Chapter 2: Flows on the Line

**General Framework:**
\[ \dot{x} = f(x), \quad x \in \mathbb{R}, \quad f: \mathbb{R} \to \mathbb{R}, \text{ differentiable.} \]

- **x** - position
- **f(x)** - velocity (really, **f** is a vector field)

We can try to solve
\[ x = \int_{x_0}^x \frac{dx}{f(s)} \, ds \]

i) We may not be able to integrate explicitly.
ii) It may not be possible to solve for \( x(t) \).

**Geometric Point of View:**

![Diagram showing flow lines and equilibrium points](image)

### Solutions to \( f(x) = 0 \) are called fixed points, equilibrium, steady states, rest states.

- \( f(x) > 0, \quad \frac{dx}{dt} > 0 \Rightarrow x(t) \) increases, particle moves right.
- \( f(x) < 0, \quad \frac{dx}{dt} < 0 \Rightarrow x(t) \) decreases, particle moves left.
- \( f(x) = 0, \quad \frac{dx}{dt} = 0, \quad x(t) = x_0 \) is a solution of \( \frac{dx}{dt} = f(x) \) with \( x(0) = x_0 \).

Fig. 2.1

![Fig. 2.2 showing stable and unstable points](image)
Note:
\[ \dot{x} = \frac{dx}{dt} = \frac{d^2x}{dx}\cdot f \]
This allows us to identify changes in concavity.

The phase portrait captures all of this behavior.

At each point in \( \mathbb{R} \) the function \( f \) assigns a tangent vector pointing left or right. Local stability can be determined graphically.

Example (Population Growth):
Infinite number of resources, no predators. We can model population growth by
\[ \frac{dP}{dt} = rP, \quad r - \text{ideal growth rate} \]
\[ P(0) = P_0 \]

Derivation:
Easy solution
\[ P = P_0 \cdot e^{rt} \]

A more realistic model must account for finite resources. As population increases it becomes harder to reproduce.

Unlimited pop growth

limited pop growth. Species in close proximity compete for resources.
We can modify the rate of reproduction

\[ P(t + \Delta t) = P(t) + r \left(1 - \frac{P(t)}{K}\right) \Delta t + P(t) = P(t) + r \left(\frac{K - P(t)}{K}\right) \Delta t + P(t) \]

\[ \frac{dP}{dt} = r \left(1 - \frac{P(t)}{K}\right) P(t) \]

\[ P(0) = P_0. \]

\[ \text{Population simply goes to carrying capacity.} \]

\[ \text{Analytical Approach:} \]

\[ x \]

\[ s'(x) < 0 \]

\[ \text{stable} \]

\[ s'(x) > 0 \]

\[ \text{unstable} \]
Take a Taylor expansion near \( x^* \):
\[
\begin{align*}
    f(x) &= f(x^*) + f'(x^*)(x-x^*) + O((x-x^*)^2) \\
    &= 0
\end{align*}
\]

Set \( y(t) = x(t) - x^* \) and omit higher order terms.

\[
\Rightarrow \frac{dy}{dt} = \frac{dx}{dt} \approx f'(x^*) y
\]

Provided, of course, that \( y \) is close to 0.

\[
\Rightarrow y(t) = y(0) \exp(f'(x^*) t)
\]

\[
\Rightarrow x(t) = x^* + y(0) \exp(f'(x^*) t)
\]

Conclusion:
1. \( x(t) \to \infty \) if \( f'(x^*) > 0 \): unstable
2. \( x(t) \to x^* \) if \( f'(x^*) < 0 \): stable

This process is called linear stability analysis and is a fundamental tool.

Example:
\[
x = x^2 - 1
\]

Fixed points \( x = \pm 1 \).

\[
f'(x) = 2x
\]

Hence,
-1 is stable, 1 is unstable.

What happens when \( f'(x) = 0 \) at an equilibrium?
Potentials (Energy or Lyapunov functions)

Overdamped motion

\[ x + \alpha x = -\frac{dV}{dx} \]

Acceleration friction potential energy gradient

If \( x \) is really large we can look at equivalent system

\[ \dot{x} = -\frac{dV}{dx} = f(x) \]

Since there is no inertia the potential energy is always decreasing

\[ \frac{dV}{dx} \leq 0. \]

\[ \frac{dV}{dx} = \frac{dx}{dt} \frac{dV}{dx} = -\frac{dV}{dx} \leq 0. \]

Proof

We need to prove that \( \frac{dV}{dx} \leq 0 \).

Consequence:
Slope Fields

Another geometric viewpoint.
\[ \dot{x} = x(1-x) \]

We can plot the slope field.

Euler's method uses slopes to construct piecewise linear approximations.

Smarter methods use more information to gain better guesses on slopes.

Explosions
\[ \dot{x} = 1 + x^2 = f(x) \]

Solution:
\[ x = \tan(x + C) \]
\[ x \to \infty \text{ as } t \to \frac{\pi}{2} - C. \]

A solution does not exist for all time even though \( f(x) \) is smooth! This is called explosive growth.

What do we know about the system
\[ \dot{x} = 1 + x^2 = f(x) \]

Then this system exhibits explosive growth.

Proof:
1. There exists \( T(x_0) > 0 \) such that \( t > T(x_0) \Rightarrow x(t) > 1 \).
2. For \( t > T(x_0) \) we have \( \dot{x} = 1 + x^{10} > 1 + x^2 \).