1 Final Exam Review Problems

1. What is the domain of the functions $f(x) = \dfrac{1}{x-3}$ and $g(x) = \dfrac{\sqrt{x+2}}{x-3}$.
   Solution: A lot of times when you are doing a domain problem you want to ask yourself where the function is undefined. For $f(x)$, we could have a problem of dividing by zero. So, we must have that $x \neq 3$. Therefore, the domain for $f(x)$ is simply:
   $$D: x \neq 3$$
   Now, for $g(x)$ we have the added problem that we cannot take the square root of a negative number. Therefore we must make sure that everything underneath the radical is positive or greater than zero. Written in math symbols this means that:
   $$\begin{aligned} x+2 &\geq 0 \\ \Rightarrow x &\geq -2. \end{aligned}$$
   Also, we have that $x \neq 3$ since we still don't want to divide by zero. Therefore, the domain for $g(x)$ is given by
   $$D: x \geq -2 \text{ and } x \neq 3.$$

2. If $f(x) = \sqrt{x}$, evaluate and completely simplify the difference quotient $\dfrac{f(x+h)-f(x)}{h}$.
   Solution: The goal with any difference quotient problem is to eliminate the h in the denominator. What makes this problem difficult is the square root. To "get rid" of the square root I am going to multiply the top and bottom of the fraction by the conjugate.
   $$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{\sqrt{x+h}-\sqrt{x}}{h} \\ &= \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \\ &= \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \frac{h}{h(\sqrt{x+h}+\sqrt{x})} \\ &= \frac{1}{\sqrt{x+h}+\sqrt{x}}. \end{aligned}$$

3. Find a possible formula for the following tables:

x	0	1	2	3	4
f(x)	7	12	17	22	27

x	0	1	2	3	4
g(x)	3	5.1	8.67	14.739	25.89151

   Solution: The first function is linear. To check if a table is a linear function check if it has a constant slope. The first table has a constant slope of 5. Also, the table has a y-intercept of 3. Therefore, the formula for the function is given by:
   $$f(x) = 5x + 3.$$

The second table is exponential. The way you check if a table is exponential is to take the quotient (i.e. divide) of consecutive y values and see if you get a constant. For this table we have that

$$
\begin{aligned}
5.1/3 &= 1.7 \\
8.67/5.1 &= 1.7 \\
14.739/8.67 &= 1.7 \\
25.89151/14.739 &= 1.7.
\end{aligned}
$$

Therefore, this table is an exponential function.

Now, lets see why this works. Suppose we have a function in the form $g(x) = ab^x$ that fits the data points of some table. Now, lets see what happens if we divide two consecutive y-values. For $x = 2$ and $x = 1$ we have that

$$g(2)/g(1) = (ab^2)/(ab^1) = b$$

Now, lets pick an arbitrary input n. The next consecutive input is $n + 1$. If we then divide there corresponding outputs we have that

$$g(n+1)/g(n) = (ab^{n+1})/(ab^n) = b.$$

It follows that no matter what two consecutive points you pick their quotient will always be the constant b.

So, if we return to our table above we have that $b = 1.7$ and $a = 3$, which is the y-intercept. So, the equation for $g(x)$ is simply

$$g(x) = 3(1.7)^x.$$

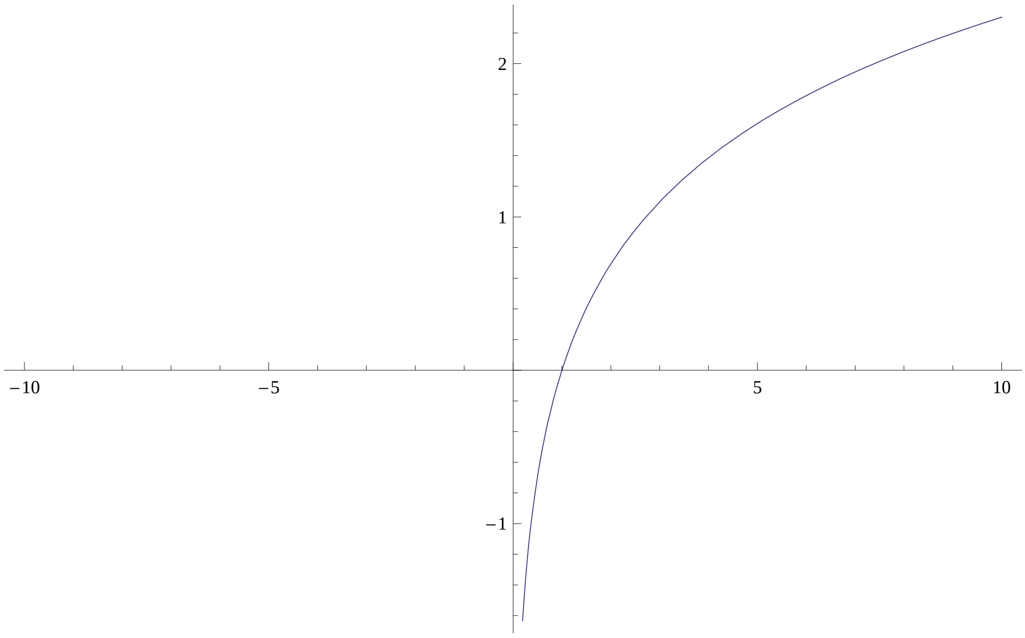
Also, recall that $b = 1 + r$ where r is the decimal representation of the percent rate of change. Therefore, every time x increases by 1, $g(x)$ increases by 70%.

Finally, lets convert our exponential function into the form $g(x) = ae^{kx}$. We already know that $a = 3$, so all we need to do is find k. Well, since this is just another way to represent the same exponential function we must have that for all values of x the following must be true:

$$
\begin{aligned}
3(1.7)^x &= 3e^{kx} \\
\Rightarrow \ln(1.7^x) &= \ln(e^{kx}) \\
\Rightarrow x\ln(1.7) &= kx \\
\Rightarrow \ln(1.7) &= k.
\end{aligned}
$$

Remember that k is called the continuous growth rate.

4. What is the range and domain of the function $f(x) = 3 + 2\ln(x - 4)$?
Solution: First, **you cannot use your calculator to do this problem**. One way to do this problem is to think about this is as a transformation of the base graph $g(x) = \ln(x)$. The graph of $g(x) = ln(x)$ is given below: Now, $ln(x)$ has a range of $(-\infty, \infty)$ and a domain of $(0, \infty)$. Now, $f(x)$ just is this graph shifted right 4, vertically expanded by a factor of 2, and then vertically shifted up 3. This means that the range gets shifted to $(-\infty, \infty)$ (no change) and the domain is shifted to $(4, \infty)$.



Another way to do this problem is to think about the inverse of $f(x)$. The inverse of $f(x)$ is given by (you should verify this)

$$f^{-1}(x) = e^{1/2(x-3)} + 4.$$

Now, the domain of $f^{-1}(x)$ is easily seen to be all real numbers since you can input any number. Now, the range of $f^{-1}(x)$ is easy to find since this is basically an exponential function that is shift up vertically 4 units. Therefore, the range is $(4, \infty)$. This means that $f(x)$ must have the opposite domain and range. In this case this means that the domain of $f(x)$ is $(4, \infty)$ and the range of $f(x)$ is $(-\infty, \infty)$.

Finally, the **only** asymptote of $f(x)$ is the line $x = 4$.

5. If $(2, 3)$ lies on the graph of $f(x)$. What point must lie on the graph of $g(x) = -2 \cdot f(x-4)+3$?
Solution: This is a simple transformation problem. The graph of $g(x)$ is related to the graph of $f(x)$ by first shifting right 4, vertically expanding by 2, reflecting across the x-axis, and then shifting up 3. Therefore, $(2, 3)$ gets translated to the point $(6, -3)$.

   Another way to think about this problem is to use our knowledge that $f(2) = 3$. So if we evaluate $g(6)$ we have that

$$g(6) = -2 \cdot f(2) + 3 = -2 \cdot 3 + 3 = -3.$$

   This means that the point $(6, -3)$ lies on the graph of $g(x)$.

6. Determine which of the following functions are odd, even, or neither.

   - $f(x) = \frac{x^2+1}{x^6-1}$
   Solution:

$$f(-x) = \frac{(-x)^2 + 1}{(-x)^6 - 1} = \frac{x^2 + 1}{x^6 - 1} = f(x).$$

     This function is even.

- $f(x) = \frac{|x|}{x-2}$.

$$f(-1) = 1/(-3) = -1/3.$$
$$f(1) = 1/(-1) = -1.$$

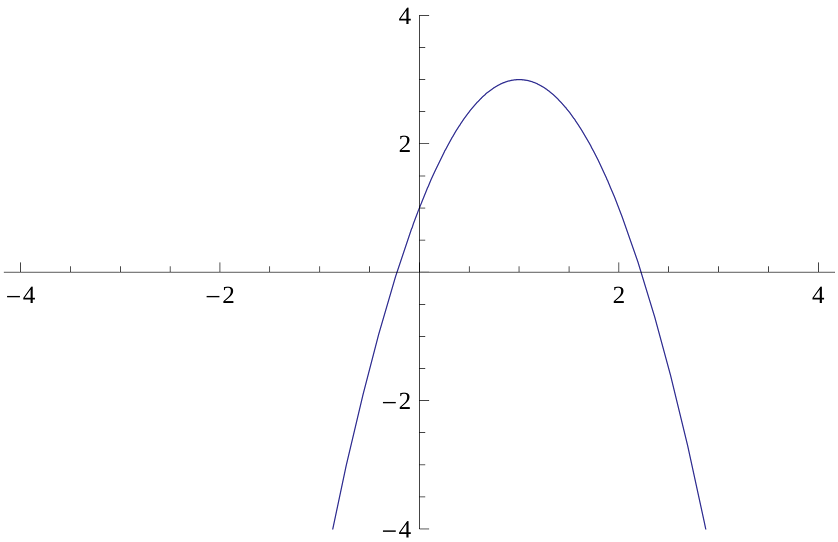
This function is neither even or odd.

- $f(x) = \sin(x)\cos(x)$.

$$f(-x) = \sin(-x)\cos(-x) = -\sin(x)\cos(x) = -f(x).$$

This function is odd.

7. What is the maximum value of the function $f(x) = -2x^2 + 4x + 1$?
Solution: If you want to find the maximum value of a function a good place to start is to graph the function to at least get an idea of what the function looks like and where the maximum value is. The graph of $f(x)$ is given below. So, we can see from the graph that the



highest point on the graph is given by the vertex. This means all we need to do is complete the square to get the vertex.

$$\begin{aligned} -2x^2 + 4x + 1 &= -2(x^2 - 2x) + 1 \\ &= -2(x^2 - 2x + 1 - 1) + 1 \\ &= -2\left[(x-1)^2 - 1\right] + 1 \\ &= -2(x-2)^2 + 2 + 1 \\ &= -2(x-2)^2 + 3 \end{aligned}$$

Therefore, the vertex is given by the coordinates $(2, 3)$. Now the maximum of a function is the highest value the function reaches. So the maximum is 3.

8. Solve the equation $\ln(3x + 2) + \ln(x - 3) - 3 = 2$.
Solution: With any log equation our first goal should be to eliminate the logarithms. To do this we have to make sure we use the properties correctly.

$$\ln(3x + 2) + \ln(x - 3) - 3 = 2$$

$$\Rightarrow \ln((3x+2)(x-3)) = 5$$
$$\Rightarrow (3x+2)(x-3) = e^5$$

Now, this is a quadratic equation so our goal is to get everything on one side of the equation and try to factor or use the quadratic formula.

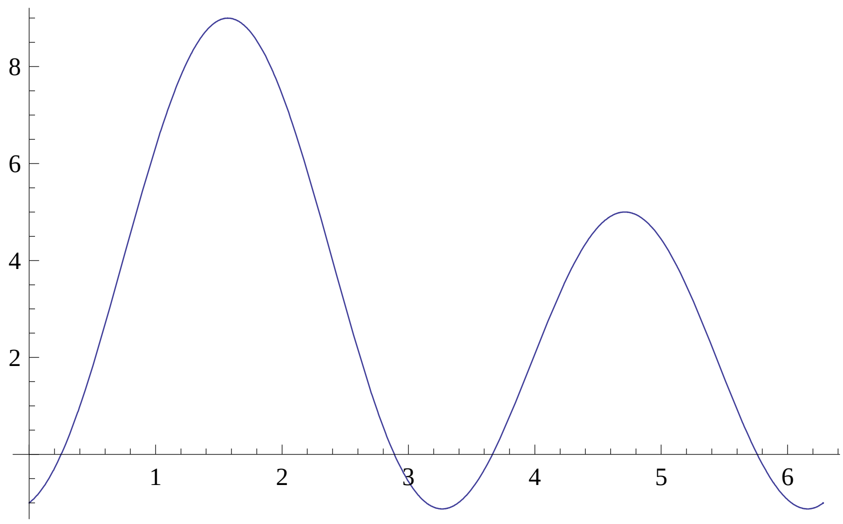
$$\Rightarrow 3x^2 - 7x - 6 = e^5$$

$$\Rightarrow 3x^2 - 7x - 6 - e^5 = 0$$
$$\Rightarrow x = \frac{7 \pm \sqrt{49 - 4 \cdot 3 \cdot (-6 - e^5)}}{6}$$
$$\Rightarrow x = \frac{7 \pm \sqrt{121 + e^5}}{6}$$

Finally, these are supposed to be solutions to the original log problem. But you will see that the logarithms above are undefined for the negative root. Therefore, the final solutions is given by:

$$x = \frac{7 + \sqrt{121 + e^5}}{6}$$

9. Find all solutions $0 < \theta < 2\pi$, satisfying $8\sin^2(\theta) + 2\sin(\theta) - 1 = 0$
Solution: One problem with solving trig equations is that they can have a ton of solutions. So it would be nice to know how many solutions there are before we try to solve the problem. One way you can figure out how many solutions there are is to graph the function $f(\theta) = 8\sin^2(\theta) + 2\sin(\theta) - 1$ on the interval $0 < \theta < 2\pi$ and count the number of times it crosses the horizontal axis. We can see from the graph that there are four zeros. Now, to actually



solve this problem you have to recognize that the equation is factorable and can be factored into the form

$$(4\sin(\theta) - 1)(2\sin(\theta) + 1) = 0$$
$$\Rightarrow 4\sin(\theta) - 1 = 0 \text{ and } 2\sin(\theta) + 1 = 0$$
$$\Rightarrow \sin(\theta) = \frac{1}{4} \text{ and } \sin(\theta) = -\frac{1}{2}$$

$$\Rightarrow \theta = \sin^{-1}\left(\frac{1}{4}\right) \text{ and } \theta = \sin^{-1}\left(-\frac{1}{2}\right)$$

$$\Rightarrow \theta = .25268 \text{ and } \theta = -\frac{\pi}{6}$$

Now, $-\frac{\pi}{6}$ is not in the interval $0 < \theta < 2\pi$ so we should add 2π to it. Therefore, we have the two solutions

$$\theta = .25268 \text{ and } \theta = \frac{11\pi}{6}.$$

To get another solution we need to think about another angle ϕ such that $\sin(\phi) = \frac{1}{4}$. Well, we know that the sine function is positive is in the second quadrant. So, using .25268 as a reference angle we have that another solution is $\pi - .25268$. If you have trouble with these type of problems, draw pictures!! Finally, we need to get one more solution. We want to find an angle ψ such that $\sin(\psi) = -1\frac{1}{2}$. Since sine is negative in the third quadrant we can use the reference angle $\frac{\pi}{6}$ to get that the fourth solution is $pi + \frac{\pi}{6}$. Therefore, the four solutions to this problem are given by:

$$.25268, \frac{11\pi}{6}, \pi - .25268 \text{ and } \frac{7\pi}{6}.$$

10. Prove the following identities:

- $\cos^4(\theta) - \sin^4(\theta) = \cos(2\theta)$.
- $\sec(2\theta) = \frac{\sec^2(\theta)}{2-\sec^2(\theta)}$.

Solution:

$$\begin{aligned}
\cos^4(\theta) - \sin^4(\theta) &= (\cos^2(\theta) - \sin^2(\theta))(\cos^2(\theta) + \sin^2(\theta)) \\
&= (\cos^2(\theta) - \sin^2(\theta)) \cdot 1 \\
&= \cos(2\theta).
\end{aligned}$$

$$\begin{aligned}
\frac{\sec^2(\theta)}{2 - \sec^2(\theta)} &= \frac{\frac{1}{\cos^2(\theta)}}{2 - \frac{1}{\cos^2(\theta)}} \\
&= \frac{\frac{1}{\cos^2(\theta)}}{\frac{2\cos^2(\theta)-1}{\cos^2(\theta)}} \\
&= \frac{1}{2\cos^2(\theta) - 1} \\
&= \frac{1}{\cos(2\theta)} \\
&= \sec(2\theta).
\end{aligned}$$

11. Suppose a population grows according to the formula $P(t) = 35e^{.7t}$.

- After how many years will the population reach 500 people?

- By what percentage does the population grow each year?

Solution: The first problem is a simple exponential equation. We need to solve the equation $500 = 35e^{.7t}$.

$$\frac{500}{35} = e^{.7t}$$
$$\Rightarrow \ln(\frac{100}{7}) = .7t$$
$$\Rightarrow t = \frac{\ln(\frac{100}{7})}{.7}$$
$$\Rightarrow t = 3.79 \text{ years.}$$

Now, to find the percentage that this population grows at we need to convert the exponential into the form ab^t. We know that $a = 35$ so we need to just find b. Setting the two equations equal to each other we have that:

$$35b^t = 35e^{.7t} = 35(e^{.7})^t$$
$$\Rightarrow b = e^{.7}$$
$$\Rightarrow b = 2.0138.$$

Therefore, the percentage the population grows in one year is 101%.

12. If $p(x) = \frac{1}{1-x}$, $g(x) = x^3 + 2$, and $h(x) = \frac{1}{x^2}$ simplify the expression $\frac{1}{p(x^2)} - h(g(x))$
Solution:

$$\begin{aligned} \frac{1}{p(x^2)} - h(g(x)) &= \frac{1}{\frac{1}{1-x^2}} - \frac{1}{(x^3+2)^2} \\ &= 1 - x^2 - \frac{1}{(x^3+2)^2} \\ &= \frac{(1-x^2)\cdot(x^3+2)^2 - 1}{(x^3+2)^2}. \end{aligned}$$

13. Find the inverses of the functions $f(x) = \dfrac{e^x+3}{e^x-2}$ and $g(x) = \dfrac{3}{2+\ln(x)}$.

Solution:

$$y = \frac{e^x+3}{e^x-2}$$
$$\Rightarrow (e^x - 2)y = e^x + 3$$
$$\Rightarrow e^x y - 2y = e^x + 3$$
$$\Rightarrow e^x y - e^x = 3 + 2y$$
$$\Rightarrow e^x(y-1) = 3 + 2y$$
$$\Rightarrow e^x = \frac{3+2y}{y-1}$$
$$x = \ln\left(\frac{3+2y}{y-1}\right).$$

$$f^{-1}(x) = \ln\left(\frac{3+2x}{x-1}\right)$$

$$\begin{aligned} y &= \frac{3}{2+\ln(x)} \\ \Rightarrow \frac{1}{y} &= \frac{2+\ln(x)}{3} \\ \Rightarrow \frac{3}{y} - 2 &= \ln(x) \\ \Rightarrow x &= e^{\frac{3-2y}{y}} \\ g^{-1}(x) &= e^{\frac{3-2x}{x}}. \end{aligned}$$

14. Express $\sin(4\theta)$ in terms of functions involving $\cos(\theta)$ and $\sin(\theta)$.

Solution:

$$\sin(4\theta) = \sin(2\cdot 2\theta) = 2\sin(2\theta)\cos(2\theta) = 4\cos(\theta)\sin(\theta)(\cos^2(\theta) - \sin^2(\theta)).$$