APMA 2811Q

Homework #1

Due: 9/25/13

1.1 Ill-posed problems

(a) Consider $I : W_0^{1,1}(0,1) \mapsto \mathbb{R}$ defined by

$$I[f] = \int_0^1 \exp\left(-f'(x)^2\right)\, dx,$$

where $W_0^{1,1}(0,1) = \left\{f \in W^{1,1}(0,1) : f(0) = f(1) = 0\right\}$. Show that I has no minimizer in $\mathcal{A}$. (This problem is not coercive or convex).

(b) Consider $I : \mathcal{A} \mapsto \mathbb{R}$ defined by

$$I[f] = \int_0^1 x f'(x)^2\, dx,$$

where $\mathcal{A} = \{f \in W^{1,2}(0,1) : f(0) = 1,\ f(1) = 0\}$. Show that I has no minimizer in $\mathcal{A}$. (This problem shows that lack of coercivity at one point is enough to guarantee non-existence of a minimum).

(c) Consider $I : \mathcal{A} \mapsto \mathbb{R}$ defined by

$$I[f] = \int_0^1 |f'(x)|\, dx,$$

where $\mathcal{A} = \{f \in W^{1,1}(0,1) : f(0) = 0,\ f(1) = 1\}$. Prove that minimizers of I are not unique. (You first need to find a potential minimizer and prove that it is indeed a minimizer).

(d) Consider $I : \mathcal{A} \mapsto \mathbb{R}$ defined by

$$I[f] = \int_{-1}^1 \left(2x - f'(x)\right)^2 f(x)^2\, dx,$$

where $\mathcal{A} = \{f \in C^\infty(-1,1) : f(-1) = 0,\ f(1) = 1\}$. Show that I has no minimum in $\mathcal{A}$. What is the correct admissible set we should have considered this problem in?

1.2 Euler-Lagrange Equations

(a) Consider $I : \mathcal{A} \mapsto \mathbb{R}$ defined by

$$I[f] = \int_0^1 \left(1 - f'(x)^2\right)^2 + \epsilon^2 \int_0^1 f''(x)^2\, dx,$$

where $\mathcal{A} = W_0^{2,2}(0,1) = \{f \in W^{2,2}(0,1) : f(0) = f(1) = f'(0) = f'(1) = 0\}$. Determine the Euler-Lagrange equations for this functional. Find at least one solution to this equation and show that it cannot be a minimum for all values of ϵ. (This is an example of a bifurcation).

(b) Consider $I : \mathcal{A}' \mapsto \mathbb{R}$ defined as above with $\mathcal{A}' = W^{2,2}(0,1)$. Determine the natural boundary conditions that must be satisfied by a smooth minimizer of this functional.

1.3 Weak-Convergence

(a) Prove that if $1 \leq p < \infty$ and $u_n \rightharpoonup u$ in $L^p([0,1])$, $v_n \to v$ in $L^q([0,1])$ with $\frac{1}{p} + \frac{1}{q} = 1$ then $u_n v_n \rightharpoonup uv$ in $L^1([0,1])$.

(b) Prove that if $u_n \rightharpoonup u$ in $L^2([0,1])$ and $u_n^2 \rightharpoonup u^2$ in $L^1([0,1])$ then $u_n \to u$ in$L^2([0,1])$.

(c) Prove that for $1 \leq p \leq \infty$ the unit ball in $L^p([0,1])$ is not strongly compact.

(d) Give an example of a bounded sequence in $L^1([0,1])$ that does not have a weakly convergent subsequence.

(e) Find a sequence of functions f_n with the property that $f_n \rightharpoonup 0$ in $L^2([0,1])$, $f_n \to 0$ in $L^{\frac{3}{2}}([0,1])$ but f_n does not converge strongly in $L^2([0,1])$.