APMA 1930M: Homework Assignment # 4

Due: 10/24/14

Problem 1 O.D.E matched asymptotic expansions

Find first order composite expansions for the following problems. Note, some of these systems are "stiff" and hence numerical solvers might have difficulty.

1.
$$\begin{cases} \epsilon y'' + (y')^2 - 1 = 0 \\ y(0) = 1 \text{ and } y(1) = 1 \end{cases}$$

2.
$$\begin{cases} \epsilon y'' + (y')^2 - 1 = 0 \\ y(0) = 1 \text{ and } y(1) = 1/2 \end{cases}$$

3.
$$\begin{cases} \epsilon y'' = -\left(x^2 - \frac{1}{4}\right) y' \\ y(0) = 1 \text{ and } y(1) = -1 \end{cases}$$

4.
$$\begin{cases} \epsilon y'' + 2xy' + (1 + \epsilon x^2)y = 0 \\ y(-1) = 1 \text{ and } y(1) = -1 \end{cases}$$

Problem 2 Non-unique solutions

The following problem boundary value problem is a canonical example of wrinkling behavior in this elastic sheets:

$$\begin{cases} \epsilon^2 \dfrac{d^4 f}{dx^4} - 2\dfrac{d}{dx}\left[\left(\left(\dfrac{df}{dx}\right)^2 - 1\right)^2 \dfrac{df}{dx}\right] = 0 \\ f(0) = f(1) = 0 \\ \left.\dfrac{df}{dx}\right|_{x=0} = \left.\dfrac{df}{dx}\right|_{x=1} = 0 \end{cases}.$$

1. Find one exact solution to this equation. (Don't overthink this.)
2. Show that other solutions exist by constructing a first order composite solution. **Hint** The outer solutions are not too hard to find. One solution in particular will have one corner layer and two boundary layers near $x = 0$ and $x = 1$.
3. Show that there are an infinite number of outer solutions.

Problem 3 Method of characteristics

Solve the following using the method of characteristics:

1. $\dfrac{\partial u}{\partial t} - 3\dfrac{\partial u}{\partial x} = 0 \;\text{ with }\; u(x,0) = \cos x$

2. $\dfrac{\partial u}{\partial t} + 4\dfrac{\partial u}{\partial x} = 0 \;\text{ with }\; u(0,t) = \sin 3t$

3. $\dfrac{\partial u}{\partial t} + x\dfrac{\partial u}{\partial x} = 1 \;\text{ with }\; u(x,0) = f(x)$

4. $\dfrac{\partial u}{\partial t} + 3t\dfrac{\partial u}{\partial x} = u(x,t) \;\text{ with }\; u(x,0) = f(x)$

Problem 4 "Simple" P.D.E.

Consider the problem

$$\epsilon\Delta\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = u - x^2y^2$$

with the following boundary conditions on the unit square: $u(x,0) = g(x)$, $u(x,1) = h(x)$, $u(0,y) = r(y)$, and $u(1,y) = s(y)$.

1. Determine the leading order outer solution.

2. Determine the leading order inner solutions near the boundary. (Do not worry about not matching boundary conditions near the corners of the square.)