1.1 Ill-posed problems
(a) Consider $I : W^{1,1}_0(0,1) \to \mathbb{R}$ defined by

$$I[f] = \int_0^1 \exp(-f'(x)^2) \, dx,$$

where $W^{1,1}_0(0,1) = \{ f \in W^{1,1}(0,1) : f(0) = f(1) = 0 \}$. Show that $I$ has no minimizer in $A$. (This problem is not coercive or convex).

(b) Consider $I : A \to \mathbb{R}$ defined by

$$I[f] = \int_0^1 xf'(x)^2 \, dx,$$

where $A = \{ f \in W^{1,2}(0,1) : f(0) = 1, f(1) = 0 \}$. Show that $I$ has no minimizer in $A$. (This problem shows that lack of coercivity at one point is enough to guarantee non-existence of a minimum).

(c) Consider $I : A \to \mathbb{R}$ defined by

$$I[f] = \int_0^1 |f'(x)| \, dx,$$

where $A = \{ f \in W^{1,1}(0,1) : f(0) = 0, f(1) = 1 \}$. Prove that minimizers of $I$ are not unique. (You first need to find a potential minimizer and prove that it is indeed a minimizer).

(d) Consider $I : A \to \mathbb{R}$ defined by

$$I[f] = \int_{-1}^1 (2x - f'(x))^2 f(x)^2 \, dx,$$

where $A = \{ f \in C^\infty(-1,1) : f(-1) = 0, f(1) = 1 \}$. Show that $I$ has no minimum in $A$. What is the correct admissible set we should have considered this problem in?

1.2 Euler-Lagrange Equations
(a) Consider $I : A \to \mathbb{R}$ defined by

$$I[f] = \int_0^1 (1 - f'(x)^2)^2 + \epsilon^2 \int_0^1 f''(x)^2 \, dx,$$

where $A = W^{2,2}_0(0,1) = \{ f \in W^{2,2}(0,1) : f(0) = f(1) = f'(0) = f'(1) = 0 \}$. Determine the Euler-Lagrange equations for this functional. Find at least one solution to this equation and show that it cannot be a minimum for all values of $\epsilon$. (This is an example of a bifurcation).

(b) Consider $I : A' \to \mathbb{R}$ defined as above with $A' = W^{2,2}(0,1)$. Determine the natural boundary conditions that must be satisfied by a smooth minimizer of this functional.
1.3 Weak-Convergence

(a) Prove that if $1 \leq p < \infty$ and $u_n \rightharpoonup u$ in $L^p([0,1])$, $v_n \rightharpoonup v$ in $L^q([0,1])$ with $\frac{1}{p} + \frac{1}{q} = 1$ then $u_nv_n \rightharpoonup uv$ in $L^1([0,1])$.

(b) Prove that if $u_n \rightharpoonup u$ in $L^2([0,1])$ and $u^2_n \rightharpoonup u^2$ in $L^1([0,1])$ then $u_n \rightharpoonup u$ in $L^2([0,1])$.

(c) Prove that for $1 \leq p \leq \infty$ the unit ball in $L^p([0,1])$ is not strongly compact.

(d) Give an example of a bounded sequence in $L^1([0,1])$ that does not have a weakly convergent subsequence.

(e) Find a sequence of functions $f_n$ with the property that $f_n \rightharpoonup 0$ in $L^2([0,1])$, $f_n \to 0$ in $L^2([0,1])$ but $f_n$ does not converge strongly in $L^2([0,1])$. 