^p} = \left(\int_\Omega |f|^p dx\right)^{1/p}.}$$

A sequence of functions $f_n$ converges strongly to $f$ in $L^p(\Omega)$ if

$$\|f_n - f\|_{L^p} \to 0$$

and we write this as $f_n \xrightarrow{L^p} f$ or $f_n \to f$.

**Example:**

$$I[f]=\int_0^1 (f'(x)^2-1)^2\,dx,\quad f(0)=f(1)=0.$$

The minimizing sequence $f_n(x)=\sqrt{\frac{1}{4}+\frac{1}{n}}-\sqrt{(x-\frac{1}{2})^2+\frac{1}{n}}$ converges strongly in $L^\infty([0,1])$ to $f(x)=\frac{1}{2}-|x|$.

* In fact we have the simple embedding result:

$$\|f_n-f\|_{L^p}=\left(\int_0^1|f_n(x)-f(x)|^p dx\right)^{1/p}\le\left(\int_0^1\|f_n-f\|^p_{L^\infty}\right)^{1/p}=\|f_n-f\|_{L^\infty}.$$

- $f_n\to f$ in $L^p([0,1])$ for $1\le p\le\infty$.
- $L^1(\Omega)\subset L^p(\Omega)\subset L^q(\Omega)\subset L^\infty(\Omega)$, $1\le p<q<\infty$.

**Weak Convergence:**

I will take a practical approach to motivate weak convergence. Suppose we want to physically measure a quantity (function) a real world method for doing this it to average over measurements.

$$\boxed{\underbrace{\text{Measurement}}_{\text{Number}}=\underbrace{L}_{\text{Probe}}(\underbrace{f}_{\text{Thing I am measuring}})}$$

The probe must be linear to make any sense:

$$L(af+g)=aL(f)+L(g).$$

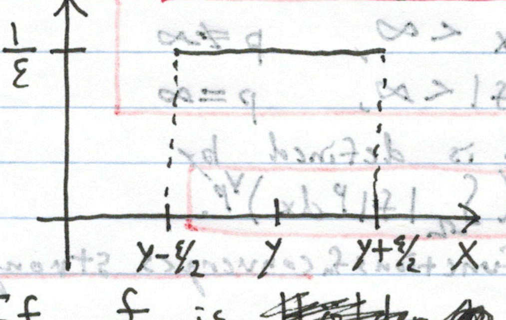
also need continuity!

If $f\in L^p(\Omega)$ a typical probe is of the form:

$$L_g(f)=\int_\Omega f\cdot g\,dx$$

For example, to determine the value of a function at a point we might consider

$$\boxed{g_\varepsilon(x)=\begin{cases}\frac{1}{\varepsilon}, & y-\frac{\varepsilon}{2}\le x\le y+\frac{\varepsilon}{2}\\ 0, & \text{o.w.}\end{cases}}$$



Then,

$$\boxed{L_{g_\varepsilon}(f(x))=\frac{1}{\varepsilon}\int_{y-\varepsilon/2}^{y+\varepsilon/2}f(x)\,dx.}$$

$$=\int_{-1/2}^{1/2}f(\varepsilon x+y)\,dx.$$

If $f$ is continuous then

$$\boxed{\lim_{\varepsilon\to0}L_{g_\varepsilon}(f(x))=f(y)}$$

(Dominated convergence gives easy proof)

**Theorem** - If $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ where $\frac{1}{p}+\frac{1}{q}=1$, then

$$\int f\cdot g\,dx \leq \int |f|\cdot|g|\,dx \leq \|f\|_{L^p}\cdot\|g\|_{L^q}.$$

This is known as Hölder's inequality. (Generalization of Cauchy-Schwarz inequality.)

**Theorem** - If $L$ is a continuous linear operator on $L^p(\Omega)$ and $1\leq p<\infty$ then there exists $g \in L^q(\Omega)$ such that

$$L(f)=\int_\Omega f\cdot g\,dx.$$

This is known as the Riesz Representation Theorem.

**Definition** - The dual space of $L^p(\Omega)$ is the set of all continuous linear operators that act on $L^p(\Omega)$. (The previous theorem tells us that the dual space of $L^p(\Omega)$ is $L^q(\Omega)$, if $1\leq p \leq \infty$)

**Definition** - For $1\leq p<\infty$ a sequence of functions $f_n$ converges weakly to $f\in L^p(\Omega)$, written $f_n \xrightarrow{L^p} f$ or simply $f_n \rightharpoonup f$ if $\forall g \in L^q(\Omega)$

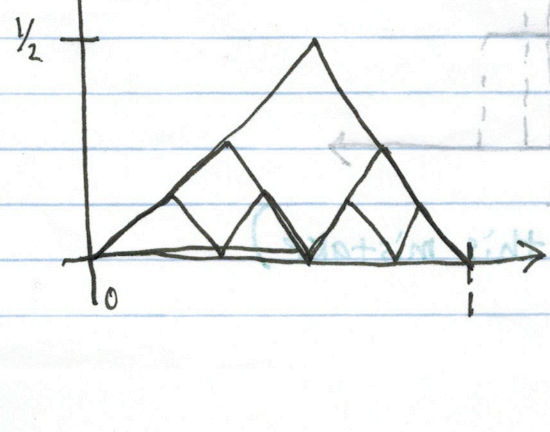
$$\int_\Omega f_n\cdot g\,dx \longrightarrow \int_\Omega f\cdot g\,dx.$$

If $p=\infty$ we say that $f_n$ converges weak-* to $f$ if for all $g\in L^1(\Omega)$

$$\int f_n\, g\,dx \longrightarrow \int_\Omega f g\,dx.$$

**Example:**

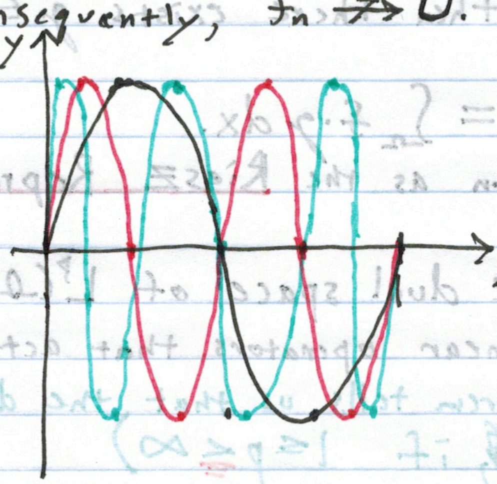
$$I[f]=\int_0^1 (f'(x)^2-1)^2\,dx+\int_0^1 f(x)^2\,dx, \quad f(0)=f(1)=0.$$




Minimizing sequence $f_n$ converges weakly to 0.

Examples (Weakly convergent, but not strongly convergent)

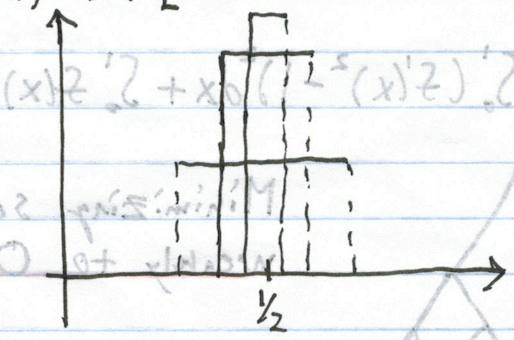
1. Oscillations - Let $f_n \in L^2([0,1])$ be defined by
$$f_n(x) = \sin(2\pi n x).$$
If $g \in L^2([0,1])$ then $\int_0^1 g(x) f_n(x)\,dx = a_n$ are the Fourier coefficients of $g(x)$. Consequently, $a_n \to 0$ as $n \to \infty$. (Riemann-Lebesgue Lemma).
That is $f_n \xrightarrow{L^2} 0$. However, $\|f_n\|_{L^2} = \frac{1}{\sqrt{2}}$.
Consequently, $f_n \not\xrightarrow{L^2} 0$.




* Rapid oscillations blur out the function.

2. Concentration - Let $f_n \in L^2([0,1])$ be defined by
$$f_n(x) = \begin{cases} 1/\sqrt{n} & \text{if } \frac{1}{2} - \frac{1}{2n} \le x \le \frac{1}{2} + \frac{1}{2n} \\ 0 & \text{o.w.} \end{cases}$$
Then, if $g \in L^2([0,1])$ we have
$$\int_0^1 f_n(x) g(x)\,dx = \frac{1}{\sqrt{n}} \int_{\frac{1}{2}-\frac{1}{2n}}^{\frac{1}{2}+\frac{1}{2n}} g(x)\,dx \le \frac{1}{\sqrt{n}} \|g\|_{L^1}.$$
Mistake!!

Therefore, $\lim_{n\to\infty} \int_0^1 f_n(x) g(x)\,dx = 0$. Consequently, $f_n \xrightarrow{L^2} 0$.
Now, $\|f_n\|_{L^2} = 1$ and we see again that $f_n \not\xrightarrow{L^2} 0$.




(Tasos corrected this mistake)

## Convergence in Averages

Weak convergence is like convergence on average. For example, suppose $f_n \xrightarrow{L^p} f$ and let $g(x) = 1$. Then,

$$\int_\Omega f_n(x)\,dx \to \int_\Omega f(x)\,dx.$$

Moreover, for any $\Omega' \subset \Omega$ we have that

$$\int_{\Omega'} f_n(x)\,dx \to \int_{\Omega'} f(x)\,dx.$$

**Properties:** Assume $f_n \to f$ in $L^p(\Omega)$. Then,

1. $f_n$ is bounded in $L^p(\Omega)$ and
$$\boxed{\|f\|_{L^p(\Omega)} \leq \liminf_{n\to\infty} \|f_n\|_{L^p}}$$
(Weak lower semi-continuity of norm)

2. If $1<p<\infty$, $f_n \xrightarrow{L^p} f$ and $\|f_n\|_{L^p} \to \|f\|_{L^p}$ then $f_n \xrightarrow{L^2} f$. (Improvement from ~~strong to weak~~ weak to strong convergence).

~~proof~~:

1. Boundedness follows from the uniform boundedness principle. To prove lower semi-continuity we prove an intermediate result: If $h \in L^p(\Omega)$ then $|h|^{p-2} h \in L^q(\Omega)$.

~~proof~~:
$$\int_\Omega \left| |h|^{p-2} \cdot h \right|^q dx = \int_\Omega \left(|h|^{p-1}\right)^{\frac{p}{p-1}} dx = \int_\Omega |h|^p dx.$$

From this result we have that

$$\cancel{\|f\|_p^p = \int_\Omega \underbrace{(|f|^{p-2} \cdot f)}_{\in L^q} \cdot \underbrace{f_n}_{\in L^p} dx}$$

$$\|f\|_{L^p}^p = \lim_{n\to\infty} \int_\Omega \underbrace{(|f|^{p-2} \cdot f)}_{\in L^q} \cdot \underbrace{f_n}_{\in L^p} dx$$

$$\leq \liminf_{n\to\infty} \| |f|^{p-1} \|_{L^q} \cdot \|f_n\|_{L^p} \quad \text{(Hölder's inequality)}$$

$$\cancel{= \liminf_{n\to\infty} \| |f|^{p-1} \|_{L^q}}$$

However, $\| |f|^{p-2} f \|_{L^q} = \|f\|_{L^p}^{p-1}$

$$\Rightarrow \|f\|_{L^p} \leq \liminf_{n\to\infty} \|f_n\|_{L^p}.$$

2. We show the result for $p=2$:
$$\|f_n - f\|_{L^2}^2 = \int_\Omega f_n^2\,dx - \int_\Omega 2 f_n \cdot f\,dx + \int_\Omega f^2\,dx$$
$$= \|f_n\|_{L^2}^2 - 2\int_\Omega f_n \cdot f\,dx + \|f\|_{L^2}^2.$$
The result follows from weak convergence. ■

Banach-Alagalou

**Compactness** - Assume $1 < p \leq \infty$ and the sequence $f_n$ is bounded in $L^p(\Omega)$. Then there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty} \subset \{f_n\}_{n \geq 1}^{\infty}$ and a function $f \in L^p(\Omega)$ with $f_{n_k} \xrightarrow{L^p} f$ if $p \neq \infty$ or $f_{n_k} \xrightarrow{*} f$ if $p = \infty$.

**Remark**: In the homework you will show that for $p=1$ the above assertion is false. This is related to why the functional

$$I[f] = \int_0^1 \sqrt{f(x)^2 + f'(x)^2}\, dx$$
$$f(0) = 0, \quad f(1) = 1$$

had no minimum in $W^{1,1}([0,1])$. → Sobolev spaces will be introduced in a bit.

## 2 Sobolev Spaces

**Notation** - $f : \Omega \to \mathbb{R}$.

1. Given a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$ we define
$$\boxed{D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}}$$
when $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

2. If $k$ is a nonnegative integer we define
$$\boxed{D^k f = \{ D^\alpha f : |\alpha| = k \}.}$$
This is a set. We define
$$\boxed{|D^k f| = \Big( \sum_{|\alpha| = k} |D^\alpha f| \Big)^{1/2}.}$$

3. Special Cases
   a.) If $k=1$:
   $$D^1 f = Df = \nabla f = (f_{x_1}, \ldots, f_{x_n}).$$
   b.) If $k=2$, we have the Hessian
   $$\boxed{D^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}}$$
   I will sometimes use the notation $\nabla^2 f$.
   c.) The Laplacian of $f$ is
   $$\boxed{\Delta f = \mathrm{tr}(D^2 f) = \sum_{i=1}^{n} u_{x_i x_i}.}$$

**Definition** - The Sobolev space $W^{k,p}(\Omega)$ consists of all locally integrable functions $f: \Omega \to \mathbb{R}$ such that for each multiindex $\alpha$ with $|\alpha| \leq k$, $D^\alpha f$ exists in the weak sense and $D^\alpha f \in L^p(\Omega)$.

- If $p=2$ we sometimes will write
$H^k(\Omega) = W^{k,2}(\Omega)$. $\to$ Hilbert space.
- If $f \in W^{k,p}(\Omega)$ we define its norm by

$$\boxed{\|f\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha|\leq k} \int_\Omega |D^\alpha f|^p dx\right)^{1/p}, & 1 \leq p < \infty \\ \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^\infty}, & p = \infty. \end{cases}}$$

- $L^p(\Omega) = W^{0,p}(\Omega)$

This norm only makes sense in dimensionless coordinates.

**Definition** - The Hölder space
$C^{k,\gamma}(\overline{\Omega})$
consists of functions $f \in C^k(\overline{\Omega})$ for which

$$\boxed{\|f\|_{C^{k,\gamma}(\Omega)} = \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^\infty(\Omega)} + \sum_{|\alpha| = k} \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^\gamma}.}$$

This norm is called the Hölder norm.

**Remark**: This space consists of functions that are almost $k+1$ differentiable. The Sobolev spaces are rougher.

**Definition** - We denote by $W_0^{k,p}(\Omega)$ as the closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$.

**Remark**: $W_0^{k,p}(\Omega)$ comprises functions $f \in W^{k,p}(\Omega)$ such that

$$\boxed{D^\alpha f = 0 \text{ on } \partial\Omega, \ \forall |\alpha| \leq k-1.}$$

$\to$ Meaning, $\forall f \in W_0^{k,p}(\Omega)$, $\exists f_n \in C_c^\infty(\Omega)$ such that $\|f_n - f\|_{W^{k,p}(\Omega)} \to 0$.

# 3. Inequalities and Embedding Theorems

**Theorem (Poincaré's Inequality)** - Suppose $f \in W_0^{1,p}(\Omega)$ for $1 \le p \le \infty$, then

$$\|f\|_{L^p(\Omega)} \le C \|\nabla f\|_{L^p(\Omega)}.$$

constant does not depend on $f$.

*proof*

Assume $\Omega = [a,b]$. Then,

$$f(x) = \int_a^x f'(y)\,dy$$

$$\Rightarrow |f(x)| \le \int_a^b |f'(y)|\,dy$$

$$\Rightarrow |f(x)|^p \le \left(\int_a^b |f'(y)|\,dy\right)^p$$

$$\le \left(\int_a^b f'(y)^p\,dy\right)(b-a)^{p/q} \quad \text{(Hölder's inequality)}$$

$$\Rightarrow \|f\|_{L^p(\Omega)} \le (b-a)^{\frac{p+q}{qp}} \|f'(y)\|_{L^p(\Omega)} = (b-a)\|f'(y)\|_{L^p(\Omega)}.$$

**Remark:** We can use dimensional analysis to obtain the scaling of $C$. In fact $C \sim L$ ($L$ is length) and is related to the diameter of the set. (How can we find the lowest value of $C$?)

**Corollary:**

If $f \in W_0^{1,p}(\Omega)$ then

$$\|f\|_{L^p(\Omega)} \le \|f\|_{W^{1,p}} \le C\|\nabla f\|_{L^p(\Omega)}$$

*proof:*

$$\|f\|_{W^{1,p}}^p = \|f\|_{L^p}^p + \|\nabla f\|_{L^p}^p \le C\|\nabla f\|_{L^p}^p.$$

**Remark:**

The point of this simple statement is that for problems with boundary conditions we can get away with the more simple norm $\|\nabla f\|_{L^p}$.

**Theorem** - If $1 \le p \le \infty$ and $f \in L^p(\Omega)$ then if $h < p$ then $f \in L^h(\Omega)$ and $\exists C$ independant of $f$ such that

$$\|f\|_{L^h(\Omega)} \le C\|f\|_{L^p(\Omega)}.$$

*proof:*

$$\int |f|^h dx \le \left(\int |f|^{h\cdot\frac{p}{h}} dx\right)^{h/p} \left(\int 1^q dx\right)^{1/q} \le C\|f\|_{L^p(\Omega)}^h.$$

$\blacksquare$

**Gagliardo-Nirenberg-Sobolev Inequality** - We are interested in finding embeddings of the form
$$\|f\|_{L^q(\Omega)} \leq C\|f\|_{W^{k,p}(\Omega)}.$$
I.e. if we know a lot of information about derivatives what do we know about the function? Our plan of attack is to first work on $\mathbb{R}^n$ and use extensions to pull back to the domain $\Omega$. We work on $\mathbb{R}^n$ so we can write $f = \int \frac{\partial f}{\partial x}$ as we did for Poincare's inequality.

We want an estimate like
$$\|f\|_{L^q(\mathbb{R}^n)} \leq C\|\nabla f\|_{L^p(\mathbb{R}^n)} \quad \text{(Like Poincare's inequality).}$$
Lets see what values of $q$ and $p$ this could possibly be true for:

$*$ Let $f_\lambda(x) = f(\lambda x)$ and assume $f \in C_c^\infty(\mathbb{R}^n)$.
$$\Rightarrow \|f\|_{L^q(\mathbb{R}^n)} \leq C\|\nabla f\|_{L^p(\mathbb{R}^n)}$$
Now,
$$\|f_\lambda\|_{L^q(\mathbb{R}^n)} = \lambda^{-n/q}\|f\|_{L^q(\mathbb{R}^n)}$$
$$\|\nabla f_\lambda\|_{L^p(\mathbb{R}^n)} = \lambda^{1-n/p}\|\nabla f\|_{L^p(\mathbb{R}^n)}$$
$$\Rightarrow \lambda^{-n/q}\|f\|_{L^q(\mathbb{R}^n)} \leq C\lambda^{1-n/p}\|\nabla f\|_{L^p(\mathbb{R}^n)}$$
In order for this inequality to be true we must have
$$\boxed{1 - \frac{n}{p} + \frac{n}{q} = 0}$$
The Sobolev conjugate of $p$ is
$$\boxed{p^* = \frac{np}{n-p}}$$

$$\boxed{\begin{array}{l}\text{Assume } 1 \leq p < n. \text{ Then, } \exists C \text{ such that } \forall f \in C_c^\infty(\mathbb{R}^n) \\ \qquad \|f\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|\nabla f\|_{L^p(\mathbb{R}^n)}\end{array}}$$

**Extension Lemma** – If $f: \Omega \to \mathbb{R}$ is smooth, ~~$\exists g \in C_c^\infty(\mathbb{R})$~~ $\exists g \in C_c^\infty(\mathbb{R}^n)$ such that $g = f$ on $\Omega$ and

$$\|g\|_{W^{1,p}(\mathbb{R}^n)} \leq C(\Omega)\|f\|_{W^{1,p}(\Omega)}.$$

By a density argument, we have $\forall f \in W^{1,p}(\Omega)$, $\exists g \in W^{1,p}(\mathbb{R}^n)$ such that $g = f$ on $\Omega$ and

$$\|g\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|f\|_{W^{1,p}(\Omega)}.$$

**Theorem** – If $1 \leq p < n$ and $f \in W^{1,p}(\Omega)$. Then $f \in L^{p^*}(\Omega)$ with

$$\boxed{\|f\|_{L^{p^*}(\Omega)} \leq C\|f\|_{W^{1,p}(\Omega)}.} \qquad \left(p^* = \frac{np}{n-p}\right)$$

*proof*:

$$\|f\|_{L^{p^*}(\Omega)} \leq \|g\|_{L^{p^*}(\mathbb{R}^n)} \leq \|\nabla g\|_{L^p(\mathbb{R}^n)} \leq \|g\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|f\|_{W^{1,p}(\Omega)}.$$

∎

**Remark**: As $p \to n$, $p^* \to \infty$ meaning $f$ lies in "better" $L^p$ spaces. What happens if $p > n$?

**Theorem** – Assume $n < p \leq \infty$ and $f \in W^{1,p}(\Omega)$, then for $\boxed{\gamma = 1 - \frac{n}{p}}$ we have $f \in C^{0, 1-\frac{n}{p}}$ with

$$\boxed{\|f\|_{C^{0,\gamma}(\Omega)} \leq C\|f\|_{W^{1,p}(\Omega)}.}$$

*idea of proof*:

Let $f_\lambda(x) = f(\lambda x)$. Assume

$$\left| \frac{f(x) - f(y)}{(x-y)^\alpha} \right| \leq C\|\nabla f\|_{L^p}$$

Now,

$$\left| \frac{f_\lambda(x) - f_\lambda(y)}{(x-y)^\alpha} \right| = \lambda^{+\alpha} \left| \frac{f(\lambda x) - f(\lambda y)}{(\lambda x - \lambda y)^\alpha} \right|$$

$$\|f_\lambda\|_{W^{1,p}(\Omega)} = \lambda^{1-n/p}\|f\|_{W^{1,p}(\Omega)}.$$

$$\Rightarrow \boxed{\alpha = 1 - \frac{n}{p}.}$$

## Theorem (Riellich-Kondrachov Compactness Theorem) -

Assume $\Omega$ is bounded in $\mathbb{R}^n$. Then,

1. $W^{1,p}(\Omega) \subset\subset L^k(\Omega)$, for all $k$ with $1 \le k < p^*$, $\frac{n}{p} > 1$,
2. $W^{1,p}(\Omega) \subset\subset L^k(\Omega)$, for all $k$ with $1 \le k < \infty$, $\frac{n}{p} = 1$,
3. $W^{1,p}(\Omega) \subset\subset C^0(\Omega)$, ~~for~~, $\frac{n}{p} < 1$.

Where $\subset\subset$ ~~means~~ denotes compactly embeded meaning the sequence is precompact in the larger space.

example:

If $f_n \in W^{1,1}(\Omega)$, where $\Omega \subset \mathbb{R}^2$ then if $\|f_n\|_{W^{1,1}} < M$ then $\exists f^*$ such that $f_n \xrightarrow{L^k} f$, where $k < p^* = 2$.

I.e. we get strong convergence of the function itself. (We also get strong converge in "larger" $L^p$ space.

example:

$$I[f] = \int_0^1 \sqrt{f(x)^2 + f'(x)^2}\, dx, \quad A = \{ f \in W^{1,1}(0,1) : f(0)=0, f(1)=1 \}.$$

We picked a minimizing sequence

$$f_n(x) = \begin{cases} 0, & 0<x<1-\frac{1}{n} \\ n(x-1)+1 & 1-\frac{1}{n}<x<1 \end{cases}$$

Now,

$$\|f_n\|_{L^1} \le \frac{1}{n} < 1 \Rightarrow \|f\|_{W^{1,1}(0,1)} < 2$$

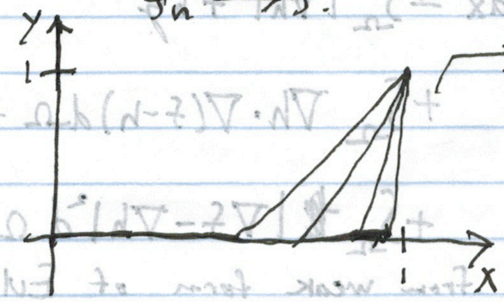
$$\left\|\frac{df_n}{dx}\right\|_{L^1} = 1.$$

Therefore, we know $\exists g \in L^1$ and $f^* \in L^1$ such that

$$f_n \xrightarrow{L^1} f, \quad \frac{df_n}{dx} \longrightarrow g \quad \text{(Banach-Alagaloo)}$$

Also, from compact embedding

$$f_n \xrightarrow{L^1} f.$$




Slopes concentrate but $f_n \xrightarrow{L^2} 0$.

## 4. Dirichlet's Principle

Let $\Omega \subset \mathbb{R}^n$ and $I : A \to \mathbb{R}$ be defined by
$$I[f] = \int_\Omega |\nabla f|^2 dx + \int_\Omega f \cdot g \, dx$$
where $A = \{ f \in W_0^{1,2}(\Omega) \}$ and $g \in L^2(\Omega)$. There exists $f^* \in A$ that minimizes $I$.

~~proof~~:

Let $f_n$ be a minimizing sequence for $I$ meaning $\lim_{n\to\infty} I[f_n] = \inf_{f \in A} I[f]$. Then, $f_n$ is bounded in $W_0^{1,2}(\Omega)$. Consequently, there exists $f^* \in W_0^{1,2}(\Omega)$ such that
$$f_n \xrightarrow{L^2} f^* \quad \text{and} \quad \nabla f \xrightarrow{L^2} \nabla f^*.$$
From weak lower semicontinuity of norms we have
$$\|\nabla f^*\|_{L^2} \le \liminf_{n\to\infty} \|\nabla f_n\|$$
We also have
$$\lim_{n\to\infty} \int_\Omega f_n \cdot g \, dx = \int_\Omega f \cdot g \, dx.$$
Putting this together we have that
$$I[f^*] \le \liminf_{n\to\infty} I[f_n] \le \lim_{n\to\infty} I[f_n] = \inf_{f\in A} I[f].$$
Consequently, $f^*$ is a minimum.

∎

**Remark:** In fact since $\|\nabla f_n\| \to \|\nabla f^*\|$ we can conclude that $f_n \xrightarrow{W^{1,2}(\Omega)} f^*$. (Important for numerics)

**Uniqueness** – In general we want something like
$I[f] = I[h] +$ 1st. variation in direction $f-h$
+ 2nd variation that's positive.

$$\int_\Omega (|\nabla f|^2 + f \cdot g) dx = \int_\Omega |\nabla h|^2 + hg$$
$$+ \int_\Omega \nabla h \cdot \nabla (f-h) \, d\Omega + \int_\Omega g (f-h) \, d\Omega$$
$$+ \int_\Omega |\nabla f - \nabla h|^2 d\Omega$$

Middle term vanishes from weak form of Euler-Lagrange equations.
$$\Rightarrow I[f] \ge I[h]$$
With equality when $f = h$. (follows from Poincare inequality).