Chapter 14: Discrete Dynamical Systems

Throughout most of this book, we have been dealing with instantaneous rates of change. In the chapter on Differential Equations, for example, we saw how a population given by a differentiable function might be determined if we know the instantaneous rate of change of the population with respect to time. In many situations in biology, however, the population only changes at discrete moments in time, such as each spring when new members of the population are born. Such situations are often described by a **discrete dynamical system,** in which the population at a certain stage is determined by the population at a previous stage. Dynamical systems are an important area of pure mathematical research as well, but in this chapter we will focus on what they tell us about population biology.

14.1: SEQUENCES

? If we know the size of a fish population this year, how can we use this information to predict the population for the next four years?

We will answer this question in Example 4 using *recursive sequences.*

A function whose domain is the set of natural numbers, such as

$$a(n) = 2n \qquad \text{for } n = 1, 2, 3, 4, \ldots$$

is a **sequence.** The sequence $a(n) = 2n$ can be written by listing its *terms,* 2, 4, 6, 8, $\cdots$, $2n \cdots$. The letter n is used instead of x as a variable to emphasize the fact that the domain includes only natural numbers. For the same reason, a is used instead of f to name the function. Sequences have many different applications; as we shall see in this chapter, one important example in the life sciences is the prediction of next year's population based on this year's population.

In our definition of sequence we used the example $a(n) = 2n$. The range values of this sequence function,

$$a(1) = 2, \qquad a(2) = 4, \qquad a(3) = 6, \ldots,$$

are called the **elements** or **terms** of the sequence. Instead of writing $a(5)$ for the fifth term of a sequence, it is customary to write a_5; for the sequence above

$$a_5 = 10.$$

In the same way, for the sequence above, $a_1 = 2$, $a_2 = 4$, $a_8 = 16$, $a_{20} = 40$, and $a_{51} = 102$.

The symbol a_n is used for the **general** or **nth term** of a sequence. For example, for the sequence 4, 7, 10, 13, 16, ... the general term might be given by $a_n = 1 + 3n$. This formula for a_n can be used to find any term of the sequence that might be needed. For example, the first three terms of the sequence are

$$a_1 = 1 + 3(1) = 4, \qquad a_2 = 1 + 3(2) = 7, \qquad \text{and} \qquad a_3 = 1 + 3(3) = 10.$$

Also, $a_8 = 25$ and $a_{12} = 37$.

EXAMPLE 1   Sequences
Find the first four terms of the sequence having general term $a_n = -4n + 2$.

Solution Replace n, in turn, with 1, 2, 3, and 4.

$$\text{If } n = 1, \quad a_1 = -4(1) + 2 = -4 + 2 = -2.$$
$$\text{If } n = 2, \quad a_2 = -4(2) + 2 = -8 + 2 = -6.$$
$$\text{If } n = 3, \quad a_3 = -4(3) + 2 = -12 + 2 = -10.$$
$$\text{If } n = 4, \quad a_4 = -4(4) + 2 = -16 + 2 = -14.$$

The first four terms of this sequence are -2, -6, -10, and -14. ■

Often sequences are given in recursive form. Rather than giving a formula for a_n as a function of n, it is given as a function of a_{n-1}, the previous term.

EXAMPLE 2   Recursive Sequences

For the sequence with $a_n = 3a_{n-1} + 2$ and $a_1 = 2$, find the next four terms.

Solution Replace n, in turn, with 2, 3, 4, and 5.

$$a_2 = 3a_1 + 2 = 3(2) + 2 = 8$$
$$a_3 = 3a_2 + 2 = 3(8) + 2 = 26$$
$$a_4 = 3a_1 + 2 = 3(26) + 2 = 80$$
$$a_5 = 3a_1 + 2 = 3(80) + 2 = 242$$

The first five terms of this sequence are 2, 8, 26, 80, and 242. ∎

The recursive sequence in the previous example could also be thought of as repeated function composition of the function $f(x) = 3x + 2$. Notice that $a_2 = f(a_1)$, $a_3 = f(a_2) = f(f(a_1))$, and so forth. To simplify the notation, let us denote $f(f(a_1))$ as $f^2(a_1)$ (not to be confused with $f^{(2)}(a_1)$, which represents the second derivative of f evaluated at a_1).

> $f^n(x)$ represents $f(f(\cdots f(x)\cdots))$, that is, f composed with itself n times. We will say that f is **iterated** n times.

EXAMPLE 3   Iterated Function

For the function $f(x) = x + 1$, find $f^2(x)$, $f^3(x)$, and $f^n(x)$.

Solution

$$\begin{aligned} f^2(x) &= f(f(x)) \\ &= f(x) + 1 \\ &= (x + 1) + 1 \\ &= x + 2 \\ f^3(x) &= f(f^2(x)) \\ &= f^2(x) + 1 \\ &= (x + 2) + 1 \\ &= x + 3 \end{aligned}$$

Continuing in this manner, we see that $f^n(x) = x + n + 1$. This result can be proven more formally with a technique known as *mathematical induction,* which is covered in many college algebra books.* ∎

One important application of sequences in the life sciences is describing the size of populations. For example, suppose a species of fish lays eggs every spring. Putting aside the issue of random variation (a rather large issue to put aside, but necessary if we want to keep things simple for now), it is reasonable to expect the population next year to be a function of the population this year. Letting a_n be the size of the population in year n, we have $a_{n+1} = f(a_n)$. If we label as year 1 the first year in which we keep track of the population, then $a_2 = f(a_1)$, $a_3 = f(a_2) = f^2(a_1)$, and $a_4 = f(a_3) = f^3(a_1)$. In general, $a_{n+1} = f(a_n) = f^n(a_1)$.

Let's consider what the function f might look like. If this year's population is too small to lay many eggs, next year's population will also be small. In fact, $f(0) = 0$; if all the fish die, they can't breed to produce any for next year. Also, if the population is too large, overcrowding may cause some fish to starve, so next year's population will be small. Somewhere in between there should be a population that's just the right size, producing a similar, nicely sized population next year.

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* For example, see *College Algebra and Trigonometry,* 8th ed., by Margaret L. Lial, John Hornsby, and David I. Schneider, Addison Wesley, 2001, pp. 606-613.

One function showing such behavior is the logistic growth model described in Sec. 11.1 on Solutions of Elementary and Separable Differential Equations. The differential equation of that model can be changed to a **difference equation,** in which a_{n+1} is given in terms of a_n:

$$a_{n+1} = f(a_n) = k\left(1 - \frac{a_n}{N}\right)a_n,$$

where k is a constant measuring the growth rate and N is the maximum size of the population. Notice that $f(0) = f(N) = 0$, and that $f(N/2) = kN/4$. Notice also that $f(x)$ is a quadratic function whose graph is a parabola opening downward, as in Figure 1.
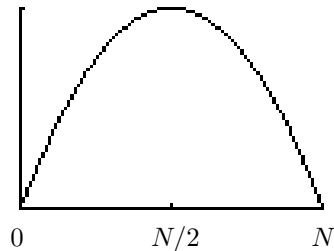

0	N/2	N

FIGURE 1

? EXAMPLE 4   Logistic Growth

Suppose a fish population grows according to the logistic growth model, with $N = 1{,}000$ and $k = 3$. Given that the population begins at $a_1 = 200$, find the population in the next four years.

Solution For this problem, $f(x) = 3(1 - x/1{,}000)x$.

$$a_2 = f(a_1) = 3(1 - 200/1000)200 = 480$$
$$a_3 = f(a_2) = 3(1 - 480/1{,}000)480 = 748.8 \approx 749$$
$$a_4 = f(a_3) = 3(1 - 749/1{,}000)749 = 563.997 \approx 564$$
$$a_5 = f(a_4) = 3(1 - 564/1{,}000)564 = 737.712 \approx 738$$

The population in the next four years is 480, 749, 564, and 738. At each step, we have rounded the population to the nearest whole number, since the number of fish must be a whole number. ∎

14.1 EXERCISES

List the first 4 terms of the sequence satisfying each of the following conditions.

1. $a_n = 5n + 2$ **2.** $a_n = -7n + 12$

3. $a_n = 2(3^n)$ **4.** $a_n = 3(2^n)$

Find the next 4 terms of the sequence satisfying each of the following conditions.

5. $a_n = 2a_{n-1} - 3$, $a_1 = 4$ **6.** $a_n = -3a_{n-1} + 11$, $a_1 = 2$

7. $a_n = -5a_{n-1} + 2$, $a_1 = -1$ **8.** $a_n = 4a_{n-1} - 7$, $a_1 = 6$

For each function, find $f^2(x)$, $f^3(x)$, and $f^n(x)$.

9. $f(x) = x + 2$ **10.** $f(x) = x - 3$

11. $f(x) = 2x$ **12.** $f(x) = -3x$

Applications

LIFE SCIENCES

13. Ricker Model Another model of population growth which has been used to model salmon is the Ricker model, given by*

$$f(x) = xe^{r(1-x/N)}.$$

a. Find $\lim\limits_{x \to \infty} f(x)$.

b. Show that $f'(x) = e^{r(1-x/N)}(1 - rx/N)$.

c. Using the result of part b, for what values of x is f increasing? decreasing? At what value of x does f have an absolute maximum?

d. Show that $f''(x) = e^{r(1-x/N)}(rx/N - 2)(r/N)$.

e. Using the result of part d, for what values of x is f concave upward? concave downward? At what value of x does f have an inflection point?

f. Using the results from parts a-e, draw a graph of this function.

✎ **g.** In your own words, describe in what ways the graph in part f is similar to the graph of the logistic growth model, and in what ways it is different.

14. Ricker Model For the Ricker model of Exercise 13 with $N = 1{,}000$, $r = 3$, and an initial population of 200, find the population for the next four years, where a_n is the population in year n and $a_{n+1} = f(a_n)$. Be sure to round the population each year to the nearest integer.

15. Ricker Model For the Ricker model of Exercise 13 with $N = 2{,}000$, $r = 2.5$, and an initial population of 500, find the population for the next four years, where a_n is the population in year n and $a_{n+1} = f(a_n)$. Be sure to round the population each year to the nearest integer.

16. Beverton-Holt Model Another model of population growth which has been used to model salmon is the Beverton-Holt model, given by†

$$f(x) = \frac{rx}{1 + x/b}.$$

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* Ricker, W.E., "Stock and Recruitment," *Journal of the Fisheries Research Board of Canada,* Vol. 11, 1957, pp. 559-623.

† Cook, R. M., A. Sinclair, and G. Stefánsson, "Potential Collapse of North Sea Cod Stocks," *Nature,* Vol. 385, Feb. 6, 1997, pp. 521-522.

a. Find $\lim\limits_{x\to\infty} f(x)$.

b. Show that $f'(x) = \dfrac{r}{(1+x/b)^2}$.

c. Using the result of part b, for what values of x is f increasing? decreasing? Does f have an absolute maximum?

d. Show that $f''(x) = \dfrac{-2r}{b(1+x/b)^3}$.

e. Using the result of part d, for what values of x is f concave upward? concave downward? Does f have an inflection point?

f. Using the results from parts a-e, draw a graph of this function.

✎ **g.** In your own words, describe in what ways the graph in part f is similar to the graph of the logistic growth model and to the Ricker model of Exercise 13, and in what ways it is different.

17. Beverton-Holt Model For the Beverton-Holt model of Exercise 16 with $b = 1{,}000$, $r = 3$, and an initial population of 500, find the population for the next four years, where a_n is the population in year n and $a_{n+1} = f(a_n)$. Be sure to round the population each year to the nearest integer.

18. Beverton-Holt Model For the Beverton-Holt model of Exercise 16 with $b = 2{,}000$, $r = 4.5$, and an initial population of 1,500, find the population for the next four years, where a_n is the population in year n and $a_{n+1} = f(a_n)$. Be sure to round the population each year to the nearest integer.

19. Shepherd Model The Shepherd model, a modification of the Beverton-Holt model of Exercise 16, is given by*

$$f(x) = \dfrac{rx}{1+(x/b)^2}.$$

For this exercise, assume $c = 2$.

a. Find $\lim\limits_{x\to\infty} f(x)$.

b. Show that $f'(x) = \dfrac{r[1-(x/b)^2]}{[1+(x/b)^2]^2}$.

c. Using the result of part b, for what values of x is f increasing? decreasing? At what value of x does f have an absolute maximum?

d. Show that $f''(x) = \dfrac{-2rx[3-(x/b)^2]}{b^2(1+x/b)^3}$.

e. Using the result of part d, for what values of x is f concave upward? concave downward? At what value of x does f have an inflection point?

f. Using the results from parts a-e, draw a graph of this function.

✎ **g.** In your own words, describe in what ways the graph in part f is similar to the graph of the logistic growth model and to the Ricker model of Exercise 13, and in what ways it is different.

20. Shepherd Model For the Shepherd model of Exercise 19 with $b = 1{,}000$, $r = 3$, and an initial population of 500, find the population for the next four years, where a_n is the population in year n and $a_{n+1} = f(a_n)$. Be sure to round the population each year to the nearest integer.

21. Shepherd Model For the Shepherd model of Exercise 19 with $b = 2{,}000$, $r = 4.5$, and an initial population of 1,500, find the population for the next four years, where a_n is the population in year n and $a_{n+1} = f(a_n)$. Be sure to round the population each year to the nearest integer.

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* Cook, R. M., A. Sinclair, and G. Stefánsson, "Potential Collapse of North Sea Cod Stocks," *Nature,* Vol. 385, Feb. 6, 1997, pp. 521-522.

14.2: EQUILIBRIUM POINTS

? If a simple model is used to describe the change in a population from one year to the next, will the behavior of the population be simple?

As we shall see in Example 2, the answer to this question is "No." Even a simple model can result in very complex behavior.

Let's return to the logistic model of the previous section:

$$f(x) = rx(1 - x/N).$$

Recall that N is the maximum size that the population can reach. To simplify matters in this section, we will set $N = 1$, so that x measures the fraction of the maximum possible population. For example, if $x = 0.75$, this means that the population is 75% of its maximum value. One advantage of this change is that we now have only one parameter, r, to worry about. Also, we no longer need to worry about rounding values of x to the nearest integer. For the rest of this section, then,

$$f(x) = rx(1 - x).$$

One question of interest to biologists is whether there is a population size that never changes, that is, for which $x_n = x_{n+1}$, so the population next year is the same as this year. Such a value of x is called an **equilibrium point.** (The term *fixed point* is also used.) We saw this concept in Section 11.5 on Nonlinear Systems of Differential Equations, where we found the equilibrium point by setting the derivative equal to 0. Here, to find any equilibrium points, set $f(x) = x$, since x is this year's population and $f(x)$ in next year's.

$$f(x) = x$$
$$rx(1 - x) = x$$
$$rx - rx^2 = x$$
$$0 = rx^2 - (r - 1)x$$
$$0 = x[rx - (r - 1)]$$
$$x = 0 \quad \text{or} \quad x = (r - 1)/r$$

The first solution is trivial. Of course, if there are no fish this year, there won't be any fish next year, assuming no fish enter from outside. The second solution is only meaningful if $r > 1$.

So what happens if $r \leq 1$? It's easy to check with a graphing calculator for a specific value of r; we will illustrate using a TI-83. For example, let $r = 0.8$. Suppose $a_1 = 0.6$. We will keep the current population size in the variable X with the command .6→ X. To update the population size, use the command .8X(1-X)→ X. Continue to hit ENTER to repeatedly update the population. You should find that

$$a_2 = 0.192,$$
$$a_3 = 0.1241088,$$
$$a_4 = 0.0869646446,$$
$$a_5 = 0.0635214362,$$

and so forth.

Notice that the population seems to be declining towards 0. We can see graphically why this is so. Figure 2 on the next page shows a TI-83 graph of $y = f(x) = 0.8x(1-x)$ and $y = x$ on the same axes. Suppose we start at $x = a_1 = 0.6$, as shown in Figure 2. Follow a vertical line to see where it intersects the graph $y = f(x)$. At this point, $y = f(a_1) = a_2 = 0.192$.

Now follow a horizontal line until it hits the graph of $y = x$. At this point, $x = a_2 = 0.192$ and we repeat the process. The first several steps of the process are shown in Figure 2, and we can see that x is approaching 0. Such a graph is called a **cobweb diagram.** We will see shortly why it has this name. Cobweb diagrams are useful in economics as well as biology.
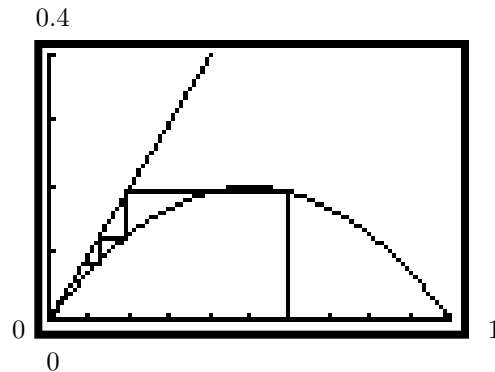

FIGURE 2

It's easy to generate graphs such as Figure 2 on a TI-83. First make sure any StatPlots are off. Then press the MODE button and change the setting from `Func` to `Seq`. Press the FORMAT button and change the setting on the first line to `Web`. Under the Y= menu, set `u(n)=.8u(n-1)(1-u(n-1))` (the "X,T,θ,n" button gives you n), and set `u(nMin)={.6}`. With the window set as in Figure 2 (and making sure PlotStart is set to 1), press GRAPH. After the graph is generated, press TRACE, and then repeatedly press the right arrow key.

Now let's see what happens when $r > 1$. There are now two fixed points, the trivial one ($x = 0$) and $x = (r - 1)/r$. For example, consider the case when $r = 2.9$. The fixed point is $(r - 1)/r = (2.9 - 1)/2.9 \approx 0.655$. If the population ever reaches 0.655, it will stay there forever, because $f(0.655) = 0.655$.

As an experiment to see what might happen for the function $f(x) = rx(1 - x)$ with $r = 2.9$ starting with a value of the population other than 0.655, let $a_1 = 0.5$. Then we can generate successive values of the sequence using our calculator as we did before. Verify that the results are as follows:

$$a_2 = 0.725,$$
$$a_3 = 0.5781875,$$
$$a_4 = 0.707271473,$$
$$a_5 = 0.6004117557.$$

The values are slowly approaching 0.655, alternating higher and lower. If we follow this sequence long enough, we eventually get as close to 0.655 as we like. For example, $a_{30} = 0.6583197205$. Once again, the cobweb diagram in Figure 3 on the next page shows why this is so. As before, we start with a value of x, namely $a_1 = 0.5$. We follow a vertical line until it intersects the graph of $y = f(x)$. We then follow a horizontal line until it intersects the graph of $y = x$. This gives us a new value whose x-coordinate equals the y-coordinate of the previous point on the graph. We then repeat the process. Notice that the resulting graph resembles a spider's web, hence the name cobweb diagram.

For situations in which the population moves toward the equilibrium point, the equilibrium point is said to be **stable.** (Some books use the term *attracting.*)

EXAMPLE 1 Equilibrium Point
Consider the logistic model with $r = 3$, that is,

$$y = f(x) = 3x(1 - x).$$

FIGURE 3

a. Find any equilibrium points.

   Solution As always for the logistic model, we have the trivial equilibrium point $x = 0$. In addition, we have $x = (r - 1)/r = (3 - 1)/3 = 2/3 \approx 0.6667$.

b. Starting with a population $a_1 = 0.5$, find the next four terms in the sequence.

   Solution

$$a_2 = f(a_1) = 3(0.5)(1 - 0.5) = 0.75$$
$$a_3 = f(a_2) = 0.5625$$
$$a_4 = f(a_3) = 0.73828125$$
$$a_5 = f(a_4) = 0.5796661377$$

c. Sketch the cobweb diagram for the sequence found in part b.

   Solution
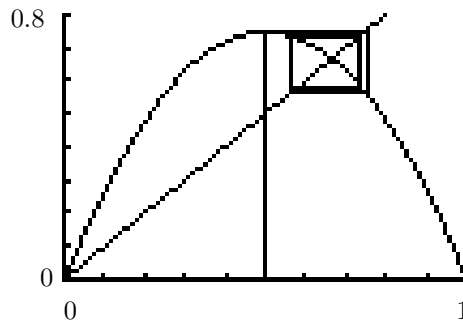
FIGURE 4

   The cobweb diagram is shown in Figure 4.

d. Determine whether the equilibrium point found in part a is stable or not.

   Solution Notice from the results in part b and the cobweb diagram in part c that the population seems to be slowly moving toward the equilibrium point. Generate more points on your own to verify this fact. Thus the equilibrium point appears to be stable. ■

? EXAMPLE 2 Equilibrium Point
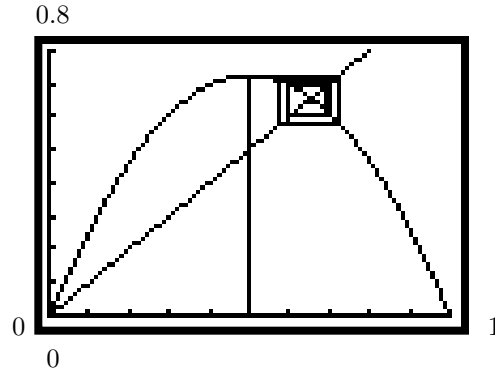Consider the logistic model with $r = 4$, that is,

$$y = f(x) = 4x(1 - x).$$

a. Find any equilibrium points.

Solution As always for the logistic model, we have the trivial equilibrium point $x = 0$. In addition, we have $x = (r - 1)/r = (4 - 1)/4 = 3/4 = 0.75$.

b. Starting with a population $a_1 = 0.6$, find the next five terms in the sequence. Repeat with $a_1 = 0.61$.

Solution

$$a_2 = f(a_1) = f(0.6) = 0.96$$
$$a_3 = f(a_2) = 0.1536$$
$$a_4 = f(a_3) = 0.52002816$$
$$a_5 = f(a_4) = 0.9983954912$$
$$a_6 = f(a_5) = 0.0064077373$$

The equilibrium point in this case is clearly not stable. With $a_1 = 0.61$, which is very close to the previous value of a_1, we might expect similar behavior.

$$a_2 = f(a_1) = f(0.61) = 0.9516$$
$$a_3 = f(a_2) = 0.18422976$$
$$a_4 = f(a_3) = 0.6011566221$$
$$a_5 = f(a_4) = 0.9590693512$$
$$a_6 = f(a_5) = 0.1570213231$$

After only five steps, the population is quite different from what it was before. The values of the population no longer approach the equilibrium point, which means that the equilibrium point is **unstable.** ∎

We encourage you to try Example 2 on your calculator and continue to press ENTER, going beyond the values we have generated so far. Any pattern that you might see emerging from the data breaks down as you continue. The numbers appear to be chaotic. In fact, such behavior is known as **chaos,** in which the numbers appear chaotic and without pattern. Changing the starting value very slightly, from 0.6 to 0.61, causes major changes later in the sequence. This is known as the **butterfly effect,** so called because the weather is just as sensitive to small changes, so that a butterfly taking off from a flower in Brazil could result in a tornado in Kansas months later (or some other dramatic meteorological event). This type of behavior in a dynamical system was first noticed by the French mathematician Henri Poincaré in the late nineteenth century. It was only in the 1960's, however, when American meteorologist E. N. Lorenz observed such behavior in a computer program running simple mathematical models for the weather, that chaos began to receive serious study.

In the next section, we will explore more definite ways for determining if an equilibrium point is stable or unstable.

14.2 EXERCISES

Find equilibrium points x, $0 \leq x \leq 1$, for each of the following functions.

1. $f(x) = 1 - |2x - 1|$

2. $f(x) = \dfrac{2}{3} - \left| \dfrac{4}{3}x - \dfrac{2}{3} \right|$

3. $f(x) = 6x^2(1 - x)$

4. $f(x) = 6x(1 - x)^2$

For each of the following functions, already studied in Exercises 1-4, find the next six values of the sequence, starting with (a) $x_1 = 0.15$, (b) $x_1 = 0.4$, (c) $x_1 = 0.65$, and (d) $x_1 = 0.85$. Then determine whether each of the equilibrium points found in the corresponding exercise above appears to be stable or unstable.

5. $f(x) = 1 - |2x - 1|$

6. $f(x) = \dfrac{2}{3} - \left| \dfrac{4}{3}x - \dfrac{2}{3} \right|$

7. $f(x) = 6x^2(1 - x)$

8. $f(x) = 6x(1 - x)^2$

For each of the following functions, already studied in Exercises 1-8, draw a cobweb diagram, starting with $x_1 = 0.4$ and iterating four times.

9. $f(x) = 1 - |2x - 1|$

10. $f(x) = \dfrac{2}{3} - \left| \dfrac{4}{3}x - \dfrac{2}{3} \right|$

11. $f(x) = 6x^2(1 - x)$

12. $f(x) = 6x(1 - x)^2$

Applications

LIFE SCIENCES

13. Ricker Model In Exercise 13 of the previous section, we considered the Ricker model, given by

$$f(x) = xe^{r(1-x)},$$

where we have changed N to 1, as in the text.

a. Find any equilibrium points.

For each of the following initial populations and values of r, find the next five values of the sequence, and determine whether the equilibrium value is stable or unstable.

b. $a_1 = 0.6$, $r = 2$

c. $a_1 = 0.6$, $r = 3$

d. Draw a cobweb diagram corresponding to the values in part b.

e. Draw a cobweb diagram corresponding to the values in part c.

14. Beverton-Holt Model In Exercise 16 of the previous section, we considered the Beverton-Holt model, given by

$$f(x) = \frac{rx}{1 + x}.$$

where we have changed b to 1, as in the text.

a. Find any equilibrium points.

For each of the following initial populations and values of r, find the next five values of the sequence, and determine whether the nontrivial equilibrium value appears to be stable or unstable.

b. $a_1 = 1.4$, $r = 2$

c. $a_1 = 2.2$, $r = 4$

d. Draw a cobweb diagram corresponding to the values in part b.

e. Draw a cobweb diagram corresponding to the values in part c.

15. Shepherd Model In Exercise 19 of the previous section, we considered the Shepherd model, given by

$$f(x) = \frac{rx}{1+x^2}.$$

where we have changed b to 1, as in the text, and let $c = 2$ as before.

a. Find any equilibrium points.

For each of the following initial populations and values of r, find the next five values of the sequence, and determine whether the nontrivial equilibrium value appears to be stable or unstable.

b. $a_1 = 4$, $r = 2$

c. $a_1 = 4$, $r = 10$

d. Draw a cobweb diagram corresponding to the values in part b.

e. Draw a cobweb diagram corresponding to the values in part c.

14.3: DETERMINING STABILITY

? Suppose a population grows according to the logistic model. How can we determine whether any equilibrium points are stable?

A result in this section will give us an answer to this question.

In the previous section, we saw that sometimes equilibrium points are stable, so that the population approaches the equilibrium point over time, and sometimes they are unstable, so that the population moves away from the equilibrium point and can even jump around chaotically. In this section, we wish to find methods for determining the nature of any equilibrium points.

As we saw back in Chapter 3, any smooth curve resembles a straight line when viewed closely enough. So let's look closely enough at an equilibrium point that the graph of $y = f(x)$ appears to be a straight line. In Figure 5, we have shown the neighborhood of the equilibrium point, with $y = f(x)$ having a negative slope that is between -1 and 0. The cobweb diagram starts by going up the longer vertical line on the left, and then spiraling in a clockwise direction toward the equilibrium point. The equilibrium point is stable because the cobweb moves toward it.
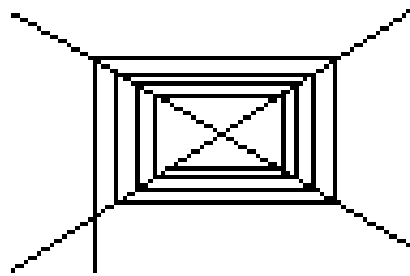

FIGURE 5

In Figure 6, we have a similar graph, but the slope of $f'(x)$ is less than -1. This time the cobweb starts by going up the longer line near the center of the diagram, and then spirals in a clockwise direction away from the equilibrium point, which must be unstable.
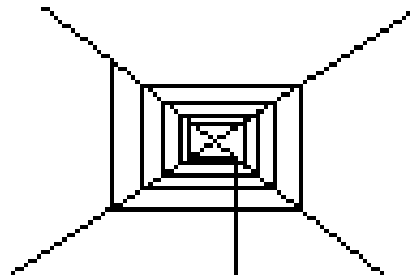
FIGURE 6

In Figure 7 on the next page, we have a function $y = f(x)$ with a positive slope that is less than 1. Notice how the population moves toward the equilibrium point. In Figure 8, the function $y = f(x)$ has a positive slope that is greater than 1, and the population moves away from the equilibrium point.

These results can be summarized by the following theorem.

STABILITY OF EQUILIBRIUM POINTS

Suppose a function $y = f(x)$ has an equilibrium point at $x = a$. Then the equilibrium point is stable if $|f'(a)| < 1$ and unstable if $|f'(a)| > 1$.
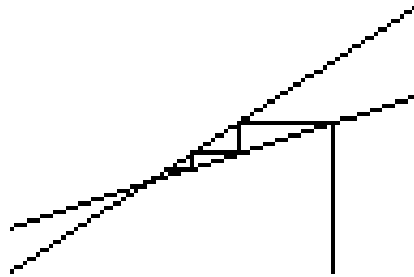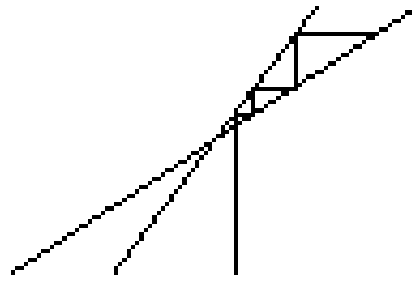

FIGURE 7 FIGURE 8

The situation is more complicated if $|f'(a)| = 1$; whether or not the equilibrium point is stable depends upon the nature of the function.

? EXAMPLE 1 Equilibrium Point

Consider the logistic model

$$y = f(x) = rx(1 - x).$$

Determine the stability of the equilibrium point $x = \dfrac{r-1}{r}$.

Solution First calculate $f'(x)$:

$$f'(x) = rx(-1) + r(1 - x) = r(1 - 2x).$$

Next, evaluate the derivative at the equilibrium point.

$$f'\left(\frac{r-1}{r}\right) = r\left(1 - 2\left(\frac{r-1}{r}\right)\right) = 2 - r.$$

Now find what values of r result in $\left|f'\left(\dfrac{r-1}{r}\right)\right| < 1$.

$$|2 - r| < 1$$
$$-1 < 2 - r < 1$$
$$-1 < r - 2 < 1 \qquad \text{Multiply by } -1 \text{ and reverse the order.}$$
$$1 < r < 3$$

This is why, with $r = 2.9$ in the example before Example 1 of the previous section, the equilibrium point was stable, but with $r = 4$ in Example 2, the equilibrium point was unstable. Notice that with $r = 3$ in Example 1, for which $|f'(x)| = 1$ at the equilibrium point, the equilibrium point was stable. This will not always be the case when $|f'(x)| = 1$ at the equilibrium point, as we will see in Exercise 7.

14.3 EXERCISES

For each of the following functions studied in exercises in the previous section, use the method of this section to determine the stability of any equilibrium points.

1. $f(x) = 1 - |2x - 1|$

2. $f(x) = \dfrac{2}{3} - \left| \dfrac{4}{3}x - \dfrac{2}{3} \right|$

3. $f(x) = 6x^2(1 - x)$

4. $f(x) = 6x(1 - x)^2$

For each of the following functions, use a graphing calculator to estimate any equilibrium points. Then, using the graphing calculator to estimate the derivative of the function at the equilibrium point, determine whether the equilibrium point is stable or unstable.

5. $f(x) = \cos x$

6. $f(x) = \sin 3x$

7. Consider the function

$$f(x) = 4x^2(1 - x).$$

a. Find any equilibrium points where $f(x) = x$.

b. Determine the derivative at each of the equilibrium points found in part a.

c. What does the theorem on the Stability of Equilibrium Points tell us about each of the equilibrium points found in part a?

d. Starting with $x_1 = 0.4$, find the next four iterations of the function.

e. Starting with $x_1 = 0.7$, find the next four iterations of the function.

f. Describe the behavior of successive iterations of the function found in parts d and e, and how this relates to the stability results found in part c.

When the function has no stable equilibrium points, many outcomes are possible, such as the chaotic behavior described in the previous section. Another outcome is a stable cycle of period 2, in which the points tend to vary between two values. Using the logistic function with each of the following values of r, and starting with $x_1 = 0.5$, iterate repeatedly until you can determine the x-values in the cycle to three decimal places.

8. $r = 3.1$

9. $r = 3.2$

Applications

LIFE SCIENCES

10. Ricker Model In Exercise 13 of the previous section, we considered the Ricker model, given by

$$f(x) = xe^{r(1-x)},$$

where r is a positive constant. Use the theorem on the Stability of Equilibrium Points to determine the stability of any equilibrium points and how they depend on r.

11. Beverton-Holt Model In Exercise 14 of the previous section, we considered the Beverton-Holt model, given by

$$f(x) = \frac{rx}{1 + x}.$$

where r is a positive constant. Use the theorem on the Stability of Equilibrium Points to determine the stability of any equilibrium points and how they depend on r.

12. Shepherd Model In Exercise 15 of the previous section, we considered the Shepherd model, given by

$$f(x) = \frac{rx}{1 + x^2}.$$

where r is a positive constant, and let $c = 2$ as before. Use the theorem on the Stability of Equilibrium Points to determine the stability of any equilibrium points and how they depend on r.

Answers to Odd Numbered Exercises

Sec. 14.1

1. 7, 12, 17, 22   **3.** 6, 18, 54, 162   **5.** 5, 7, 11, 19   **7.** 7, -33, 167, -833
9. $x + 4$, $x + 6$, $x + 2n$ **11.** $4x$, $8x$, $2^n x$ **13. a.** 0 **c.** Increasing for $0 \leq x < N/r$, decreasing for $x > N/r$, maximum at $x = N/r$. **e.** Concave downward for $0 \leq x < 2N/r$, concave upward for $x > 2N/r$, inflection point at $x = 2N/r$.
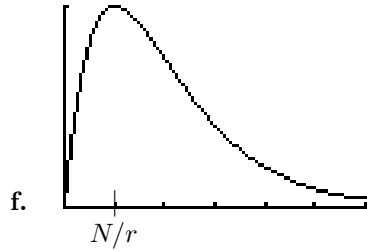
f. **15.** 3260, 675, 3537, 518   **17.** 1000, 1500, 1800, 1929

19. a. 0 **c.** Increasing for $0 \leq x < b$, decreasing for $x > b$, maximum at $x = b$. **e.** Concave downward for $0 \leq x < \sqrt{3}b$, concave upward for $x > \sqrt{3}b$, inflection point at $x = \sqrt{3}b$.
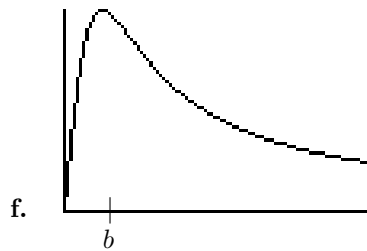
f. **21.** 4320, 3431, 3916, 3646

Sec. 14.2

1. 0, 2/3   **3.** 0, $(3 \pm \sqrt{3})/6$ **5.** 0.3, 0.6, 0.8, 0.4, 0.8, 0.4; 0.8, 0.4, 0.8, 0.4, 0.8, 0.4; 0.7, 0.6, 0.8, 0.4, 0.8, 0.4; 0.3, 0.6, 0.8, 0.4, 0.8, 0.4; 0 and 2/3 are unstable.   **7.** 0.11475, 0.0699395082, 0.0272965396, 0.0043485744, $1.129672023 \times 10^{-4}$, $7.656088294 \times 10^{-8}$; 0.576, 0.844038144, 0.6666437209, 0.8888888857, 0.5267489824, 0.7878621148; 0.88725, 0.5325492985, 0.7954386698, 0.7765835552, 0.8084306426, 0.7512125347; 0.65025, 0.8872983937, 0.5323787927, 0.7952193576, 0.7769875109, 0.807808673; 0 is stable; $(3 + \sqrt{3})/6$ and $(3 - \sqrt{3})/6$ are unstable.
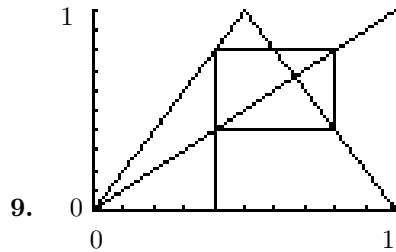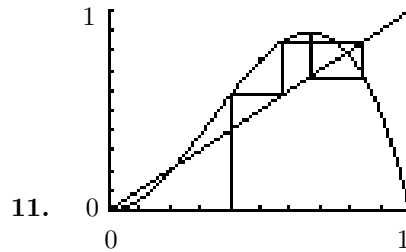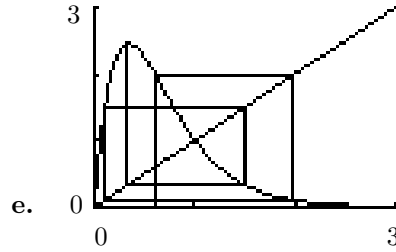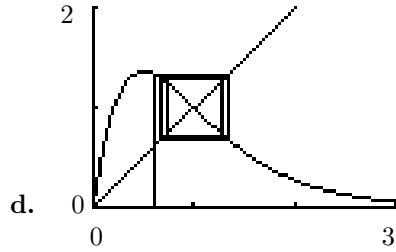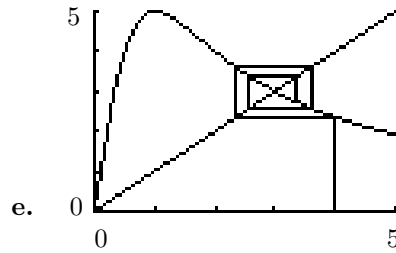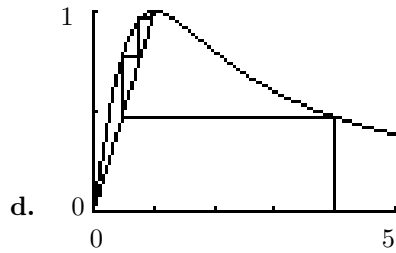
9.   **11.**

13. a. 0, 1   **b.** 1.335325, 0.682854, 1.287649, 0.724351, 1.257115; stable   **c.** 1.992070, 0.101567, 1.504202, 0.331428, 2.462978; unstable

d.

e.

15. a. 0, $\sqrt{r-1}$ **b.**   0.470588, 0.770538, 0.966963, 0.999436, 0.9999998; stable
c.   2.352941, 3.599788, 2.578927, 3.370766, 2.726702; stable

d.

e.

Sec. 14.3

1.   0 and 2/3 are unstable.   **3.**   0 is stable; $(3+\sqrt{3})/6$ and $(3-\sqrt{3})/6$ are unstable.
5.   0.73908513 is stable   **7. a.**   0, 1/2   **b.**   0, 1   **c.**   0 is stable; we do not know
about 1/2.   **d.**   0.384, 0.363331584, 0.3361859827, 0.300099736   **e.**   0.588, 0.569786112,
0.5586864474, 0.5509897614   **f.**   Values of x less than 0.5 move toward 0, while values of
x greater than 0.5 move toward 0.5.   **9.**   0.513, 0.799   **11.** $x = 0$ is stable if $0 < r < 1$
and unstable if $r > 1$; $x = r - 1$ only exists if $r > 1$, and it is stable.