Here is the example I butchered in class today. Thanks go to Tasos for reminding me to change variables and Dane for providing a second proof that dug into the intricacies of this problem. Let \( f_n(x) \in L^2([-1, 1]) \) be defined by

\[
  f_n(x) = \begin{cases} 
    \sqrt{n} & \text{if } -\frac{1}{n} \leq x \leq \frac{1}{n} \\
    0 & \text{o.w.}
  \end{cases}
\]  

We will show that \( f_n \rightharpoonup 0 \) in \( L^2 \) but \( f_n \) does not converge strongly. We have \( \forall g \in L^2 \) that

\[
  \int_{-1}^{1} f_n(x)g(x) \, dx = \sqrt{n} \int_{-\frac{1}{n}}^{\frac{1}{n}} g(x) \, dx = \frac{1}{\sqrt{n}} \int_{-1}^{1} g\left(\frac{u}{n}\right) \, du \leq \frac{1}{\sqrt{n}} \|g\|_{L^1}.
\]

Taking the limit as \( n \to \infty \) gives the result. The change of variables \( u = nx \) correctly “zooms” in on interval the mass of \( f \) is being concentrated on. This is a simple example of how to rescale near where mass is being concentrated.