Lecture 2: Variational Derivative

Goals:

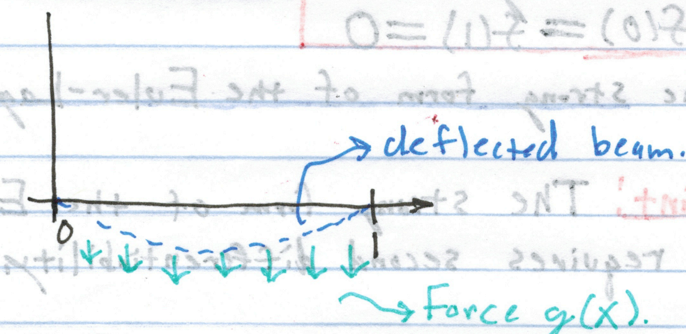
1. Derive Euler-Lagrange equations.
2. Discuss Dirichlet's principle.
3. Example from minimal surfaces.
4. Discuss "natural" boundary conditions.

Key Points:

1. How to derive Euler-Lagrange equations.

1 Formal Recipe

Elastic beam



$I: A \to \mathbb{R}$ is defined by

$$I[f] = \int_0^1 [f'(x)^2 + g(x)\cdot f(x)]\,dx$$

where $A = \{ f \in C^\infty(\mathbb{R}) : f(0) = f(1) = 0\}$.

1. Assume that $f(x) \in A$ is a minimum.
2. Let $\eta \in C^\infty$ be a perturbation of f meaning $\forall t \in (0,1)$ $(f + t\eta) \in A$. This means $\eta(0) = \eta(1) = 0$.
3. Since f is a minimum it follows that
$$\frac{d}{dt}\left(I[f+t\eta]\right)\Big|_{t=0} = 0.$$
4. $I[f+t\eta] = \int_0^1 [(f'(x) + t\eta'(x))^2 + g(x)(f(x) + t\eta(x))]\,dx$
$$\Rightarrow \frac{dI}{dt} = \int_0^1 \left[2(f'(x) + t\eta'(x))\eta'(x) + g(x)\eta(x) \right] dx$$
$$\Rightarrow \frac{dI}{dt}\Big|_{t=0} = \int_0^1 [2f'(x)\eta'(x) + g(x)\eta(x)]\,dx.$$

Why can we differentiate inside the integral?

5. The weak form of the Euler-Lagrange equations is

$$\boxed{\begin{array}{c}\int_0^1 [2f'(x)\eta'(x) + g(x)\eta(x)]\,dx = 0 \\ \forall \eta \in C_c^\infty(\mathbb{R}) \text{ satisfying } \eta(0)=\eta(1)=0.\end{array}}$$

These are necessary conditions for a minimizer.

6. Integrate by parts:

$$\int_0^1 [-2f''(x) + g(x)]\,\eta(x)\,dx = 0.$$

Since this is true for all $\eta(x) \in C_c^\infty(\mathbb{R})$ we can conclude that

$$\boxed{\begin{array}{c}-2f''(x) + g(x) = 0 \\ f(0) = f(1) = 0\end{array}}$$

This is the strong form of the Euler-Lagrange equations.

Critical Point: The strong form of the Euler-Lagrange equations requires second differentiability.

7. Lets solve if $g(x) = g$ a constant.

~~$f(x) = \frac{1}{4}$~~

$f(x) = -\frac{1}{4} g x(x-1)$. → Very simple model of deflection.

Elastic beam with thickness

$I: A \to \mathbb{R}$ is defined by

$$I[f] = \int_0^1 f'(x)^2\,dx + h^2 \int_0^1 f''(x)^2\,dx + \int_0^1 g(x) f(x)\,dx$$

thickness of beam

$f(0) = f(1) = 0$

$f'(0) = f'(1) = 0$ → Clamped boundary conditions.

1. Assume $f(x)$ is a minimum. Let $\eta(x)$ be a perturbation meaning $\eta(0) = \eta(1) = \eta'(0) = \eta'(1) = 0$.
2. Now,

$$\frac{d}{dt} I[f + t\eta]\Big|_{t=0} = \int_0^1 [2f'(x)\eta'(x) + 2h^2 f''(x)\eta''(x) + g(x)\eta(x)]\,dx$$

3. Integrating by parts:

$$\boxed{\begin{gathered}-2f''(x) + 2h^2 f^{iv}(x) + g(x) = 0 \\ f(0) = f(1) = 0,\ f'(0) = f'(1) = 0.\end{gathered}}$$

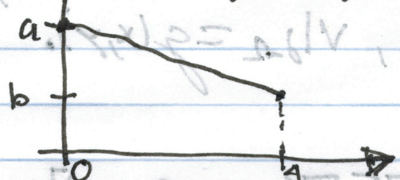
Elastic beam with thickness (no clamping)

$I : A \to \mathbb{R}$ is defined by

$$I[f] = \int_0^1 [f'(x)^2 + h^2 f''(x)^2 + g(x)f(x)]\,dx.$$

~~$A = \{f \in C^\infty(\mathbb{R}) : f(0) = f(1) = 0\}$.~~

$$A = \{f \in C^\infty(0,1) : f(0) = a,\ f(1) = b\}.$$



Now we assume a perturbation satisfying $\eta(0) = \eta(1) = 0$ so that $f + t\eta \in A$. Then,

$$0 = \frac{d}{dt} I[f + t\eta]\Big|_{t=0} = -2f''(x)h^2\eta'(x)\Big|_0^1 + \int_0^1 [-2f''(x) + 2h^2 f^{iv}(x) + g(x)]\eta(x)\,dx$$

$$\Rightarrow \quad -2f''(x) + 2h^2 f^{iv}(x) + g(x) = 0$$

$$f(0) = a,\ f(1) = b$$

$$f''(0) = f''(1) = 0 \quad > \text{Natural Boundary Conditions.}$$

Dirichlet's Principle

$I : A \to \mathbb{R}$ is defined by

$$I[f] = \int_\Omega |\nabla f|^2\,dA + \int_\Omega fg\,dA,$$

where $A = \{f \in C^\infty(\mathbb{R}^n) : f(x) = h(x) \text{ on } \partial\Omega\}$. Let η be a perturbation satisfying $f + t\eta \in A$, i.e. $\eta = 0$ on $\partial\Omega$. Then,

$$\frac{d}{dt} I[f + t\eta] = \frac{d}{dt}\left(\int_\Omega |\nabla f + t\nabla\eta|^2\,dA + \int_\Omega (f + t\eta)g\,dA\right)$$

$$= \int_\Omega 2\nabla\eta \cdot (\nabla f + t\nabla\eta)\,dA + \int_\Omega (\eta g)\,dA$$

$$\Rightarrow \quad \boxed{\int_\Omega 2\nabla\eta \cdot \nabla f\,dA + \int_\Omega \eta g\,dA = 0.} \to \text{weak form.}$$

Apply Gauss-Green Theorem:
$$\int_{\partial\Omega} \eta \nabla f \cdot \vec{n}\, ds - \int_{\Omega} (2\Delta f - g)\eta\, dA = 0$$
$$\Rightarrow \boxed{\begin{aligned} \Delta f &= \tfrac{1}{2} g \\ f|_{\partial\Omega} &= 0. \end{aligned}} \rightarrow \text{Strong-form}$$

Parametric Example

$I : A \to \mathbb{R}$ defined by
$$I[F] = \int_{\Omega} [(u_x + v_y)^2 + (u_y + v_x)^2]\, dx\, dy$$
where $F : \Omega \to \mathbb{R}^2$ is C^∞ and $F(x,y) = (u(x,y), v(x,y))$ and $u|_{\partial\Omega} = f(x,y)$, $v|_{\partial\Omega} = g(x,y)$.

Define,
$$\frac{\delta E}{\delta u} = \lim_{t\to 0} \frac{d}{dt} I[F(u + t\delta u, v)].$$
This is the Gateaux differential in the direction δu.

Calculating,
$$\frac{\delta E}{\delta u} = \int_{\Omega} [2(u_x + v_y)\delta u_x + 2(u_y + v_x)\delta u_y]\, dx\, dy$$
$$= \int_{\Omega} [2(u_x + v_y, u_y + v_x) \cdot \nabla \delta u]\, dx\, dy$$
$$= \int_{\partial\Omega} \delta u \cdot 2(u_x + v_y, u_y + v_x) \cdot \vec{n} - \int_{\Omega} 2(u_{xx} + 2v_{xy} + u_{yy})\delta u\, dx\, dy$$

$$\Rightarrow u_{xx} + 2v_{xy} + u_{yy} = 0,$$

Likewise,
$$v_{xx} + 2u_{xy} + v_{yy} = 0$$

$$\Rightarrow \boxed{\begin{aligned} u_{xx} + 2v_{xy} + u_{yy} &= 0 \\ v_{xx} + 2u_{xy} + v_{yy} &= 0 \\ u|_{\partial\Omega} &= f(x,y) \\ v|_{\partial\Omega} &= g(x,y) \end{aligned}}$$