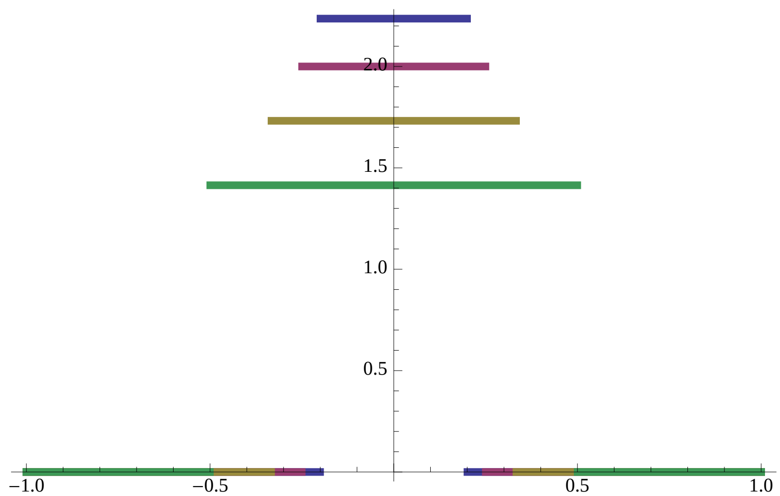
Here is the example I butchered in class today. Thanks go to Tasos for reminding me to change variables and Dane for providing a second proof that dug into the intricacies of this problem. Let $f_n(x) \in L^2([-1,1])$ be defined by

$$f_n(x) = \begin{cases} \sqrt{n} & \text{if } -\frac{1}{n} \le x \le \frac{1}{n} \\ 0 & \text{o.w.} \end{cases} \tag{1}$$



We will show that $f_n \rightharpoonup 0$ in L^2 but f_n does not converge strongly. We have $\forall g \in L^2$ that

$$\begin{aligned} \int_{-1}^{1} f_n(x) g(x)\, dx &= \sqrt{n} \int_{-\frac{1}{n}}^{\frac{1}{n}} g(x)\, dx \\ &= \frac{1}{\sqrt{n}} \int_{-1}^{1} g\left(\frac{u}{n}\right)\, du \\ &\le \frac{1}{\sqrt{n}} \|g\|_{L^1}. \end{aligned}$$

Taking the limit as $n \to \infty$ gives the result. The change of variables $u = nx$ correctly "zooms" in on interval the mass of f is being concentrated on. This is a simple example of how to rescale near where mass is being concentrated.