If we assume \[ U = U_0 + \epsilon U_1 + \ldots \]
\[ \Rightarrow (1 + (h_0')^2) \frac{d^2 U_0}{ds^2} + (x^2 + \beta^2) \frac{dU_0}{ds} = 0 \]
\[ \Rightarrow U_0 = A(r) + B(r) \exp \left( -\left( \frac{x^2 + \beta^2}{1 + (h_0')^2} \right) r \right) \]

Boundary conditions imply:
\[ U_0(0, r) = g \]
\[ U_0(r, \infty) = \int_0^r g(x, \kappa) \, dx \]

The solution can be pieced together as usual.

**Burger's Equation**

Suppose we want to model the density \( g(x, t) \) of cars on a stretch of highway.

\[ g(x, t) \] is traffic density \( (\# \text{ cars/length}) \)
\[ q(x, t) \] is traffic flow \( (\# \text{ cars/hour}) \)

N of cars in \((a, b)\) is
\[ N = \int_a^b g(x, t) \, dx \]
\[ \Rightarrow \frac{dN}{dt} = q(a) - q(b) \] (Conservation Law)
\[ \Rightarrow \int_a^b \frac{dg(x, t)}{dt} \, dx = q(a) - q(b) \]
\[ \Rightarrow \int_a^b \frac{d}{dt} g(x, t) \, dx = \int_a^b \frac{dq}{dx} \, dx \]

Take limit as \((b-a) \to 0\) we get
\[ \frac{dg}{dt} = -\frac{dq}{dx} \]
We need a constitutive law.

Let \( u \) be the velocity of cars.

\[ \Rightarrow q = s \cdot u. \]

Assume \( u = u(s) \) velocity only depends on density.

\[ \Rightarrow \frac{d}{dt} s + q'(s) \frac{d}{dx} s = 0. \]

This is like advection with a changing speed. The function \( q'(s) \) is the density wave velocity.

Simple Traffic Model.

1. \( u \) should be decreasing.
2. \( u(0) = U_{\text{max}} \) (speed limit).
3. \( u(s_{\text{max}}) = 0 \) (traffic jam).

\[ U_{\text{max}} \]

\[ U = -s_{\text{max}} s + U_{\text{max}} \]

\[ \Rightarrow q = -s_{\text{max}} s^2 + U_{\text{max}} s. \]

This gives us:

\[ \frac{d}{dt} s + U_{\text{max}} \left(1 - \frac{2s}{s_{\text{max}}} \right) \frac{d}{dx} s = 0. \]

Red light; green light:

\[ \frac{d}{dt} s + \left(1 - 2s \right) \frac{d}{dx} s = 0. \]

\[ s(x, 0) = \begin{cases} -1, & x < 0 \\ 0, & x > 0. \end{cases} \]

Characteristics:

\[ \frac{dx}{dt} = 1 - 2s, \quad \frac{ds}{dt} = 0. \]
What happens at $x=0$? $s$ can take any value so the slope rotates from $-1$ to $1$. This is known as a rarefaction wave.

Green light > red light:

$$\frac{\partial s}{\partial t} + (1 - 2s) \frac{\partial s}{\partial x} = 0$$

$s(x,0) = 0, \quad x < 0,$

$s(0,t) = 1, \quad t > 0$

Characteristics:

$$\frac{dx}{dt} = 1 - 2s, \quad \frac{ds}{dt} = 0$$

$$\Rightarrow x = (1 - 2s)t + x_0, \quad s = s_0$$

or

$$x = \frac{1}{1 - 2s_0} (x + t_0), \quad s = s_0$$
The characteristics intersect. How can we make sense of this?

Shock Waves. Consider the general P.D.E.

\[
\frac{ds}{dt} + c(s) \frac{ds}{dx} = 0
\]
Let's look at
\[ \frac{dg}{dt} + g \frac{dg}{dx} = 0 \]

\[ g(x, t) \text{ profile at later time} \]

\[ g(x, s, t) \text{ Multivalued function.} \]

Physically, there cannot be a multivalued solution.

At a value \( x_s(t) \) there should be a jump or a shock. Since \( g \) is conserved to the left and to the right of the shock, we must have

\[ g(x_s^-, t) [u(x_s^-, t) - \frac{dx_s}{dt}] = g(x_s^+, t) [u(x_s^+, t) - \frac{dx_s}{dt}] \]

\[ \Rightarrow \frac{dx_s}{dt} = g(x_s^+, t) - g(x_s^-, t) = \frac{1}{g} \left( \frac{\text{Rankine-Hugoniot condition}}{g(x_s^+, t) - g(x_s^-, t)} \right) \]

\[ \begin{cases} \frac{\partial g}{\partial t} + 2g \frac{\partial g}{\partial x} = 0 \end{cases} \]

\[ g(x, 0) = \begin{cases} \begin{cases} 4 & x < 0 \\ 3 & x > 0 \end{cases} \end{cases} \]
The characteristics are:

\[ x = 2g_0 t + x_0 \]

For our system we know \( q'(s) = 2s \Rightarrow q(s) = s^2 \).

The R-K Hugoniot condition implies

\[ \frac{dx}{dt} = q(4) - q(3) = 7 \]

\[ 4 - 3 \]

\[ \Rightarrow x_s(t) = \frac{7}{4} t + d \quad (d = 0). \]

\[ \text{shock wave.} \]

Alternate Derivation of Rankine-Hugoniot Condition:

We have the conservation law

\[ \frac{d}{dt} \int_{0}^{b} g(x,t) \, dx = q(a,t) - q(b,t) \]

Suppose we have a shock located at \( x_s(t) \).

\[ \frac{d}{dt} \left[ \int_{a}^{x_s(t)} g(x,t) \, dx + \int_{x_s(t)}^{b} g(x,t) \, dx \right] = q(a,t) - q(b,t). \]
\[
\Rightarrow \int_{x(x)}^{b} \frac{ds}{dx} \, dx + \frac{dx}{dt} \cdot g(x, t) + \int_{x(x)}^{b} \frac{ds}{dt} \, dx = \frac{dx}{dt} \cdot g(x, t)
\]
\[
= g(a, t) - g(b, t).
\]

Now away from the shock
\[
\frac{dx}{dt} = -\frac{\partial g}{\partial x}
\]

\[
\Rightarrow g(a, t) - g(x^-, t) + \frac{dx}{dt} \cdot g(x^-, t) + g(x^+, t) - g(b, t) - \frac{dx}{dt} \cdot g(x^+, t) = g(a, t) - g(b, t)
\]

\[
\Rightarrow \frac{dx}{dt} = \frac{g(x^-, t) - g(x^+, t)}{g(x^-, t) - g(x^+, t)}
\]

(Ranked - Huijsen +
Condition

**Viscosity Solutions.**

One point of view is that discontinuous solutions cannot describe reality. There must be some small effect that removes non-smoothness.

\[\nu_t + \nu \nu_x = \nu \nu_{xx} \quad \text{(Burger's equation)}\]

**Conservation Law**

**Viscosity**

(diffusion, friction, etc.)

**Example**

\[\nu(x, t) = \begin{cases} 1, & x < 0 \\ 0, & x > 0 \end{cases}\]

**Outer Solution:**

\[\nu_t + \nu_x + \nu \cdot \frac{dx}{dt} \nu_x = 0\]

**Characteristics:**

\[\frac{dx}{dt} = \nu_0 \quad \Rightarrow x(t) = \nu_0 t + c\]

\[\frac{d\nu}{dt} = 0 \quad \Rightarrow \nu(0) = c\]
We know the characteristics intersect. We must add an inner layer to correct non-second differentiability.

Let

\[ X = x - x_s(t) \]

\[ \frac{d}{dx} = \frac{\partial X}{\partial x} \frac{d}{dx} + \frac{1}{2} \left( \frac{\partial X}{\partial x} \right)^2 \]

\[ \frac{\partial_x U_0}{\partial x} = e^{-x} s'(x) \frac{d}{dx} U_0 + e^{-x} U_0 \frac{d}{dx} U_0 = e^{1-x} \frac{d}{dx} U_0 \]

Letting \( x = 1 \) we obtain:

\[-s'(x) \frac{d}{dx} U_0 + U_0 \frac{d}{dx} U_0 = \frac{d^2}{dx^2} U_0 \]

\[ \Rightarrow -s'(x) U_0 + \frac{1}{2} U_0^2 = \frac{d}{dx} U_0 + A(x) \]

\[
\lim_{X \to -\infty} U_0 = 0 \quad \text{and} \quad \lim_{X \to \infty} U_0 = 1
\]

\[ \Rightarrow 0 = A(x) \]

\[-s'(x) + \frac{1}{2} = 0 \]

\[ \Rightarrow s'(x) = \frac{1}{2} \quad \text{(Rankine-Hugoniot condition)} \]

\[ \Rightarrow s(x) = \frac{1}{2} x \]
So we now need to solve:

$$\frac{1}{2} U_0 + \frac{1}{2} U_0^2 = \frac{dU}{dX}$$

$$\Rightarrow U_0(x, t) = \frac{1}{1 + B(t) e^{-x/2}}$$

$B(t)$ has to be determined at $O(ε)$.