APMA: 0200
Homework #4

Due Date: October 9, 2015

1. (Nondimensionalizing the logistic equation) Consider the logistic equation
$$\frac{dN}{dt} = rN\left(1 - \frac{N}{\kappa}\right)$$
with initial condition $N(0) = N_0$.
   (a) This system has three dimensional parameters, r, κ, and N_0. Find the dimensions of each of these parameters.
   (b) By letting $x = N/\kappa$ and $\tau = rt$ show that this system can be written in dimensionless form:
$$\frac{dx}{d\tau} = x(1-x) \text{ and } x(0) = x_0.$$
   (c) Find a different nondimensionalization in terms of u and τ, where u is chosen such that the initial condition always satisfies $u(0) = 1$.

2. (Model of an Epidemic) In a pioneering work in epidemiology, Kermack and McKendrick (1927) purposed the following model for the evolution of an epidemic. Suppose the population can be divided into three classes:
   i $x(t) =$ the number of healthy people.
   ii $y(t) =$ the number of infected people.
   iii $z(t) =$ the number of infected people who have died.

   The model for the evolution of the disease is the following:
$$\frac{dx}{dt} = -kxy,$$
$$\frac{dy}{dt} = kxy - ly,$$
$$\frac{dz}{dt} = ly,$$
where $k > 0$ and $l > 0$ are parameters and $x, y, z \geq 0$.
   (a) Explain why this is a sensible model for the spread of a disease. Think carefully about what each formula for the derivative means.
   (b) Show that $x + y + z = N$ is a constant. **Hint:** Calculating $\frac{dN}{dt}$ might be useful.
   (c) Use the equation for $\frac{dx}{dt}$ and $\frac{dz}{dt}$ to show that
$$x(t) = x_0 e^{-kz(t)/l},$$
where $x(0) = x_0$.

(d) Show that z satisfies the following differential equation:

$$\frac{dz}{dt} = l\left[N - z - x_0 e^{-kz/l}\right].$$

(e) Show that this equation can be nondimensionalized to

$$\frac{du}{d\tau} = a - bu - e^{-u}.$$

(f) Show that $a \geq 1$ and $b > 0$.

(g) Determine the number of fixed points u^* and classify their stability. What does this analysis tell you about the long term behavior of the epidemic.

(h) Show that the maximum of $\frac{du}{dt}$ occurs at the same time as the maximum of both $\frac{dz}{dt}$ and $y(t)$. At this time their are more sick people and a higher daily death rate than at any other time.

(i) Show that if $b < 1$ then $\frac{du}{dt}$ is increasing at $t = 0$ and reaches a maximum at some time $t_{\text{peak}} > 0$. Thus, things get worse before they get better. The term **epidemic** is reserved for this case.