APMA 2811Q

Homework #2

Due: 10/25/13

2.1 Convexity and Existence

(a) Let $\mathcal{A} = \{ f \in W^{1,\infty}(0,1) : f(0) = a \text{ and } f(1) = b \}$ and define $I : \mathcal{A} \mapsto \mathbb{R}$ by

$$I[f] = \int_0^1 W(f'(x))\, dx,$$

where W is a non-strictly convex function – i.e. it has at least one affine part – satisfying the coercivity condition $\displaystyle\lim_{|z|\to\infty} \frac{W(z)}{|z|} = \infty$. Prove that there exists $a, b \in \mathbb{R}$ for which I has an infinite number of minimizers over $\mathcal{A}$.

Hint: This is similar to problem 1.c from the last homework.

(b) This exercise will work through in detail how to derive a *variational inequality* for a constrained minimization problem. Define,

$$\mathcal{A} = \{ f \in W_0^{1,2} : |\nabla f| \leq 1 \ a.e. \}$$

and $I : \mathcal{A} \mapsto \mathbb{R}$ by

$$I[f] = \int_0^1 \frac{1}{2} |\nabla f(x)|^2 - f(x)g(x)\, dx,$$

where $g \in L^2(\Omega)$.

(1) Prove in detail using the *direct method* the existence of a minimum for this problem.

(2) Prove that the minimizer for this problem is unique.

Hint: The inequality $2a \cdot b \leq |a|^2 + |b^2| - |a - b|^2$ might be useful.

(3) Prove that if $f^* \in \mathcal{A}$ minimizes I then for all $f \in \mathcal{A}$:

$$\int_\Omega \nabla f^*(x) \cdot \nabla \left(f(x) - f^*(x) \right) dx \geq \int_\Omega g(x) \left(f(x) - f^*(x) \right) dx.$$

Hint: The derivation of this inequality is very similar to the derivation of the Euler-Lagrange equations, however you must use the convexity of $\mathcal{A}$ to define the perturbation.

2.2 Duality

(a) Consider the problem $\displaystyle\min_{f \in W^{1,2}(\Omega)} I[f]$ where $I : W^{1,2}(\Omega) \mapsto \mathbb{R}$ is defined by

$$I[f] = \int_\Omega \left(W(\nabla f) + f(x)g(x) \right) dx$$

with

$$W(p) = \begin{cases} 2|p| & |p| \leq 1 \\ 1 + |p|^2 & |p| \geq 1 \end{cases},$$

and $g \in L^2$. What is the dual problem in this case? (Note that the Lagrangian in this case is convex but not differentiable so it is not clear that calculating the E-L equations makes any sense).

(b) Consider the problem $\min\limits_{f \in \mathcal{A}} I[f]$ with $I : \mathcal{A} \mapsto \mathbb{R}$ defined by $I[f] = \int_\Omega \frac{1}{2} |\nabla f|^2 \, dx$ where $\mathcal{A} = \{f \in W^{1,2}(\Omega) : f = g \text{ on } \partial\Omega\}$. Show that the dual to this problem is

$$\max_{\nabla \cdot \sigma = 0} \int_{\partial\Omega} (\sigma \cdot n) g(x) \, ds - \frac{1}{2} \int_\Omega |\sigma|^2 \, dx.$$

(c) Let $\mathcal{A} = \{\sigma \in W^{1,1}(\Omega) : \nabla \cdot \sigma = F \text{ in } \Omega \text{ and } \sigma \cdot n = f \text{ on } \partial\Omega\}$ and $\mathcal{B} = \big\{u \in W^{1,\infty}(\Omega) : \|\nabla u\|_{L^\infty} \leq 1\big\}$. Show that the problems

$$(P) \quad \min_{\sigma \in \mathcal{A}} \int_\Omega |\sigma| \, dx \quad \text{and} \quad (D) \quad \max_{u \in \mathcal{B}} \int_\Omega \left(u(x) f(x) \, ds - \int_\Omega u(x) F(x) \, dx \right)$$

are a dual pair if $\int_\Omega F(x) \, dx = \int_{\partial\Omega} f(x) \, dx$.