Calculus of Variations, Fall 2013
APMA 2811Q

John Gemmer, john_gemmer@brown.edu,
www.math.arizona.edu/~jgemmer
Office: 182 George Street, Room 328, **Phone:** (401) 863-2114

Office Hours: Tuesday 9:00-11:00, Thursday 9:00-11:00
Class Meeting Times: MWF: 12:00-1:00
Class Location: Math Building Room 205

Recommended Prerequisites: *Real Analysis* APMA 2110 (MATH 2210) and *Partial Differential Equations* APMA 2230 (MATH 2370). Please see the instructor if you do not need meet these prerequisites.

Course Description: An introduction to modern techniques in the calculus of variations. Topics covered will include: existence of solutions and the direct method, Euler-Lagrange equations and necessary and sufficient conditions, one-dimensional problems, multidimensional nonconvex problems, relaxation and quasiconvexity, Gamma-convergence, Young's measures, and singular perturbations. The emphasis of the course will be equal parts theory and applications with numerous examples drawn from topics in nonlinear elasticity, pattern formation, wrinkling thin elastic sheets, martensitic phase transitions, minimal surfaces, differential geometry and optimal control. Mathematical concepts such as Sobolev spaces and weak convergence will be covered as needed.

Course Rationale: The calculus of variations is an indispensable tool in modern science and a rich area of research in applied and pure mathematics. Many problems in the field arise from questions in physics and engineering with concrete answers coming from analysis. The purpose of this course is for students to learn how to apply modern variational methods to research problems in either applied mathematics, pure mathematics or the physical sciences. Moreover, since modern research in the calculus of variations is intrinsically interdisciplinary a secondary purpose of this course will be to introduce techniques to students outside of their field. For example, for mathematics students this will include some modeling while for engineering and physics students this will include learning how to use techniques from applied analysis.

Course Goals: Upon completion of this course students will be able to do the following:

1. Apply modern techniques in the calculus of variations and functional analysis to specific problems within their research.
2. Analyze, interpret and synthesize current scientific literature within the context of their respective discipline.
3. Analyze, interpret and synthesize current scientific literature outside of their field.

Class Delivery: The course material will be delivered through lectures. Evaluation of the students understanding of the material will be assessed through written homework assignments and a semester term paper project.

Textbooks: The course will be self-contained and all lectures will be supported by handwritten lecture notes posted on my website. However, the course notes themselves will mainly be drawn from the following books:

1. Evans, L. C. *Partial Differential Equations.* Vol. 19. American Mathematical Society, 1998. (We will cover topics from chapter 8 in detail).
2. Jost, Jürgen, and Xianqing Li-Jost. *Calculus of variations*. Vol. 64. Cambridge University Press, 1998.
3. Braides, Andrea. *Gamma-convergence for Beginners*. Vol. 22. Oxford University Press, 2002.
4. Müller, Stefan. "Variational models for microstructure and phase transitions." *Calculus of variations and geometric evolution problems*. Springer Berlin Heidelberg, 1999. 85-210.

I will indicate in my notes and lectures which books are relevant to the topics covered. Additional useful references include:

1. Dacorogna, Bernard, et al. *Introduction to the Calculus of Variations*. London: Imperial College Press, 2004. (The first chapter of this book provides an introduction – along with worked out exercises – to many of the mathematical concepts we will use through the course).
2. Dacorogna, Bernard. *Direct methods in the calculus of variations*. Vol. 78. Springer, 2008. (This book is a comprehensive treatment of lower semicontinuity and relaxation in higher dimensional settings).
3. Evans, Lawrence. *Weak convergence methods for nonlinear partial differential equations*. 1990. (This book provides an introduction to some modern techniques in the analysis of nonlinear PDEs.)
4. Buttazzo, Giuseppe, Mariano Giaquinta, and Stefan Hildebrandt. *One-dimensional variational problems: an introduction*. Vol. 15. Oxford University Press, 1998. (Some of the examples I will cover in the course will be taken from this book).
5. Young, Laurence C. *Lectures on the calculus of variations and optimal control theory*. Vol. 304. Amer Mathematical Society, 1980. (This is an excellent book that gives a very lively account of the calculus of variations from a personal perspective.)

Course Policies:

- **Grading:**
  This course will be graded on a S/NC grade option. Please register for this option. A satisfactory grade will be obtained based on evaluation of the following:
  - Homework: 20%
  - Outline of term paper: 20%
  - Rough draft of term paper: 20%
  - Final draft of term paper: 20%
  - Final presentation: 20%

- **Homework:**
  Homework will be assigned every two to three weeks and will correspond to each main topic in the course. The homework assignments will be designed to facilitate the student's understanding of the material covered in the lectures. Specifically, typical assignments will fill in the ``gaps" in the lecture. Collaboration with other students is strongly encouraged but each student must write up their own work individually.

- **Term Paper:**
  The student's progress towards completion of the course goals will be predominately evaluated through a term paper project. The term paper itself is an expository description of a current research problem that uses variational techniques. Sample term paper topics that were specifically chosen to coincide with material covered in this course are included at the end of this document. However, if a student has their own idea for a topic that will use the techniques from this course, they are encouraged to meet with me to discuss designing their own project. At the end of the semester a twenty minute presentation will be given by the student on their term paper topic.

- **Feedback:** Feedback on the student's progress towards satisfactory completion of the project will be evaluated as follows:
  - Outline (Due 10/11/13): A one to three page outline of the topic the student has selected.
  - Rough Draft (Due 11/26/13)
  - Final Draft (Due 12/12/13)
  - Final Presentation: The final presentations will occur during the final exam period.

  All material submitted for evaluation will be returned within a week.

- **Attendance/Participation:** Attendance and participation is not mandatory.

- **Late Work:** Late work is not accepted without a valid excuse.

Tentative Course Schedule:

1. Example problems and mathematical preliminaries (1-2 weeks).
2. The direct method and necessary and sufficient conditions (2 weeks).
3. One dimensional variational problems (2 weeks).
4. Multidimensional problems (2 weeks).
5. Non-convex problems and relaxation (2 weeks).
6. Gamma-convergence (2 weeks).
7. Singular perturbation and length scale of microstructure (1-2 weeks).
8. Young's measures (1 week).

Potential Term Paper Topics:

1. **Minimizers that do not satisfy the Euler-Lagrange equations:**
   - Ball, John M., and Victor J. Mizel. "One-dimensional variational problems whose minimizers do not satisfy the Euler-Lagrange equation." *Analysis and Thermomechanics*. Springer Berlin Heidelberg, 1987. 285-348.

2. **Ridge singularities in crumpled paper:**
   - Venkataramani, Shankar C. "Lower bounds for the energy in a crumpled elastic sheet—a minimal ridge." *Nonlinearity* 17.1 (2004): 301.
   - Lobkovsky, Alexander E. "Boundary layer analysis of the ridge singularity in a thin plate." *Physical Review E* 53.4 (1996): 3750.

3. **Blistering of Thin Elastic Sheets:**
   - Jin, Weimin, and Peter Sternberg. "Energy estimates for the von Kármán model of thin-film blistering." *Journal of Mathematical Physics* 42 (2001): 192.
   - Jin, Weimin, and Peter Sternberg. "In-plane displacements in thin-film blistering." *Proceedings of the Royal Society of Edinburgh: Section A Mathematics* 132.04 (2002): 911-930.

4. **Differential growth and incompatible elasticity:**
   - Lewicka, Marta. "Reduced theories in nonlinear elasticity." *Nonlinear Conservation Laws and Applications*. Springer US, 2011. 393-403.
   - Liang, Haiyi, and L. Mahadevan. "The shape of a long leaf." *Proceedings of the National Academy of Sciences* 106.52 (2009): 22049-22054.

5. **Differential growth and the hyperbolic plane:**
   - Klein, Yael, Shankar Venkataramani, and Eran Sharon. "Experimental study of shape transitions and

energy scaling in thin non-euclidean plates." *Physical Review Letters* 106.11 (2011): 118303.
- Gemmer, John A., and Shankar C. Venkataramani. "Shape selection in non-Euclidean plates." *Physica D: Nonlinear Phenomena* 240.19 (2011): 1536-1552.
- Gemmer, J. A., and S. C. Venkataramani. "Defects and boundary layers in non-Euclidean plates." *Nonlinearity* 25.12 (2012): 3553.

6. **Compressed thin film bonded to a compliant substrate and herringbone patterns:**
   - Audoly, Basile, and Arezki Boudaoud. "Buckling of a stiff film bound to a compliant substrate—Part I:: Formulation, linear stability of cylindrical patterns, secondary bifurcations." *Journal of the Mechanics and Physics of Solids* 56.7 (2008): 2401-2421.
   - Audoly, Basile, and Arezki Boudaoud. "Buckling of a stiff film bound to a compliant substrate—Part II:: A global scenario for the formation of herringbone pattern." *Journal of the Mechanics and Physics of Solids* 56.7 (2008): 2422-2443.
   - Audoly, Basile, and Arezki Boudaoud. "Buckling of a stiff film bound to a compliant substrate—Part III:: Herringbone solutions at large buckling parameter." *Journal of the Mechanics and Physics of Solids* 56.7 (2008): 2444-2458.
   - Kohn, Robert V., and Hoai-Minh Nguyen. "Analysis of a compressed thin film bonded to a compliant substrate: the energy scaling law." *Journal of Nonlinear Science* (2013): 1-20.

7. **Domain branching in ferromagnets:**
   - Kohn, Robert V. "Energy-driven pattern formation." *Proceedings oh the International Congress of Mathematicians: Madrid, August 22-30, 2006: invited lectures.* 2006.
   - Choksi, Rustum, Robert V. Kohn, and Felix Otto. "Domain branching in uniaxial ferromagnets: a scaling law for the minimum energy." *Communications in mathematical physics* 201.1 (1999): 61-79.

8. **Geometry of twinning near an austenite-twinned-martensite interface.**
   - Kohn, Robert V. "Energy-driven pattern formation." *Proceedings oh the International Congress of Mathematicians: Madrid, August 22-30, 2006: invited lectures.* 2006.
   - Conti, Sergio. "Branched microstructures: Scaling and asymptotic self-similarity." *Communications on Pure and Applied Mathematics* 53.11 (2000): 1448-1474.

9. **Singularities in convective pattern formation:**
   - Ercolani, N. M., et al. "The geometry of the phase diffusion equation." *Journal of Nonlinear Science* 10.2 (2000): 223-274.
   - Ercolani, N. M., and S. C. Venkataramani. "A variational theory for point defects in patterns." *Journal of nonlinear science* 19.3 (2009): 267-300.

10. **Phase selection through singular perturbation:**
    - Müller, Stefan. "Minimizing sequences for nonconvex functionals, phase transitions and singular perturbations." *Problems involving change of type.* Springer Berlin Heidelberg, 1990. 31-44.
    - Müller, Stefan. "Singular perturbations as a selection criterion for periodic minimizing sequences." *Calculus of Variations and Partial Differential Equations* 1.2 (1993): 169-204.

11. **Notions of convexity in higher dimensions:**
    - Alibert, Jean-Jacques, and Bernard Dacorogna. "An example of a quasiconvex function that is not polyconvex in two dimensions." *Archive for rational mechanics and analysis* 117.2 (1992): 155-166.
    - Sverak, Vladimir. "Rank-one convexity does not imply quasiconvexity." *Proc. Royal Soc. Edinburgh A*. Vol. 120. 1992.

12. **Newton's problem:**
    - Buttazzo, Giuseppe, and Bernhard Kawohl. "On Newton’s problem of minimal resistance." *The*

Mathematical Intelligencer 15.4 (1993): 7-12.
- Brock, Friedemann, V. Ferone, and Bernhard Kawohl. "A symmetry problem in the calculus of variations." *Calculus of Variations and Partial Differential Equations* 4.6 (1996): 593-599.
- Lachand-Robert, T., and M. A. Peletier. "An example of non-convex minimization and an application to Newton's problem of the body of least resistance." *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*. Vol. 18. No. 2. Elsevier Masson, 2001.

13. **Optimal design and relaxation of variational problems:**
    - Kohn, Robert V., and Gilbert Strang. "Optimal design and relaxation of variational problems, I." *Communications on Pure and Applied Mathematics* 39.1 (1986): 113-137.

14. **Nematic elastomers:**
    - DeSimone, Antonio. "Coarse-grained models of materials with non-convex free-energy: two case studies." *Computer methods in applied mechanics and engineering* 193.48 (2004): 5129-5141.
    - DeSimone, Antonio. "Electro-mechanical response of nematic elastomers: an introduction." *Mechanics and Electrodynamics of Magneto-and Electro-elastic Materials.* Springer Vienna, 2011. 231-266.
    - DeSimone, Antonio, and Georg Dolzmann. "Macroscopic Response of Nematic Elastomers via Relaxation of a Class of SO (3)-Invariant Energies." *Archive for rational mechanics and analysis* 161.3 (2002): 181-204.

15. **Ginzburg-Landau energy and vortices:**
    - Baldo, Sisto, et al. "Asymptotics for Ginzburg-Landau energies in 3-D condensed matter physics." *Communications in Applied and Industrial Mathematics* 2.1 (2011).

16. **Computational microstructure:**
    - Carstensen, Carsten. "Course B: Modelling and simulation of microstructure evolution."
    - Carstensen, Carsten, and Tomáš Roubíček. "Numerical approximation of young measures in non-convex variational problems." *Numerische Mathematik* 84.3 (2000): 395-415.

17. **Reduction of dimension in elasticity:**
    - Acerbi, Emilio, Giuseppe Buttazzo, and Danilo Percivale. "A variational definition of the strain energy for an elastic string." *Journal of Elasticity* 25.2 (1991): 137-148.

18. **Passage from discrete to continuum mechanics:**
    - Braides, Andrea, Maria Stella Gelli, and Mario Sigalotti. "The passage from non-convex discrete systems to variational problems in Sobolev spaces: the one-dimensional case." (2002).

19. **Calculus of variations from an optimal control perspective:**
    - McShane, E. J. "The calculus of variations from the beginning through optimal control theory." *SIAM journal on control and optimization* 27.5 (1989): 916-939

20. **Cavitation in nonlinear elasticity:**
    - Sivaloganathan, J. "A field theory approach to stability of radial equilibria in nonlinear elasticity." *Math. Proc. Camb. Phil. Soc*. Vol. 99. 1986.

21. **Modeling curly hair:**
    - Lazarus, A., J. T. Miller, and P. M. Reis. "Continuation of equilibria and stability of slender elastic rods using an asymptotic numerical method." *Journal of the Mechanics and Physics of Solids* (2013).
    - Lazarus, Arnaud, et al. "Contorting a heavy and naturally curved elastic rod." *Soft Matter* (2013).