Chapter 4: Introduction

Classical physics—Derive a system of differential equations and solve them. The entire evolution of the system is determined from initial conditions.

Example—An object is placed in a cold (hot) environment at a fixed temperature $T_E$.

If the temperature difference between the object and its environment changes at a rate proportional to the temperature difference, what is a function for the temperature of the object?

$$\frac{dT}{dt} = K(T_E - T) \quad \text{(Newton's Law of Cooling)}$$

$p(t) = T_e - T_0$ (Units of inverse time)

Let $\Delta T = T - T_E$ and $\Delta T_0 = T_0 - T_E$. Then,

$$\Rightarrow \frac{d\Delta T}{dt} = -K \Delta T$$

$$\Rightarrow \Delta T = \Delta T_0 e^{-Kt}$$

*Exact quantitative prediction of temperature.*
Example: Now suppose the object and its environment influence each other:

\[
\frac{dT}{dt} = -K \left( T - T_E \right), \quad T(0) = T_0
\]
\[
\frac{dT_E}{dt} = K \left( T - T_E \right), \quad T_E(0) = T_{E_0}
\]

Again, let \( \Delta T = T - T_E \) and \( H = T + T_E \). Then,

\[
\frac{dH}{dt} = (K - K_1) \Delta T, \quad H(0) = T_0 + T_{E_0} = H_0
\]
\[
\frac{d\Delta T}{dt} = -K_1 \Delta T, \quad \Delta T(0) = T_0 - T_{E_0} = \Delta T_0
\]
\[
\Rightarrow \Delta T = \Delta T_0 e^{-K_1 t}
\]
\[
\frac{dH}{dt} = (K_2 - K_1) \Delta T_0 e^{-K_1 t}
\]
\[
\Rightarrow H(t) = \left( \frac{K_2 - K_1}{K_2 K_1} \right) \Delta T_0 e^{-K_1 t} (e^{K_1 t} - 1) + H_0
\]

Notation - A general framework for (continuous) dynamical systems is the system of equations

\[
\begin{align*}
X_1 &= \mathbf{f}_1(x_1, \ldots, x_n) \\
& \quad \text{(overdot denotes differentiation in time)} \\
\vdots \\
X_n &= \mathbf{f}_n(x_1, \ldots, x_n)
\end{align*}
\]

Such a system is called an \( n \)-dimensional dynamical system.

Linearity - If the functions \( \mathbf{f}_i \) are linear then the system can be solved exactly using linear algebra, and complex solutions can be built up as a superposition.

Nonlinear - If any \( \mathbf{f}_i \) are nonlinear then it may not be solvable exactly and more complex behavior can arise.

1-D       2-D       3-D and beyond
stable, unstable, oscillations stable, unstable, oscillations, chaos!
Example:
Nonlinear version of Newton's law of cooling
\[ \Delta T = -K \Delta T - \alpha \Delta T^3 \]
\[ \Delta T(0) = T_0 \]
Note: If we think of \( \Delta T = f(\Delta T) \) then this is the first two nonzero terms in a Taylor expansion.

We can try separating variables again:
\[ -t = \int T_0^{\Delta T} \frac{1}{ks + \alpha s^3} \, ds \]
\[ = \int T_0^{\Delta T} \left( \frac{1}{ks} + \frac{K}{\alpha s^2} \right) \, ds \]
\[ = \frac{1}{k} \ln \left( \frac{\Delta T}{T_0} \right) + \frac{K}{\alpha} \sqrt{\frac{K}{\alpha} \left( \frac{\Delta T}{T_0} \right) - 1} \]

To get a complete solution we would need to invert.

Poincaré took a different approach
\[ \Delta T = 0 \text{ when } \Delta T = 0 \]
i.e. the object stops cooling when \( \Delta T = 0 \). Before that \( \Delta T \) is always negative implying \( \Delta T \) is monotone decreasing.

All initial conditions end up at the endpoint
Poincaré had the insight to study flows instead of solutions.
Example

Pendulum

Classic Form
\[ \ddot{\theta} + \sin(\theta) = 0 \]
\[ \theta(0) = \theta_0 \]
\[ \dot{\theta}(0) = V_0 \]

Equivalent Form
\[ \ddot{\theta} = \dot{V} \]
\[ \dot{\theta}(0) = \dot{\theta}_0, \quad V(0) = V_0 \]

The equation is non-linear, so it is difficult to solve.

Can we answer some questions about this system:

\[ \frac{dV}{dt} = -\sin(\theta) \]

\[ \Rightarrow \int V \, dV = \int_{\theta_0}^{\theta} \sin(\theta) \, d\theta \]

\[ \Rightarrow \frac{1}{2} V^2 - \frac{1}{2} V_0^2 = \cos(\theta) - \cos(\theta_0) \]

\[ V = \pm \sqrt{V_0^2 + 2 \cos(\theta) - 2 \cos(\theta_0)} \]

\[ \frac{d\theta}{dt} = \pm \sqrt{V_0^2 + 2 \cos(\theta) - 2 \cos(\theta_0)} \]

If \( V_0^2 - 2 \cos(\theta_0) > 2 \) then no oscillations possible.

Oscillations

No oscillations.