Lecture 4: Modeling with Differential Equations.

Example 1:
Differential equations can model polar states:
- religion
- party affiliation
- left/right handed
- language
- party affiliation.

Let $x$ denote fraction of population that is republican and $1-x$ denote the fraction of population that is democrat $y$.

Note:
$$x + (1-x) = 1 \rightarrow \text{total population.}$$
$$\text{dep} \quad \text{dem} = y$$

$$\frac{dx}{dt} = -P_{xy}x + P_{yx}(1-x)$$
$$\text{frac. that} \quad \text{frac. that}$$
$$\text{switch} \quad \text{switch to \ rep.}$$

$$P_{yx} = s \times a$$

and $$P_{xy} = (1-s)(1-x)^q$$

status
- i.e. how awesome the party is

$a > 1$, scale parameter

This function measures:
- quantifies that a party is more popular for a larger number of members.
Let's analyze what this model predicts:

\[ \frac{dx}{dt} = s x^a (1-x) - (1-s)(1-x)^a x \]

\[ = x(1-x) \left( s x^{a-1} - (1-s)(1-x)^{a-1} \right) \]

\[ = x(1-x) g(x). = f(x). \]

Let's try sketching a graph:

\[ g(x) \]

\[ s \]

\[ -1-s \]

\[ \text{Note:} \]

\[ g'(x) = s(a-1)x^{a-2} + (1-s)(1-x)^{a-2} a - 1 > 0 \]

This model predicts that the population will all become democrats or republicans.
Stability analyzed via:

\[ f'(x) = ax^{a-1}(1-x) - sx^{a-1}(1-s)(1-x)^{a-1} + (1-s)(1-x)^a \]

\[ \Rightarrow f'(0) = -(1-s), \]

\[ f'(1) = -s. \]

Hence, 0, and 1 are unstable \(\Rightarrow x_2\) has to be stable.

Example 2:

Modeling spread of infectious diseases

- \(S\) - population that is susceptible
- \(I\) - population that is infected

Modeling:

- When a susceptible encounters an infected there is a chance the susceptible becomes infected.
- Infected can recover and return to being susceptible.
- Infected can die.

\[
\frac{ds}{dt} = -\alpha IS + \beta I
\]

\[
\frac{dI}{dt} = \alpha IS - \beta I - \gamma I
\]

\[
\frac{dR}{dt} = \gamma I
\]

For now, assume \(\gamma = 0\) (Nobody dies)

\[
\Rightarrow \frac{ds}{dt} = -\alpha IS + \beta I
\]

\[
\frac{dI}{dt} = \alpha IS - \beta I.
\]
Now, \[
\frac{dI}{dt} + \frac{dS}{dt} = 0
\]

The total population is \( P = I + S \). We have found \( \frac{dP}{dt} = 0 \). (Total population remains the same)

It follows that

\[
I + S = P_0
\]

\[
\Rightarrow I = P_0 - S
\]

The equation for \( S \) becomes;

\[
\frac{dS}{dt} = -\alpha (P_0 - S) S + \beta (P_0 - S)
\]

\[
= (P_0 - S)(\beta - \alpha S)
\]

If \( \beta / \alpha < P_0 \) then the disease remains endemic.

* \( \beta / \alpha \to \text{ratio of recovery rate to infection rate} \)

**If \( \beta / \alpha > P_0 \) then everybody recovers.**
Example 3:
\[
\frac{dN}{dt} = rN - aN(N-b)^2
\]
\[
= rN - a b^2 N( N/b - 1 )^2
\]
We assume that this models a population.
- \( rN \) - exponential growth
- \(-aN(N-b)\) - term that penalizes growth for small and large values of \( N \).

**Dimensional Analysis**

**Variables:**
- \([N]\) - population
- \([t]\) - time

**Parameters:**
- \([r]\) - time\(^{-1}\)
- \([b]\) - population
- \([a]\) - population\(^2\) time

**Change of variables:**
\[
x = \frac{N}{\alpha}, \quad [\alpha] = \text{population}
\]
\[
\tau = \frac{t}{\beta}, \quad [\beta] = \text{time}
\]
\[
\Rightarrow N = \alpha x.
\]
\[
\Rightarrow \frac{dN}{dt} = \frac{dx}{dt} \frac{d}{dx}(\alpha x) = \frac{d\tau}{dt} d\tau (\alpha x) = \alpha \beta \frac{d\alpha x}{d\tau}
\]
Set \( x = b, \beta = r^{-1}. \)

\[
\Rightarrow br \frac{dx}{dt} = rb x - ab^3 x(x-1)^2
\]

\[
\Rightarrow \frac{dx}{dt} = x - \frac{ab^3}{r} x(x-1)^2
\]

Set \( \delta = ab^3/r \) we have that:

\[
\frac{dx}{dt} = x - \delta x(x-1)^2
\]

**Analysis**

**Fixed points:**

\( x = 0, \quad 1 - \delta (x-1)^2 = 0 \)

\[
\Rightarrow +1 = x - 1
\]

\[
\Rightarrow x = 1 \pm \sqrt[2]{\delta}
\]

We have three fixed points:

We want to determine when \( 1 - x \delta > 0 \)

\[
\Rightarrow \sqrt[2]{\delta} - 1 > 0
\]

\[
\Rightarrow \delta > 1.
\]

**Case 1:** \( \delta < 1 \)

**Case 2:** \( \delta > 1 \)

Population could die. (Sensitive to change in parameters)

Population survives
Solution Curves:

Case 1: Logistic

Case 2: Logistic or extinction.

Bifurcation curve:

Plot fixed points vs parameter. Indicate stability by solid lines and unstable fixed points by dashed curves.
Example 4: Insect Outbreak

\[ \frac{dN}{dt} = rN(1 - \frac{N}{K}) - p(N) \]

\[ p(N) = \frac{BN^2}{A^2 + N^2} \]

\[ \frac{dN}{dt} = rN(1 - \frac{N}{K}) - \frac{BN^2}{A^2 + N^2} \]

**Dimensional Analysis**

**Variables**:

- \([N]\) - population
- \([t]\) - time
- \([\frac{dN}{dt}]\) - population/time

**Parameters**:

- \([r]\) - time\(^{-1}\)
- \([K]\) - population
- \([B]\) - time\(^{-1}\) population\(^{-1}\)
- \([A]\) - population
This motivates rescaling by
\[ x = \frac{N}{A}, \quad \tau = \frac{B}{A} t \]

\[ \Rightarrow N = Ax, \quad \frac{dN}{dt} = \frac{d}{dt} Ax = B \frac{dx}{d\tau} \]

Therefore, the system reduces to:
\[ \frac{dx}{dt} = \alpha x \left(1 - \frac{x}{\beta}ight) - \frac{x^2}{1 + x^2} = f(x) \]

\[ \text{Fixed Points.} \]
\[ \text{One obvious fixed point is } x = 0. \text{ Calculating we have that} \]
\[ f'(x) = \alpha - 2x \frac{\beta x - (1 + x^2) 2x - x^2 (2x)}{(1 + x)^2} \]

\[ \Rightarrow f'(0) = \alpha. \]

Consequently, \( x = 0 \) is always unstable.

To determine the other fixed points we take a graphical approach. Let \( g(x) = \alpha \left(1 - \frac{x}{\beta}\right) - \frac{x}{1 + x^2} \). We try to analyze when \( \alpha \left(1 - \frac{x}{\beta}\right) = \frac{x}{1 + x^2} \). We do this graphically.
To determine when case 1 switches to case 2 we look where the line passes tangentially to $\frac{x}{1+x^2}$. This gives us the conditions:

\[ a.) \quad \frac{-\alpha}{\beta} = \frac{1+x^2-2x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2}, \]

\[ b.) \quad \alpha x (1-x/\beta) = \frac{x^2}{1+x^2}. \]