Calculus of Variations, Fall 2013
APMA 2811Q
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Office: 182 George Street, Room 328, Phone: (401) 863-2114
Office Hours: Tuesday 9:00-11:00, Thursday 9:00-11:00
Class Meeting Times: MWF: 12:00-1:00
Class Location: Math Building Room 205

Recommended Prerequisites: Real Analysis APMA 2110 (MATH 2210) and Partial Differential Equations APMA 2230 (MATH 2370). Please see the instructor if you do not need meet these prerequisites.

Course Description: An introduction to modern techniques in the calculus of variations. Topics covered will include: existence of solutions and the direct method, Euler-Lagrange equations and necessary and sufficient conditions, one-dimensional problems, multidimensional nonconvex problems, relaxation and quasiconvexity, Gamma-convergence, Young's measures, and singular perturbations. The emphasis of the course will be equal parts theory and applications with numerous examples drawn from topics in nonlinear elasticity, pattern formation, wrinkling thin elastic sheets, martensitic phase transitions, minimal surfaces, differential geometry and optimal control. Mathematical concepts such as Sobolev spaces and weak convergence will be covered as needed.

Course Rationale: The calculus of variations is an indispensable tool in modern science and a rich area of research in applied and pure mathematics. Many problems in the field arise from questions in physics and engineering with concrete answers coming from analysis. The purpose of this course is for students to learn how to apply modern variational methods to research problems in either applied mathematics, pure mathematics or the physical sciences. Moreover, since modern research in the calculus of variations is intrinsically interdisciplinary a secondary purpose of this course will be to introduce techniques to students outside of their field. For example, for mathematics students this will include some modeling while for engineering and physics students this will include learning how to use techniques from applied analysis.

Course Goals: Upon completion of this course students will be able to do the following:
1. Apply modern techniques in the calculus of variations and functional analysis to specific problems within their research.
2. Analyze, interpret and synthesize current scientific literature within the context of their respective discipline.
3. Analyze, interpret and synthesize current scientific literature outside of their field.

Class Delivery: The course material will be delivered through lectures. Evaluation of the students understanding of the material will be assessed through written homework assignments and a semester term paper project.

Textbooks: The course will be self-contained and all lectures will be supported by handwritten lecture notes posted on my website. However, the course notes themselves will mainly be drawn from the following books:

I will indicate in my notes and lectures which books are relevant to the topics covered. Additional useful references include:

5. Young, Laurence C. *Lectures on the calculus of variations and optimal control theory*. Vol. 304. Amer Mathematical Society, 1980. (This is an excellent book that gives a very lively account of the calculus of variations from a personal perspective.)

**Course Policies:**

- **Grading:**
  This course will be graded on a S/NC grade option. Please register for this option. A satisfactory grade will be obtained based on evaluation of the following:
  - Homework: 20%
  - Outline of term paper: 20%
  - Rough draft of term paper: 20%
  - Final draft of term paper: 20%
  - Final presentation: 20%

- **Homework:**
  Homework will be assigned every two to three weeks and will correspond to each main topic in the course. The homework assignments will be designed to facilitate the student's understanding of the material covered in the lectures. Specifically, typical assignments will fill in the "gaps" in the lecture. Collaboration with other students is strongly encouraged but each student must write up their own work individually.

- **Term Paper:**
  The student's progress towards completion of the course goals will be predominately evaluated through a term paper project. The term paper itself is an expository description of a current research problem that uses variational techniques. Sample term paper topics that were specifically chosen to coincide with material covered in this course are included at the end of this document. However, if a student has their own idea for a topic that will use the techniques from this course, they are encouraged to meet with me to discuss designing their own project. At the end of the semester a twenty minute presentation will be given by the student on their term paper topic.
• **Feedback**: Feedback on the student's progress towards satisfactory completion of the project will be evaluated as follows:
  ◦ Outline (Due 10/11/13): A one to three page outline of the topic the student has selected.
  ◦ Rough Draft (Due 11/26/13)
  ◦ Final Draft (Due 12/12/13)
  ◦ Final Presentation: The final presentations will occur during the final exam period.
All material submitted for evaluation will be returned within a week.

• **Attendance/Participation**: Attendance and participation is not mandatory.

• **Late Work**: Late work is not accepted without a valid excuse.

**Tentative Course Schedule:**
1. Example problems and mathematical preliminaries (1-2 weeks).
2. The direct method and necessary and sufficient conditions (2 weeks).
3. One dimensional variational problems (2 weeks).
4. Multidimensional problems (2 weeks).
5. Non-convex problems and relaxation (2 weeks).
6. Gamma-convergence (2 weeks).
7. Singular perturbation and length scale of microstructure (1-2 weeks).
8. Young's measures (1 week).

**Potential Term Paper Topics:**

1. **Minimizers that do not satisfy the Euler-Lagrange equations:**

2. **Ridge singularities in crumpled paper:**

3. **Blistering of Thin Elastic Sheets:**

4. **Differential growth and incompatible elasticity:**

5. **Differential growth and the hyperbolic plane:**
   • Klein, Yael, Shankar Venkataramani, and Eran Sharon. "Experimental study of shape transitions and


6. Compressed thin film bonded to a compliant substrate and herringbone patterns:

7. Domain branching in ferromagnets:


9. Singularities in convective pattern formation:

10. Phase selection through singular perturbation:

11. Notions of convexity in higher dimensions:

12. Newton's problem:
    • Buttazzo, Giuseppe, and Bernhard Kawohl. "On Newton’s problem of minimal resistance." The


13. **Optimal design and relaxation of variational problems:**


14. **Nematic elastomers:**


15. **Ginzburg-Landau energy and vortices:**


16. **Computational microstructure:**

- Carstensen, Carsten. "Course B: Modelling and simulation of microstructure evolution."

17. **Reduction of dimension in elasticity:**


18. **Passage from discrete to continuum mechanics:**


19. **Calculus of variations from an optimal control perspective:**

- McShane, E. J. "The calculus of variations from the beginning through optimal control theory." *SIAM journal on control and optimization* 27.5 (1989): 916-939

20. **Cavitation in nonlinear elasticity:**


21. **Modeling curly hair:**