Lecture 3.1: Tools From Analysis

Goals:
1. Definition of weak convergence
2. Sobolev spaces
3. Properties of weak convergence
4. Direct method applied to Dirichlet's principle

Key points:
1. Compactness in weak topologies
2. Sobolev embedding theorems

Sources:
1. Evans - PDEs
2. Evans - Weak convergence methods for nonlinear PDEs

1. Modes of Convergence

Notation:
- \( \Omega \rightarrow \) bounded domain in \( \mathbb{R}^n \); generally, we assume simply connected with smooth boundary.
- \( \partial \Omega \rightarrow \) boundary of \( \Omega \).

Definition - Let \( 1 \leq p \leq \infty \). The space \( L^p(\Omega) \) is the space of Lebesgue measurable functions \( f : \Omega \rightarrow \mathbb{R}^n \) such that

\[
\begin{align*}
\text{ess sup}_{\Omega} |f| &< \infty, \quad p = \infty \\
\int_{\Omega} |f|^p \, dx &< \infty, \quad p < \infty
\end{align*}
\]

The \( L^p \)-norm of \( f \) is defined by

\[
\|f\|_{L^p} = \left( \int_{\Omega} |f|^p \, dx \right)^{1/p}.
\]

A sequence of functions \( f_n \) converges strongly to \( f \) in \( L^p(\Omega) \) if

\[
\|f_n - f\|_{L^p} \to 0;
\]

and we write this as \( f_n \xrightarrow{L^p} f \) or \( f_n \to f \).
Example:
\[ I[\varphi] = \int_0^\infty (\varphi'(x) - 1)^2 \, dx, \quad \varphi(0) = \varphi(1) = 0 \]
The minimizing sequence \( f_n(x) = \sqrt{\frac{1}{4x} + \frac{1}{n}} - \sqrt{(x - \frac{1}{2})^2 + \frac{1}{n}} \)
converges strongly in \( L^p([0,1]) \) to \( f(x) = \frac{1}{2} - 1/x \).

* In fact we have the simple embedding result:
\[ \| f_n - f \|_{L^p} = \left( \int_0^1 |f_n(x) - f(x)|^p \, dx \right)^{1/p} \leq \left( \frac{1}{p} \| f_n - f \|_{L^p(L^\infty)}^p \right)^{1/p} = \| f_n - f \|_{L^\infty} \]
\[ f_n \to f \quad \mathrm{in} \quad L^p([0,1]) \quad \mathrm{for} \quad 1 \leq p < \infty . \]
\[ L^1([0,1]) \subset L^p([0,1]) \subset C([0,1]) \subset L^\infty([0,1]), \quad 1 \leq p < q < \infty . \]

**Weak Convergence:**
I will take a practical approach to motivate weak convergence. Suppose we want to physically measure a quantity (function) a real world method for doing this is to average over measurements.

**Measurement** = \( L(f) \)

**Number** \( \rightarrow \) **Probe** \( \rightarrow \) **Thing I am measuring**.

The probe must be linear to make any sense:
\[ L(af + g) = aL(f) + L(g) . \]

If \( f \in L^p([0,1]) \) a typical probe is of the form:
\[ L_\varphi(f) = \int_0^1 \varphi(x) \, df(x) . \]

For example, to determine the value of a function at a point we might consider:
\[ \varphi_\varepsilon(x) = \begin{cases} \frac{1}{\varepsilon}, & \varepsilon - \frac{\varepsilon}{2} < x \leq \varepsilon + \frac{\varepsilon}{2} \\ 0, & \text{o.w.} \end{cases} \]

Then,
\[ L_{\varphi_\varepsilon}(f(x)) = \frac{1}{\varepsilon} \int_{\varepsilon - \frac{\varepsilon}{2}}^{\varepsilon + \frac{\varepsilon}{2}} f(x) \, dx . \]

If \( f \) is \( \text{Hölder continuous} \) then
\[ \lim_{\varepsilon \to 0} L_{\varphi_\varepsilon}(f(x)) = f(\cdot) \]
(Dominated convergence gives easy proof)
**Theorem:** If \( f \in L^p(\Omega) \) and \( g \in L^{q}(\Omega) \) where \( \frac{1}{p} + \frac{1}{q} = 1 \), then
\[
\int_{\Omega} f \cdot g \, dx = \int_{\Omega} f \cdot \frac{1}{q} \cdot g \, dx \leq \left( \int_{\Omega} |f|^p \, dx \right)^{1/p} \left( \int_{\Omega} |g|^q \, dx \right)^{1/q}.
\]
This is known as Hölder's inequality (generalization of Cauchy-Schwarz inequality).

**Theorem:** If \( L \) is a continuous linear operator \( L^p(\Omega) \) and \( 1 \leq p < \infty \) then there exists \( g \in L^q(\Omega) \) such that
\[
L(f) = \int_{\Omega} f \cdot g \, dx.
\]
This is known as the Riesz Representation Theorem.

**Definition:** The dual space of \( L^p(\Omega) \) is the set of all continuous linear operators that act on \( L^p(\Omega) \). (The previous theorem tells us that the dual space of \( L^p(\Omega) \) is \( L^q(\Omega) \) if \( 1 \leq p < \infty \).)

**Definition:** For \( 1 \leq p < \infty \) a sequence of functions \( f_n \) converges weakly to \( f \in L^p(\Omega) \) if for all \( g \in L^q(\Omega) \)
\[
\int_{\Omega} f_n \cdot g \, dx \to \int_{\Omega} f \cdot g \, dx.
\]
If \( p = \infty \) we say that \( f_n \) converges weak-* to \( f \) if for all \( g \in L^1(\Omega) \)
\[
\int_{\Omega} f_n \cdot g \, dx \to \int_{\Omega} f \cdot g \, dx.
\]

**Example:**
\[
I[f] = \int_{0}^{1} (S'(x)^2 - 1)^2 \, dx + \int_{0}^{1} S(x)^2 \, dx, \quad S(0) = S(1) = 0.
\]
Minimizing sequence \( f_n \) converges weakly to 0.
Examples (Weakly Convergent, but not strongly convergent)

1. Oscillations - Let $f_n \in L^2([0,1])$ be defined by
   $f_n(x) = \sin(2\pi n x)$.
   If $g \in L^2([0,1])$ then $\int_0^1 g(x) f_n(x) \, dx = a_n$ are the Fourier coefficients of $g(x)$. Consequently, $a_n \to 0$ as $n \to \infty$. (Riemann-Lebesgue Lemma).
   That is, $f_n \not\to 0$. However, $\|f_n\|_L^2 = \frac{1}{n}$.
   Consequently, $f_n \not\to 0$.

   * Rapid oscillations blur out the function.

2. Concentration - Let $f_n \in L^2([0,1])$ be defined by
   $f_n(x) = \left\{ \begin{array}{ll} \frac{1}{\sqrt{n}} & \text{if } \frac{1}{2} - \frac{1}{2n} \leq x \leq \frac{1}{2} + \frac{1}{2n} \\ 0 & \text{o.w.} \end{array} \right.$

   Then, if $g \in L^2([0,1])$ we have
   $\int_0^1 f_n(x) g(x) \, dx = \int_0^1 \frac{1}{\sqrt{n}} g(x) \, dx = \frac{1}{\sqrt{n}} \|g\|_L^2$.

   Therefore, $\lim_{n \to \infty} \int_0^1 f_n(x) g(x) \, dx = 0$. Consequently, $f_n \not\to 0$.

   Now, $\|f_n\|_L^2 = 1$ and we see again that $f_n \not\to 0$.

   (Tasos corrected this mistake)
Convergence in Averages

Weak convergence is like convergence on average. For example, suppose \( f_n \rightarrow f \) and let \( g(x) = 1 \). Then,
\[
\int \frac{f_n - f}{x} \, dx \rightarrow \int \frac{f - f}{x} \, dx.
\]
Moreover, for any \( \alpha < \beta \) we have that
\[
\int_{\alpha}^{\beta} f_n(x) \, dx \rightarrow \int_{\alpha}^{\beta} f(x) \, dx.
\]

**Properties:** Assume \( f_n \rightarrow f \) in \( L^p(\Omega) \). Then:
1. \( f_n \) is bounded in \( L^p(\Omega) \) and \( \|f_n\|_{L^p} \) is uniformly bounded.

\[ \|f\|_{L^p} = \lim_{n \to \infty} \|f_n\|_{L^p} \] (Weak lower semi-continuity of norm)

2. If \( 1 < p < \infty \), \( f_n \rightarrow f \) and \( \|f_n\|_p \rightarrow \|f\|_p \) then \( f_n \rightharpoonup f \). (Improvement from strong to weak convergence)

**Proof**

1. Boundedness follows from the uniform boundedness principle.

To prove lower semi-continuity we prove an intermediate result: If \( h \in L^p(\Omega) \) then \( v_{1/p} \chi_{B(x,1)} \in L^p(\Omega) \).

\[
\int |h|^{p-2} h \chi_{B(x,1)} \, dx = \int |h|^{p-2} h \, dx = \int |h|^{p-1} \, dx.
\]
From this result we have that
\[
\|f\|_p^p \leq \liminf_{n \to \infty} \|f_n\|_p^p \leq \limsup_{n \to \infty} \|f_n\|_p^p \leq \|f\|_p^p.
\]

Hölder's inequality

However, \( \|f\|_p^{p-2} \|f\|_p \leq \|f\|_p^{p-1} \cdot \|f_n\|_p \).

\[
\Rightarrow \|f\|_p \leq \liminf_{n \to \infty} \|f_n\|_p \leq \|f\|_p.
\]

2. We show the result for \( p = 2 \):
\[
\|f_n - f\|_2^2 = \int f_n^2 - 2 f_n f + f^2 \, dx = \|f_n\|_2^2 - 2 \|f_n f\|_2 + \|f\|_2^2.
\]
The result follows from weak convergence.

\[ \square \]
Compactness - Assume $1 < p \leq \infty$ and the sequence $s_n$ is bounded in $L^p(\Omega)$. Then there exists a subsequence $s_{n_k}, s_{n_k} \to s$ in $L^p(\Omega)$, and a function $f \in L^p(\Omega)$ with $s_{n_k} \rightharpoonup f$ if $p \neq \infty$, or $s_{n_k} \overset{L^p}{\to} f$ if $p = \infty$.

Remark: In the homework, you will show that $p = 1$ the above assertion is false. This is related to why the functional
\[ I[f] = \int_0^1 \sqrt{s(x)^2 + f(x)^2} \, dx \]
has no minimum in $W^{1,1}(\Omega, \Omega)$. Sobolev spaces will be introduced in a bit.

2. Sobolev Spaces

Notation - $\mathcal{F} : \Omega \to \mathbb{R}$

1. Given a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, we define
\[ D^\alpha \mathcal{F} = \frac{\partial^{|\alpha|} \mathcal{F}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \]
when $|\alpha| = \alpha_1 + \cdots + \alpha_n$

2. If $K$ is a non-negative integer, we define
\[ \mathcal{D}^K \mathcal{F} = \{ D^\alpha \mathcal{F} : |\alpha| = K \} \]
This is a set. We define
\[ \| \mathcal{F} \|_{K} = \left( \sum_{|\alpha| = K} \| D^\alpha \mathcal{F} \|^2 \right)^{1/2} \]

3. Special Cases
   a) If $K = 1$,
      \[ D^1 \mathcal{F} = D \mathcal{F} = \nabla \mathcal{F} = (s_{x_1}, \ldots, s_{x_n}) \]
   b) If $K = 2$, we have the Hessian
      \[ D^2 \mathcal{F} = \begin{pmatrix} s_{x_1x_1} & \cdots & s_{x_1x_n} \\ \vdots & \ddots & \vdots \\ s_{x_nx_1} & \cdots & s_{x_nx_n} \end{pmatrix} \]
   c) The Laplacian of $\mathcal{F} - \Delta \mathcal{F} = \sum_{i,j} s_{x_ix_j}$.
Definition: The Sobolev space $W^{k,p}(\Omega)$ consists of all locally integrable functions $f: \Omega \to \mathbb{R}$ such that for each multi-index $\alpha$ with $|\alpha| \leq k$, $D^\alpha f \in L^p(\Omega)$.

- If $p = 2$ we sometimes will write $H^k(\Omega) = W^{k,2}(\Omega)$, $\Rightarrow$ Hilbert space.
- If $f \in W^{k,p}(\Omega)$ we define its norm by
\[
\|f\|_{W^{k,p}(\Omega)} = \left( \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^p(\Omega)}^p \right)^{1/p}, \quad 1 \leq p < \infty,
\]
\[
\|f\|_{L^p(\Omega)} = \|D^0 f\|_{L^p(\Omega)} = \left( \sum_{|\alpha| = 0} \|D^\alpha f\|_{L^p(\Omega)}^p \right)^{1/p}, \quad p = \infty.
\]

This norm only makes sense in dimension less coordinates.

**Definition:** The Hölder space $C^{k,\beta}(\Omega)$ consists of functions $f \in C^k(\Omega)$ for which
\[
\|f\|_{C^{k,\beta}(\Omega)} = \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^\infty(\Omega)} + \sum_{|\alpha| = k} \sup_{|x-y| \leq 1} \left| D^\alpha f(x) - D^\alpha f(y) \right|.
\]

This norm is called the Hölder norm.

**Remark:** This space consists of functions that are almost $k + 1$ differentiable. The Sobolev spaces are rougher.

**Definition:** We denote by $W_0^{k,p}(\Omega)$ as the closure of $C^\infty_0(\Omega)$ in $W^{k,p}(\Omega)$.

**Remark:** $W^{k,p}(\Omega)$ comprises functions $f \in W^{k,p}(\Omega)$ such that
\[
D^\alpha f = 0 \text{ on } \partial \Omega, \quad \forall |\alpha| \leq k - 1.
\]

Meaning, $W^{k,p}(\Omega)$ is the closure of $C^\infty_0(\Omega)$ in $W^{k,p}(\Omega)$ such that
\[
\|f_n - f\|_{W^{k,p}(\Omega)} \to 0.
\]
3. Inequalities and Embedding Theorems

Theorem (Poincaré's Inequality). Suppose $f \in W_0^1(\Omega)$ for $1 \leq p \leq \infty$, then
\[ \| f \|_{L^p(\Omega)} \leq C \| \nabla f \|_{L^p(\Omega)}. \]

**Proof.** Assume $\Omega = [a, b]$. Then,
\[ f(x) = \int_a^b f(y) \, dy, \]
\[ \Rightarrow \| f(x) \| \leq \int_a^b | f(y) | \, dy, \]
\[ \Rightarrow \| f(x) \|_p \leq (\int_a^b (f(y))^p \, dy)^{1/p} \leq (b-a)^{1/p} (\int_a^b (f(y))^p \, dy)^{1/p}, \] (Hölder's inequality)
\[ \Rightarrow \| f \|_{L^p(\Omega)} \leq (b-a)^{1/p} \| f \|_{L^p(\Omega)}. \]

**Remark:** We can use dimensional analysis to obtain the scaling of $C$. In fact, $C \propto L$ ($L$ is length) and is related to the diameter of the set. (How can we find the lowest value of $C$?)

**Corollary:**
If $f \in W_0^1(\Omega)$ then
\[ \| f \|_{W_0^1(\Omega)} \leq C \| \nabla f \|_{L^p(\Omega)} \]

**Proof:**
\[ \| f \|_{W_0^1(\Omega)} = \| f \|_{L^p(\Omega)} + \| \nabla f \|_{L^p(\Omega)} \leq C \| \nabla f \|_{L^p(\Omega)}. \]

**Remark:**
The point of this simple statement is that for problems with boundary conditions, we can get away with the more simple norm $\| \nabla f \|_{L^p(\Omega)}$.

**Theorem:** If $1 \leq p \leq \infty$ and $f \in L^p(\Omega)$ then if $h < p$
\[ f \in L^h(\Omega) \text{ and } \exists C \text{ independent of } f \text{ such that} \]
\[ \| f \|_{L^h(\Omega)} \leq C \| f \|_{L^p(\Omega)}. \]

**Proof:**
\[ \int \| f \|_h^h \, dx \leq (\int \| f \|_h^h)^{1/p} (\int \| f \|_p^p)^{1/h} \leq C \| f \|_{L^p(\Omega)}. \]
Cagliando-Nirenberg-Sobolev Inequality. We are interested in finding embeddings of the form
\[ \| f \|_{L^p(\Omega)} \leq C \| \nabla f \|_{L^q(\Omega)}. \]
I.e., if we know a lot of information about derivatives, what do we know about the function? Our plan of attack is to first work on \( \mathbb{R}^n \) and use extensions to pull back to the domain \( \Omega \). We work on \( \mathbb{R}^n \) so we can write \( f = f \frac{\partial}{\partial x} \) as we did for Poincare's inequality.

We want an estimate like
\[ \| f \|_{L^q(\mathbb{R}^n)} \leq C \| \nabla f \|_{L^p(\mathbb{R}^n)}. \]
(like Poincare's inequality)

Let's see what values of \( q \) and \( p \) this could possibly be true for:

1. Let \( f_\lambda(x) = f(\lambda x) \) and assume \( f \in C^\infty_c(\mathbb{R}^n) \),
\[ \Rightarrow \| f_\lambda \|_{L^q(\mathbb{R}^n)} \leq C \| \nabla f_\lambda \|_{L^p(\mathbb{R}^n)}. \]

Now,
\[ \| f_\lambda \|_{L^q(\mathbb{R}^n)} = \lambda^{\frac{n}{q}} \| f \|_{L^q(\mathbb{R}^n)}, \]
\[ \| \nabla f_\lambda \|_{L^p(\mathbb{R}^n)} = \lambda^{-\frac{n}{p}} \| \nabla f \|_{L^p(\mathbb{R}^n)}. \]
\[ \Rightarrow \lambda^{-\frac{n}{q} + \frac{n}{p}} \| f \|_{L^q(\mathbb{R}^n)} \leq C \lambda^{-\frac{n}{p}} \| \nabla f \|_{L^p(\mathbb{R}^n)}. \]

In order for this inequality to be true we must have
\[ 1 - \frac{n}{q} + \frac{n}{p} = 0. \]

The Sobolev conjugate of \( p \) is
\[ p^* = \frac{np}{n-p}. \]

Assume \( 1 \leq p \leq n \). Then there is \( \phi \in C^\infty_c(\mathbb{R}^n) \) such that \( \forall f \in C^\infty_c(\mathbb{R}^n) \)
\[ \| f \|_{L^p(\mathbb{R}^n)} \leq C \| \nabla f \|_{L^p(\mathbb{R}^n)}. \]
Extension Lemma: If \( f : \Omega \to \mathbb{R} \) is smooth, \( f \in C^\infty(\mathbb{R}^n) \) such that \( f = 0 \) on \( \Omega \) and \( \| f \|_{W^{1,p}(\mathbb{R}^n)} \leq C \| f \|_{W^{1,p}(\Omega)} \).

By a density argument, we have \( \forall \varepsilon > 0 \), \( \exists g \in \mathcal{E}_c^{\infty}(\mathbb{R}^n) \) such that \( g = f \) on \( \Omega \) and \( \| g \|_{W^{1,p}(\mathbb{R}^n)} \leq \varepsilon \| f \|_{W^{1,p}(\mathbb{R}^n)} + C \| f \|_{W^{1,p}(\Omega)} \).

Theorem: If \( 1 \leq p < n \) and \( f \in W^{1,p}(\Omega) \). Then \( f \in L^p(\Omega) \) with

\[
\| f \|_{L^p(\Omega)} \leq C \| f \|_{W^{1,p}(\Omega)}.
\]

\( (p^* = \frac{np}{n-p} ) \)

\[
\| f \|_{L^{p^*}(\Omega)} \leq \| f \|_{W^{1,p}(\Omega)} \leq \| f \|_{L^p(\Omega)} \leq C \| f \|_{W^{1,p}(\Omega)}.
\]

Remark: As \( p \to n \), \( p^* \to \infty \) meaning \( f \) lies in "better" \( L^p \) spaces. What happens if \( p = n \)?

Theorem: Assume \( n < p < \infty \) and \( f \in W^{1,p}(\Omega) \), then for \( y = \frac{y}{\| \eta \|_{L^p(\Omega)}} \)

we have \( f \in C^{\alpha,1-rac{1}{p}}(\Omega) \) with

\[
\| f \|_{C^{\alpha,1-rac{1}{p}}(\Omega)} \leq C \| f \|_{W^{1,p}(\Omega)}.
\]

\( \ellag \) of proved.

Let \( \frac{f}{\eta}(x) = f(\lambda x) \). Assume

\[
\frac{f(x) - f(y)}{(x-y)^k} \leq C \| f \|_{L^p(\Omega)}
\]

Now,

\[
\left| \frac{f(\lambda x) - f(\lambda y)}{(\lambda x - \lambda y)^k} \right| = \lambda^{1-k} \left| \frac{f(x) - f(y)}{(x-y)^k} \right|
\]

\[
\| f \|_{L^p(\Omega)} = \lambda^{1-k} \| f \|_{L^p(\Omega)}
\]

\[
\Rightarrow \quad x = 1 - \frac{k}{p}.
\]
Theorem (Rellich-Kondrachov Compactness Theorem)

Assume $\Omega$ is bounded in $\mathbb{R}^n$. Then,
1. $W^{1,p}(\Omega) \subset \mathcal{L}^k(\Omega)$, for all $k$ with $1 \leq k < p^*$, \( \frac{n}{p} > 1 \),
2. $W^{1,p}(\Omega) \subset \mathcal{L}^k(\Omega)$, for all $k$ with $1 \leq k < \infty$, \( \frac{n}{p} = 1 \),
3. $W^{1,p}(\Omega) \subset \mathcal{E}^0(\Omega)$, for $\frac{n}{p} < 1$.

Where $\subset$ denotes compactly embedded meaning the sequence is precompact in the larger space.

Example:

If $f_n \in W^{1,1}(\Omega)$, where $\Omega \subset \mathbb{R}^2$, then if $\|f_n\|_{W^{1,1}} \leq M$ then $\exists f^*$ such that $f_n \xrightarrow{\text{w}} f^*$, where $K < p^* = 2$.

I.e. we get strong convergence of the function itself. We also get strong convergence in "larger" $L^1\text{-space}.

Example:

\[ \mathcal{I}[f] = \int_0^1 \sqrt{s(x)^2 + s'(x)^2} \, dx, \quad A = \{ f \in W^{1,1}(0,1) : s(0) = 0, s(1) = 1 \} \]

We picked a minimizing sequence
\[ f_n(x) = \begin{cases} \frac{1}{n} & \text{if } 0 < x < 1 - \frac{1}{n} \\ n(x-1) + 1 & \text{if } 1 - \frac{1}{n} < x < 1 \end{cases} \]

Now,
\[ \|f_n\|_{L^1} \leq \frac{1}{n} < 1 \Rightarrow \|f_n\|_{W^{1,1}(0,1)} < 2 \]
\[ \|\frac{d}{dx}f_n\|_{L^1} = 1. \]

Therefore, we know $f \in A$, and $f^* \in A$, such that $f_n \xrightarrow{\text{w}} f^*$, \( f \xrightarrow{\text{w}} f^* (\text{Banach-Alagelov}) \)

Also, from compact embedding
\[ f_n \xrightarrow{\text{w}} f, \quad f \xrightarrow{\text{w}} f^* \]

\[ \begin{array}{c}
\text{Slopes concentrate but} \\
\text{not compactly} \\
\text{in } W^{1,1} \\
\end{array} \]

\[ f_n \xrightarrow{\text{w}} 0. \]

Let \( Q \subset \mathbb{R}^n \) and \( \mathbb{I} : A \rightarrow \mathbb{R} \) be defined by

\[
\mathbb{I}[f] = \int_Q \sqrt{1 + \nabla f^2} \, dx + \int_Q f \, g \, dx
\]

where \( A = \{ f \in W^{1,2}(Q) \} \) and \( g \in L^1(Q) \). There exists \( f^* \in A \) that minimizes \( \mathbb{I} \).

Proof:

Let \( f_n \) be a minimizing sequence for \( \mathbb{I} \), meaning\( \lim_{n \to \infty} \mathbb{I}[f_n] = \inf \mathbb{I}[f] \). Then, \( f_n \) is bounded in \( W^{1,2}(Q) \).

Consequently, there exists \( f^* \in W^{1,2}(Q) \) such that

\[
f_n \rightharpoonup f^*, \quad \text{and} \quad \nabla f_n \rightharpoonup \nabla f^*.
\]

From weak lower semicontinuity of norms we have

\[
\| \nabla f^* \|_{L^2} = \liminf_{n \to \infty} \| \nabla f_n \|_{L^2}.
\]

We also have

\[
\lim_{n \to \infty} \int_Q f_n \cdot g \, dx = \int_Q f \cdot g \, dx.
\]

Putting this together we have that\( \mathbb{I}[f^*] \leq \liminf_{n \to \infty} \mathbb{I}[f_n] = \inf \mathbb{I}[f] \).

Consequently, \( f^* \) is a minimum.

Remark: In fact, since \( \| \nabla f_n \| \to \| \nabla f^* \| \), we can conclude that \( f_n \overset{W^{1,2}}{\to} f^* \) (Important for numerics).

Uniqueness - In general we want something like

\[
\mathbb{I}[f] = \mathbb{I}[h] + 1 \text{st. variation in direction } f-h
\]

+ 2nd variation that is positive.

\[
\int_Q \left( \nabla |f|^2 + f \cdot g \right) \, dx = \int_Q \nabla |h|^2 + h \cdot g \, dx
\]

+ \( \int_Q \nabla h \cdot \nabla (f-h) \, dx + \int_Q g (f-h) \, dx \)

+ \( \int_Q \nabla f - \nabla h \, dx \)

Middle term vanishes from weak form of Euler-Lagrange equations.

\( \Rightarrow \mathbb{I}[f] \geq \mathbb{I}[h] \)

With equality when \( f = h \). (Follows from Poincaré inequality.)