APMA 2811Q

Homework #1

Due: 9/25/13

1.1 Ill-posed problems

(a) Consider $I : W_0^{1,1}(0,1) \mapsto \mathbb{R}$ defined by

$$I[f] = \int_0^1 \exp\left(-f'(x)^2\right)\, dx,$$

where $W_0^{1,1}(0,1) = \left\{f \in W^{1,1}(0,1) : f(0) = f(1) = 0\right\}$. Show that I has no minimizer in $\mathcal{A}$. (This problem is not coercive or convex).

Proof. Clearly, from strict positivity of the function $g(x) = \exp(-x^2)$ it follows that $\forall f \in \mathcal{A}$ that $I[f] > 0$. Now consider the sequence of functions defined by

$$f_n(x) = \begin{cases} nx, & 0 < x < \frac{1}{2} \\ -nx + n, & \frac{1}{2} < x < 1 \end{cases}.$$

Calculating it follows that

$$I[f_n] = \int_0^1 e^{-n^2}\, dx = e^{-n^2}$$

and hence $\lim_{n\to\infty} I[f_n] = 0$. Therefore, $\forall f \in \mathcal{A}$ there exists $N \in \mathbb{N}$ such that $I[f] > I[f_N] > 0$ proving there is no minimum in $\mathcal{A}$. $\square$

Remark: Notice that the minimizing sequence need not converge to anything. In fact, this is to be expected since the problem is neither convex nor coercive.

(b) Consider $I : \mathcal{A} \mapsto \mathbb{R}$ defined by

$$I[f] = \int_0^1 x f'(x)^2\, dx,$$

where $\mathcal{A} = \{f \in W^{1,2}(0,1) : f(0) = 1,\ f(1) = 0\}$. Show that I has no minimizer in $\mathcal{A}$. (This problem shows that lack of coercivity at one point is enough to guarantee non-existence of a minimum).

Proof. Clearly $\forall f \in \mathcal{A}$, $I[f] \geq 0$. Let f_n be defined by

$$f_n(x) = \begin{cases} 1 & 0 < x < \frac{1}{n} \\ -\frac{\ln(x)}{\ln(n)} & \frac{1}{n} < x < 1 \end{cases}.$$

Therefore,

$$I[f_n] = \int_{\frac{1}{n}}^1 \frac{1}{x \ln(n)^2}\, dx = \frac{1}{\ln(n)}$$

and consequently $\lim_{n\to\infty} I[f_n] = 0$. Now, suppose $\exists f \in \mathcal{A}$ such that $I[f] = 0$. Then $f'(x) = 0$ *a.e.* which is not compatible with the boundary conditions. $\square$

Remark: The construction of the minimizing sequence is not trivial. The reason is if you try to confine the derivative to a small region then from dimensional analysis the value of I in this region will scale like an $\mathcal{O}(1)$ quantity. Instead, what I did was concentrate $f(x) = 1$ into a small region and then spread out the derivative over the interval $(0,1)$.

(c) Consider $I : \mathcal{A} \mapsto \mathbb{R}$ defined by

$$I[f] = \int_0^1 |f'(x)|\, dx,$$

where $\mathcal{A} = \{f \in W^{1,1}(0,1) : f(0) = 0,\ f(1) = 1\}$. Prove that minimizers of I are not unique. (You first need to find a potential minimizer and prove that it is indeed a minimizer).

Proof. From the Fundamental Theorem of Calculus it follows that $\forall f \in \mathcal{A}$, $I[f] \geq 1$. The lower bound is obtained by any smooth monotone increasing function and hence the minimizer is not unique. □

(d) Consider $I : \mathcal{A} \mapsto \mathbb{R}$ defined by

$$I[f] = \int_{-1}^1 \left(2x - f'(x)\right)^2 f(x)^2\, dx,$$

where $\mathcal{A} = \{f \in C^\infty(-1,1) : f(-1) = 0,\ f(1) = 1\}$. Show that I has no minimum in $\mathcal{A}$. What is the correct admissible set we should have considered this problem in?

Proof. Clearly, $I[f] \geq 0$. Moreover, this lower bound is obtained by the non-smooth function f defined by

$$f(x) = \begin{cases} 0 & -1 < x < 0 \\ x^2 & 0 < x < 1 \end{cases}.$$

To obtain a minimizing sequence take any function $f_n \in \mathcal{A}$ satisfying $f_n \to f$ strongly in $W^{1,2}(0,1)$.

Now, suppose $\exists g \in \mathcal{A}$ such that $I[g] = 0$ *a.e.*. Then for every x in $(0,1)$ we have that $g'(x) = 2x$ or $g(x) = 0$ which for our boundary conditions cannot be satisfied by a smooth function. The correct space we should have considered is $W^{1,2}(0,1)$. □

1.2 Euler-Lagrange Equations

(a) Consider $I : \mathcal{A} \mapsto \mathbb{R}$ defined by

$$I[f] = \int_0^1 \left(1 - f'(x)^2\right)^2 + \epsilon^2 \int_0^1 f''(x)^2\, dx,$$

where $\mathcal{A} = W_0^{2,2}(0,1) = \{f \in W^{2,2}(0,1) : f(0) = f(1) = f'(0) = f'(1) = 0\}$. Determine the Euler-Lagrange equations for this functional. Find at least one solution to this equation and show that it cannot be a minimum for all values of ϵ. (This is an example of a bifurcation).

Proof. The formal calculation yields

$$\begin{aligned}\delta I[f] &= -\int_0^1 4(1 - f'(x)^2) f'(x)(\delta f(x))'\, dx + 2\epsilon^2 \int_0^1 f''(x)(\delta f(x))''\, dx \\ &= \int_0^1 \left(4\left(1 - f'(x)^2\right) f'(x) + 2\epsilon^2 \int_0^1 f^{(iv)}(x)\right) \delta f(x)\, dx.\end{aligned}$$

Consequently, the Euler-Lagrange equations are

$$\epsilon^2 f^{(iv)}(x) + 2\frac{d}{dx}\left[\left(1 - f'(x)^2\right) f'(x)\right] = 0.$$

One obvious solution is the function $f_1(x) = 0$. To show that this cannot be a minimizer for all values of ϵ we will rewrite the functional as

$$I[f] = I_1[f] + \epsilon^2 I_2[f].$$

$f_1(x)$ minimizes $I_2[f]$ alone. For large values of ϵ where I_2 is dominate over I_1 we expect f_1 to be a minimizer. As ϵ decreases I_1 dominates over I_2 and we expect the minimizer to look something like

$$f_2(x) = -\left|x - \frac{1}{2}\right| + \frac{1}{2}.$$

However, f_2 does not satisfy our boundary conditions and is too rough. We need to smooth out the corners of the function. For simplicity we will only smooth out near $x = 0$ and argue from symmetry. Define,

$$g_w(x) = \begin{cases} \frac{x^2}{2w^2} & 0 < x < w \\ 0 & w < x < 1 \end{cases}$$

Then $I[g_w] \leq 2w + \frac{\epsilon^2}{w}$. Minimizing over the choice of w we find that $w = \sqrt{2}\epsilon$. Now, there are three corners so we get the upper bound that

$$\inf_{f\in\mathcal{A}} I[f] \leq 3\left(2\sqrt{2} + \frac{1}{\sqrt{2}}\right)\epsilon.$$

Consequently, for ϵ small enough it follows that f_1 cannot be a minimizer since $I[f_1] = 1$. $\square$

(b) Consider $I : \mathcal{A}' \mapsto \mathbb{R}$ defined as above with $\mathcal{A}' = W^{2,2}(0,1)$. Determine the natural boundary conditions that must be satisfied by a smooth minimizer of this functional.

Proof. The formal calculation yields

$$\begin{aligned}\delta I[f] &= -\int_0^1 4(1 - f'(x)^2)f'(x)(\delta f(x))'\,dx + 2\epsilon^2 \int_0^1 f''(x)(\delta f(x))''\,dx \\ &= \int_0^1 \left(4\frac{d}{dx}\left[\left(1 - f'(x)^2\right) f'(x)\right] + 2\epsilon^2 \int_0^1 f^{(iv)}(x)\right) \delta f(x)\,dx \\ &\quad + 4\left(1 - f'(x)^2\right) f'(x)\delta f(x)\Big|_0^1 + 2\epsilon^2 f''(x)\left(\delta f(x)\right)'\Big|_0^1 - 2\,\epsilon^2 f^{(iii)}(x)\delta f(x)\Big|_0^1\end{aligned}$$

Consequently, the natural boundary conditions are:

$$\begin{aligned}-4\left(1 - f'(0)^2\right) f'(0) &= 2f^{(iii)}(0) \\ -4\left(1 - f'(1)^2\right) f'(1) &= 2f^{(iii)}(1) \\ f''(0) &= 0 \\ f''(1) &= 0.\end{aligned}$$

$\square$

1.3 Weak-Convergence

(a) Prove that if $1 \leq p < \infty$ and $u_n \rightharpoonup u$ in $L^p([0,1])$, $v_n \to v$ in $L^q([0,1])$ with $\frac{1}{p} + \frac{1}{q} = 1$ then $u_n v_n \rightharpoonup uv$ in $L^1([0,1])$.

Proof. Let $g \in L^\infty([0,1])$. Then,

$$\begin{aligned}\left|\int_0^1 (u_n v_n - uv) g \, dx\right| &= \left|\int_0^1 g u_n (v_n - v) \, dx + \int_0^1 g v (u_n - u) \, dx\right| \\ &\leq \int_0^1 |g||u_n||(v_n - v)| \, dx + \left|\int_0^1 g v (u_n - u) \, dx\right| \\ &\leq \|g\|_{L^\infty} M \|v_n - v\|_{L^q} + \left|\int_0^1 g v (u_n - u)\right| \, dx,\end{aligned}$$

where $M = \sup_n \|u_n\|_{L^q} < \infty$ by boundedness of weakly convergent sequences. Since $gv \in L^2([0,1])$ the result follows from taking the limit. □

(b) Prove that if $u_n \rightharpoonup u$ in $L^2([0,1])$ and $u_n^2 \rightharpoonup u^2$ in $L^1([0,1])$ then $u_n \to u$ in$L^2([0,1])$.

Proof.

$$\|u_n - u\|_{L^2}^2 = \|u_n\|_{L^2}^2 - 2\int_0^1 u_n u \, dx + \|u\|_{L^2}^2.$$

Since $1 \in L^2([0,1])$ it follows that $\|u_n\|_{L^2}^2 \to \|u\|_{L^2}^2$. The results thus follows from taking the limit. □

(c) Prove that for $1 \leq p \leq \infty$ the unit ball in $L^p([0,1])$ is not strongly compact.

Proof. Let $f_n(x) = \sin(2\pi n x)$. Clearly for all n, $\|f_n\|_{L^p} \leq 1$. Moreover, $\|f_n\|_{L^1} = \frac{2}{\pi}$ and since the L^p norms are monotone increasing in p it follows for all p that $\|f_n\|_{L^p} \geq \frac{2}{\pi}$. Now, for q satisfying $\frac{1}{q} + \frac{1}{p} = 1$ it follows for all $g \in L^q \cap L^2$ that

$$\int_0^1 f_n(x) g(x) \, dx = a_n,$$

where a_n are the coefficients in the sine Fourier series of g. Hence, $a_n \to 0$ and consequently if f_n has a strongly convergent subsequence it must converge to zero. However, from the bounds above this is a contradiction. □

(d) Give an example of a bounded sequence in $L^1([0,1])$ that does not have a weakly convergent subsequence.

Proof. The delta sequence f_n defined by

$$f_n(x) = \begin{cases} n & \frac{1}{2} - \frac{1}{2n} < x < \frac{1}{2} + \frac{1}{2n} \\ 0 & o.w. \end{cases}$$

does not weakly converge to an L^1 function. □

(e) Find a sequence of functions f_n with the property that $f_n \rightharpoonup 0$ in $L^2([0,1])$, $f_n \to 0$ in $L^{\frac{3}{2}}([0,1])$ but f_n does not converge strongly in $L^2([0,1])$.

Proof. The sequence of functions defined by

$$f_n(x) = \begin{cases} \sqrt{n} & \frac{1}{2} - \frac{1}{2n} < x < \frac{1}{2} + \frac{1}{2n} \\ 0 & o.w. \end{cases}$$

works. $\square$