1.1 Ill-posed problems

(a) Consider $I : W_{0}^{1,1}(0,1) \mapsto \mathbb{R}$ defined by

$$I[f] = \int_{0}^{1} \exp(-f'(x)^2) \, dx,$$

where $W_{0}^{1,1}(0,1) = \{ f \in W^{1,1}(0,1) : f(0) = f(1) = 0 \}$. Show that $I$ has no minimizer in $\mathcal{A}$. (This problem is not coercive or convex).

**Proof.** Clearly, from strict positivity of the function $g(x) = \exp(-x^2)$ it follows that $\forall f \in \mathcal{A}$ that $I[f] > 0$. Now consider the sequence of functions defined by

$$f_n(x) = \begin{cases} nx, & 0 < x < \frac{1}{2} \\ -nx + n, & \frac{1}{2} < x < 1 \end{cases}.$$

Calculating it follows that

$$I[f_n] = \int_{0}^{1} e^{-n^2} \, dx = e^{-n^2}$$

and hence $\lim_{n \to \infty} I[f_n] = 0$. Therefore, $\forall f \in \mathcal{A}$ there exists $N \in \mathbb{N}$ such that $I[f] > I[f_N] > 0$ proving there is no minimum in $\mathcal{A}$.

**Remark:** Notice that the minimizing sequence need not converge to anything. In fact, this is to be expected since the problem is neither convex nor coercive.

(b) Consider $I : \mathcal{A} \mapsto \mathbb{R}$ defined by

$$I[f] = \int_{0}^{1} x f'(x)^2 \, dx,$$

where $\mathcal{A} = \{ f \in W^{1,2}(0,1) : f(0) = 1, f(1) = 0 \}$. Show that $I$ has no minimizer in $\mathcal{A}$. (This problem shows that lack of coercivity at one point is enough to guarantee non-existence of a minimum).

**Proof.** Clearly $\forall f \in \mathcal{A}$, $I[f] \geq 0$. Let $f_n$ be defined by

$$f_n(x) = \begin{cases} \frac{1}{n} \ln(x/n), & 0 < x < \frac{1}{n} \\ \frac{1}{n} \ln(n), & \frac{1}{n} < x < 1 \end{cases}.$$

Therefore,

$$I[f_n] = \int_{0}^{1} \frac{1}{x \ln(n)} \, dx = \frac{1}{\ln(n)}$$

and consequently $\lim_{n \to \infty} I[f_n] = 0$. Now, suppose $\exists f \in \mathcal{A}$ such that $I[f] = 0$. Then $f'(x) = 0$ a.e. which is not compatible with the boundary conditions.
Remark: The construction of the minimizing sequence is not trivial. The reason is if you try to confine the derivative to a small region then from dimensional analysis the value of \(I\) in this region will scale like an \(O(1)\) quantity. Instead, what I did was concentrate \(f(x) = 1\) into a small region and then spread out the derivative over the interval \((0, 1)\).

(c) Consider \(I : \mathcal{A} \rightarrow \mathbb{R}\) defined by
\[
I[f] = \int_0^1 |f'(x)| \, dx,
\]
where \(\mathcal{A} = \{ f \in W^{1,1}(0, 1) : f(0) = 0, f(1) = 1 \}\). Prove that minimizers of \(I\) are not unique. (You first need to find a potential minimizer and prove that it is indeed a minimizer).

Proof. From the Fundamental Theorem of Calculus it follows that \(\forall f \in \mathcal{A}, I[f] \geq 1\). The lower bound is obtained by any smooth monotone increasing function and hence the minimizer is not unique.

(d) Consider \(I : \mathcal{A} \rightarrow \mathbb{R}\) defined by
\[
I[f] = \int_{-1}^1 (2x - f'(x))^2 f(x)^2 \, dx,
\]
where \(\mathcal{A} = \{ f \in C^\infty(-1, 1) : f(-1) = 0, f(1) = 1 \}\). Show that \(I\) has no minimum in \(\mathcal{A}\). What is the correct admissible set we should have considered this problem in?

Proof. Clearly, \(I[f] \geq 0\). Moreover, this lower bound is obtained by the non-smooth function \(f\) defined by
\[
f(x) = \begin{cases} 
0 & -1 < x < 0 \\
x^2 & 0 < x < 1 
\end{cases}
\]
To obtain a minimizing sequence take any function \(f_n \in \mathcal{A}\) satisfying \(f_n \rightarrow f\) strongly in \(W^{1,2}(0, 1)\).

Now, suppose \(\exists g \in \mathcal{A}\) such that \(I[g] = 0\) a.e.. Then for every \(x\) in \((0, 1)\) we have that \(g'(x) = 2x\) or \(g(x) = 0\) which for our boundary conditions cannot be satisfied by a smooth function. The correct space we should have considered is \(W^{1,2}(0, 1)\).

1.2 Euler-Lagrange Equations

(a) Consider \(I : \mathcal{A} \rightarrow \mathbb{R}\) defined by
\[
I[f] = \int_0^1 (1 - f'(x)^2)^2 + \epsilon^2 \int_0^1 f''(x)^2 \, dx,
\]
where \(\mathcal{A} = W^{2,2}_0(0, 1) = \{ f \in W^{2,2}(0, 1) : f(0) = f(1) = f'(0) = f'(1) = 0 \}\). Determine the Euler-Lagrange equations for this functional. Find at least one solution to this equation and show that it cannot be a minimum for all values of \(\epsilon\). (This is an example of a bifurcation).

Proof. The formal calculation yields
\[
\delta I[f] = -\int_0^1 4(1 - f'(x)^2) f'(x) (\delta f(x))' \, dx + 2\epsilon^2 \int_0^1 f''(x) (\delta f(x))'' \, dx
= \int_0^1 \left( 4(1 - f'(x)^2) f'(x) + 2\epsilon^2 \int_0^1 f^{(iv)}(x) \right) \delta f(x) \, dx.
\]
Consequently, the Euler-Lagrange equations are
\[ \epsilon^2 f^{(iv)}(x) + 2 \frac{d}{dx} \left[ (1 - f'(x)^2) f'(x) \right] = 0. \]

One obvious solution is the function \( f_1(x) = 0 \). To show that this cannot be a minimizer for all values of \( \epsilon \) we will rewrite the functional as
\[ I[f] = I_1[f] + \epsilon^2 I_2[f]. \]
f\(_1\)(x) minimizes \( I_2[f] \) alone. For large values of \( \epsilon \) where \( I_2 \) is dominate over \( I_1 \) we expect \( f_1 \) to be a minimizer. As \( \epsilon \) decreases \( I_1 \) dominates over \( I_2 \) and we expect the minimizer to look something like
\[ f_2(x) = -\left| x - \frac{1}{2} \right| + \frac{1}{2}. \]

However, \( f_2 \) does not satisfy our boundary conditions and is too rough. We need to smooth out the corners of the function. For simplicity we will only smooth out near \( x = 0 \) and argue from symmetry. Define,
\[ g_w(x) = \begin{cases} x^2 & 0 < x < w \\ 0 & w < x < 1 \end{cases} \]

Then \( I[g_w] \leq 2w + \frac{\epsilon^2}{w} \). Minimizing over the choice of \( w \) we find that \( w = \sqrt{2} \epsilon \). Now, there are three corners so we get the upper bound that
\[ \inf_{f \in \mathcal{A}} I[f] \leq 3 \left( 2\sqrt{2} + \frac{1}{\sqrt{2}} \right) \epsilon. \]
Consequently, for \( \epsilon \) small enough it follows that \( f_1 \) cannot be a minimizer since \( I[f_1] = 1 \).

(b) Consider \( I : \mathcal{A}' \to \mathbb{R} \) defined as above with \( \mathcal{A}' = W^{2,2}(0,1) \). Determine the natural boundary conditions that must be satisfied by a smooth minimizer of this functional.

**Proof.** The formal calculation yields
\[
\delta I[f] = -\int_0^1 4(1 - f'(x)^2) f'(x)(\delta f(x))' \, dx + 2\epsilon^2 \int_0^1 f''(x)(\delta f(x))'' \, dx \\
= \int_0^1 \left( 4 \frac{d}{dx} \left[ (1 - f'(x)^2) f'(x) \right] + 2\epsilon^2 \int_0^1 f^{(iv)}(x) \delta f(x) \, dx \\
+ 4 \left( 1 - f'(x)^2 \right) f'(x) \delta f(x) \right|_0^1 + 2\epsilon^2 f''(x)(\delta f(x))' \right|_0^1 - 2 \epsilon^2 f^{(iii)}(x) \delta f(x) \right|_0^1
\]
Consequently, the natural boundary conditions are:
\[
-4 \left( 1 - f'(0)^2 \right) f'(0) = 2f^{(iii)}(0) \\
-4 \left( 1 - f'(1)^2 \right) f'(1) = 2f^{(iii)}(1) \\
\quad f''(0) = 0 \\
\quad f''(1) = 0.
\]
1.3 Weak-Convergence

(a) Prove that if $1 \leq p < \infty$ and $u_n \rightharpoonup u$ in $L^p([0,1])$, $v_n \to v$ in $L^q([0,1])$ with $\frac{1}{p} + \frac{1}{q} = 1$ then $u_nv_n \rightharpoonup uv$ in $L^1([0,1])$.

Proof. Let $g \in L^\infty([0,1])$. Then,

$$\left| \int_0^1 (u_nv_n - uv)g \, dx \right| = \left| \int_0^1 gu_n(v_n - v) \, dx + \int_0^1 gv(u_n - u) \, dx \right|$$

$$\leq \int_0^1 |g| |u_n||v_n - v| \, dx + \int_0^1 |gv(u_n - u)| \, dx$$

$$\leq \|g\|_{L^\infty} M \|v_n - v\|_{L^q} + \left| \int_0^1 gv(u_n - u) \, dx \right|,$$

where $M = \sup_n \|u_n\|_{L^q} < \infty$ by boundedness of weakly convergent sequences. Since $gv \in L^2([0,1])$ the result follows from taking the limit.

(b) Prove that if $u_n \to u$ in $L^2([0,1])$ and $u_n^2 \to u^2$ in $L^1([0,1])$ then $u_n \to u$ in $L^2([0,1])$.

Proof. 

$$\|u_n - u\|_{L^2}^2 = \|u_n\|_{L^2}^2 - 2 \int_0^1 u_n u \, dx + \|u\|_{L^2}^2.$$

Since $1 \in L^2([0,1])$ it follows that $\|u_n\|_{L^2}^2 \to \|u\|_{L^2}^2$. The results thus follows from taking the limit.

(c) Prove that for $1 \leq p \leq \infty$ the unit ball in $L^p([0,1])$ is not strongly compact.

Proof. Let $f_n(x) = \sin(2\pi nx)$. Clearly for all $n$, $\|f_n\|_{L^p} \leq 1$. Moreover, $\|f_n\|_{L^1} = \frac{2}{\pi}$ and since the $L^p$ norms are monotone increasing in $p$ it follows for all $p$ that $\|f_n\|_{L^p} \geq \frac{2}{\pi}$. Now, for $q$ satisfying $\frac{1}{q} + \frac{1}{p} = 1$ it follows for all $g \in L^q \cap L^2$ that

$$\int_0^1 f_n(x)g(x) \, dx = a_n,$$

where $a_n$ are the coefficients in the sine Fourier series of $g$. Hence, $a_n \to 0$ and consequently if $f_n$ has a strongly convergent subsequence it must converge to zero. However, from the bounds above this is a contradiction.

(d) Give an example of a bounded sequence in $L^1([0,1])$ that does not have a weakly convergent subsequence.

Proof. The delta sequence $f_n$ defined by

$$f_n(x) = \begin{cases} n & \frac{1}{2} - \frac{1}{2n} < x < \frac{1}{2} + \frac{1}{2n} \\ 0 & \text{otherwise} \end{cases}$$

does not weakly converge to an $L^1$ function.

(e) Find a sequence of functions $f_n$ with the property that $f_n \to 0$ in $L^2([0,1])$, $f_n \to 0$ in $L^2([0,1])$ but $f_n$ does not converge strongly in $L^2([0,1])$. 

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Proof. The sequence of functions defined by

\[ f_n(x) = \begin{cases} \sqrt{n} & \frac{1}{2} - \frac{1}{2n} < x < \frac{1}{2} + \frac{1}{2n} \\ 0 & o.w. \end{cases} \]

works. \qed