HARNACK INEQUALITY FOR TIME-DEPENDENT LINEARIZED PARABOLIC MONGE-AMPÈRE EQUATION

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ABSTRACT. We prove a Harnack inequality for nonnegative solutions of linearized parabolic Monge-Ampère equations

$$-\frac{u_t}{\phi_t} - \mathrm{tr}((D^2\phi)^{-1}D^2u) = 0,$$

in terms of a variant of parabolic sections associated with ϕ, where ϕ satisfies $\lambda \le -\phi_t \det D^2\phi \le \Lambda$ and $C_1 \le -\phi_t \le C_2$.

1. INTRODUCTION

In this paper we study linearized parabolic Monge-Ampère equations as follows:

$$L_\phi(u) = -\frac{u_t}{\phi_t} - \mathrm{tr}((D^2\phi)^{-1}D^2u) = 0 \quad \text{in} \quad \mathbb{R}\times\mathbb{R}^d, \tag{1.1}$$

where u is nonnegative, $u_t = \frac{\partial u}{\partial t}$, D^2u denotes the Hessian of u, $(D^2\phi)^{-1}$ is the inverse of the Hessian of ϕ, and $\mathrm{tr}(A)$ is the trace of a matrix A. We assume that ϕ is a strictly parabolic convex function in $\mathbb{R}\times\mathbb{R}^d$ and satisfies the following inequalities in $\mathbb{R}^{d+1}$,

$$0 < \lambda \le -\phi_t \mathrm{det} D^2\phi \le \Lambda < \infty, \qquad 0 < C_1 \le -\phi_t \le C_2 < \infty, \tag{1.2}$$

where λ, Λ, C_1, and C_2 are positive constants. We call d, λ, Λ, C_1, and C_2 structure conditions and the constant depending only on structure conditions are called universal constant. The parabolic convexity is defined in Section 2.

The purpose of this paper is to establish a Harnack inequality for nonnegative solutions of (1.1) and (1.2) in terms of a variant of parabolic sections of ϕ:

$$\tilde{Q}_{c_0}(z_0,h) = (t_0 - c_0h, t_0] \times S_\phi(x_0, \frac{1}{2}h|t_0),$$

where c_0 is a positive parameter, $z_0 = (t_0,x_0) \in \mathbb{R}^{d+1}$, $h > 0$, and

$$S_\phi(x_0,h|t_0) = \{x : \phi(t_0,x) \le \phi(t_0,x_0) + \nabla\phi(t_0,x_0)\cdot(x-x_0) + h\},$$

which is the section in the x variable with height h at time t_0. Throughout this paper, we assume that all the functions are sufficiently smooth. It is easily seen that our results do not depend on the smoothness of u. The main result is stated as follows.

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Theorem 1.1. *Assume that ϕ satisfies* (1.2) *and $u \in C^2$ nonnegative solves* (1.1) *in $\mathbb{R}^{d+1}$. Then there exists a universal constant C such that*

$$\sup_{Q_-} u \leq C \inf_{Q_+} u,$$

where

$$Q_- = \tilde{Q}_{c_0}((-\frac{c_0}{2}, x_0), \frac{1}{16}),$$
$$Q_+ = \tilde{Q}_{c_0}((T_0, x_0), r_0),$$

and $x_0 \in \mathbb{R}^d$, T_0, c_0, and r_0 are positive universal constants with $T_0 > c_0 r_0$.

We make the following remark regarding Theorem 1.1.

Remark 1.2. First, for simplicity, in our theorem above we assume that u satisfies (1.1) in $\mathbb{R}^{d+1}$. Actually we only need to consider (1.1) in a bounded domain Ω and prove the Harnack inequality for the corresponding cylindrical domains Q_- and Q_+. This can be easily seen by scaling and translation of the coordinates. Second, the assumption that $C_1 \leq -\phi_t \leq C_2$ is crucial in our proof. We are able to apply several geometric properties of the sections in x variable to our proof under this assumption. To our best knowledge, the case without assuming the upper bound and lower bound of ϕ_t remains open.

By a standard argument, we obtain the following corollary, which implies the Hölder continuity.

Corollary 1.3. *Assume that u is a solution of* (1.1) *and* (1.2)*. For any $z_0 \in \mathbb{R}^{d+1}$ and $0 < \rho < R$, we have*

$$osc_{\tilde{Q}_{c_0}(z_0,\rho)} u \leq C(\frac{\rho}{R})^\alpha osc_{\tilde{Q}_{c_0}(z_0,R)} u,$$

where C, α are universal constants.

Our result extends Huang [11] to a more general setting. In [11] Huang obtains the Harnack inequality under the condition that $\phi = -t + \psi$, where ψ is a convex function in $\mathbb{R}^d$ and the corresponding Monge-Ampère measure satisfies the μ_∞ condition, see (1.3). Instead of $\phi_t = -1$, we assume $0 < C_1 \leq -\phi_t \leq C_2$ in our paper. The extension from constant to bounded away from zero is nontrivial because it is much more delicate to choose a proper variant of parabolic section. We discuss the difficulties in detail later.

Our work builds upon several previous results on the elliptic and parabolic Monge-Ampère equations. We borrow some ideas from the study of linearized Monge-Ampère equations. Let us mention some previous work as follows.

For the Monge-Ampère equation

$$\det D^2 u = f,$$

Caffarelli [1] introduced cross-sections of solutions to the Monge-Ampère equation which play the same role as balls for uniformly elliptic equations.

Let ψ be a smooth convex function in $\mathbb{R}^d$. For any $x \in \mathbb{R}^d$ and $h \geq 0$, we define the section with center x and height h as follows:

$$S_\psi(x,h) = \{y \in \mathbb{R}^d : \psi(y) \leq \psi(x) + \nabla\psi(x)\cdot(y-x) + h\}.$$

If there is no confusion, we omit ψ in the definition of the sections. We review several properties of sections in Section 2. For more related work and the development of the Monge-Ampère equation, we refer the reader to [2, 3, 4, 9, 7, 16, 17], and the references therein.

For the parabolic Monge-Ampère equation, Gutièrrez and Huang [10] proved $W^{2,p}$ estimates for

$$-u_t \mathrm{det} D^2 u = f,$$

with some suitable conditions on f. Besides the parabolic Monge-Ampère equation mentioned above, Krylov introduced some other types of parabolic Monge-Ampère operators in [12]. Moreover, Daskalopoulos and Savin [6] obtained a $C^{1,\alpha}$ estimate for the following parabolic Monge-Ampère equation:

$$u_t = b(t,x)(\mathrm{det} D^2 u)^p,$$

where $p > 0$ and $\lambda \leq b \leq \Lambda$.

For the linearized Monge-Ampère equations, Caffarelli and Gutiérrez [5] established the Harnack inequality in terms of sections for nonnegative solutions of the following linearized Monge-Ampère equations

$$\mathrm{tr}((D^2\phi)^{-1}D^2 u) = 0 \quad \text{in} \quad \mathbb{R}^d,$$

where ϕ is a convex function and the corresponding Monge-Ampère measure satisfies the μ_∞ condition. Specifically, let

$$d\mu = \mathrm{det} D^2\phi \, dx$$

in the Alexandrov sense, see [9, Chapter 1], and μ satisfies the following condition: Given $\delta_1 \in (0,1]$, there exists $\delta_2 \in (0,1]$, so that, for all sections S and measurable set $E \subset S$,

$$\frac{|E|}{|S|} < \delta_2 \quad \text{implies} \quad \frac{\mu(E)}{\mu(S)} < \delta_1, \tag{1.3}$$

where $|\cdot|$ is the Lebesgue measure. The μ_∞ condition above implies the following doubling property: There exist constants $C > 0, 0 < \alpha < 1$ such that

$$\mu(S(x,h)) \leq C\mu(\alpha S(x,h)), \tag{1.4}$$

for every section $S(x,h)$, where $\alpha S(x,h)$ denotes the α-dilation of the set $S(x,h)$ with respect to the center of mass, which is used later in this paper. For more recent work related to the linearized Monge-Ampère equation near the boundary, we refer the reader to [13, 14].

The proof of Theorem 1.1 uses some ideas in [5] and [11]. However, unlike the elliptic case in [5], we have to overcome the difficulties caused by the t dependence of the solution u, which is also the general reason that estimates for parabolic equations are subtler than for elliptic equations. On the other hand, we cannot use the method in [20], since the equation lacks the property of uniform ellipticity. Therefore we work on the variant of parabolic sections. Furthermore, the assumption in [11] is $\phi_t = -1$, so the parabolic section used there is a cylindrical domain $(t_0 - \frac{r}{2}, t_0 + \frac{r}{2}) \times S(x_0, r)$. Notice that $S(x_0, r)$ does not evolve with respect to time. But in our case, the evolvement in t of the variant of parabolic sections makes the estimate much more delicate. We need to compare the sections in x variable at different time levels. To this end, we apply a conclusion in [10]

$$S(x, h|t_1) \subset S(x, \theta h|t_2),$$

for certain t_1, t_2, and h, where $\theta > 1$ is a universal constant. After choosing small parameter c_0, we show the following *critical density argument*: For any $h > 0$, if

$$\inf_{\tilde{Q}_{c_0}(z_1, \theta h)} u \leq 1,$$

then there exists a large universal constant M such that

$$|\tilde{Q}_{c_0}(z_0, h) \cap \{u \leq M\}| \geq \varepsilon |\tilde{Q}_{c_0}(z_0, h)|,$$

where $z_0 = (t_0, x_0)$, $z_1 = (t_0 + c_0\theta h, x_0)$, and ε is a small universal constant. Moreover, the Calderón-Zygmund decomposition holds in terms of the following cylindrical domains for $\delta \leq c_0$

$$\tilde{Q}^{\delta}_{c_0}(z, h) = (t - c_0 h, t + \delta h) \times S(x, \frac{1}{2}h|t).$$

For δ sufficiently small, we are able to combine the Calderón-Zygmund decomposition with our critical density argument to show a power decay of the distribution function of the solution u. In this way we derive a weak Harnack inequality and then the Harnack inequality.

The paper is organized as follows. In the next section, we introduce some preliminary results and properties that we use in our proof. In Section 3, we prove the critical density argument and some necessary preparations for the estimate of the distribution function. In Section 4, we prove the Calderón-Zygmund decomposition. Finally, we show the Harnack inequality in Section 5.

2. PRELIMINARY

In this section, we recall some basic properties of (parabolic) convex solutions to the (parabolic) Monge-Ampère equation.

We say a function ϕ parabolic convex if ϕ is nonincreasing with respect to t for each x and convex in x for each t. Furthermore, if ϕ is strictly decreasing with respect to t and strictly convex with respect to x, then ϕ is strictly parabolic convex. Denote $B(0, r)$ to be the Euclidean ball in $\mathbb{R}^d$ with center 0 and radius r. We say an open bounded convex set $S \subset \mathbb{R}^d$ is normalized if

$$B(0, \alpha_d) \subset S \subset B(0, 1),$$

where $\alpha_d \in (0,1)$ is a constant depending only on d. We mention the following lemma due to F. John: Let S be a convex set in $\mathbb{R}^d$ with nonempty interior. Then there exists an (invertible) orientation preserving affine transformation $T : \mathbb{R}^d \to \mathbb{R}^d$ which normalizes S, i.e.,

$$B(0, \alpha_d) \subset T(S) \subset B(0, 1). \tag{2.5}$$

In particular,

$$\frac{\alpha_d^d \omega_d}{|S|} \leq \det T \leq \frac{\omega_d}{|S|},$$

where $|S|$ is the Lebesgue measure of S and $\omega_d = |B(0,1)|$.

Next, we collect some properties of the sections to the strictly convex functions. Let ψ be a strictly convex function in $\mathbb{R}^d$ and $d\mu = \det D^2\psi \, dx$ be the corresponding Monge-Ampère measure. Assume that μ satisfies the doubling condition (1.4) with $\alpha = 1/2$. As shown in [9, Chapter 3], $S_\psi(x,h)$ satisfies some strong geometric properties and we briefly recall several of them as follows:

(i) There exists a universal constant $\theta > 1$ such that, if $S(x,h) \cap S(y,h) \neq \emptyset$, then $S(y,h) \subset S(x, \theta h)$, for any $x, y \in \mathbb{R}^d, h > 0$. This is also called the engulfing property.

(ii) Let $S(x_0, r_0)$ and $S(x_1, r_1)$ be sections with $r_0 \leq r_1$ such that

$$S(x_0, r_0) \cap S(x_1, r_1) \neq \emptyset,$$

and T be an affine transformation that normalizes $S(x_1, r_1)$, then there exist universal constants K_1, K_2, K_3, and ε such that

$$B(Tx_0, K_2 \frac{r_0}{r_1}) \subset T(S(x_0, r_0)) \subset B(Tx_0, K_1 (\frac{r_0}{r_1})^\varepsilon),$$

and $Tx_0 \in B(0, K_3)$.

(iii) Let $S(x_0, 1)$ be a section. There exist constants $p, C > 0$, such that for $0 < r < s < 1$ and $x \in S(x_0, r)$,

$$S(x, C(s-r)^p) \subset S(x_0, s).$$

(iv) There exist $0 < \tau, \lambda < 1$ such that for all $x_0 \in \mathbb{R}^d$ and $r > 0$

$$S(x_0, \tau r) \subset \lambda S(x_0, r).$$

(v) Assuming $0 < \lambda < 1$, for any $r > 0$, we have

$$\lambda S(x_0, r) \subset S(x_0, (1 - (1-\lambda)\frac{\alpha_d}{2})r),$$

where α_d is the constant in (2.5).

We state the following lemma and corollary of [9, Chapter 3].

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open convex set. Assume that ψ is a convex function in Ω such that $\psi \leq 0$ on $\partial\Omega$. If $x \in \Omega$ and $l(y) = \psi(x) + p \cdot (y - x)$ is a supporting hyperplane to ψ at $(x, \psi(x))$, then*

$$|p| \leq \frac{-\psi(x)}{dist(x, \partial\Omega)}.$$

Corollary 2.2. *Let ψ be convex in Ω with $0 < k \leq \det D^2\psi \leq K$, where k, K are positive constants. Assuming $S_\psi(x_0, h) \subset \Omega$, for any h small, we have*

$$|S_\psi(x_0, h)| \approx h^{d/2}.$$

For an open connected domain $Q \subset \mathbb{R}^{d+1}$, we define the parabolic boundary of Q to be a set of all points $X_0 = (t_0, x_0) \in \partial Q$, such that there exists a continuous function $x = x(t)$ on an interval $[t_0, t_0 + \delta)$ with values in $\mathbb{R}^d$ satisfying $x(t_0) = x_0$ and $(t, x(t)) \in Q$ for all $t \in (t_0, t_0 + \delta)$. Here $x = x(t)$ and $\delta > 0$ depend on X_0. For any $Q \subset \mathbb{R}^{d+1}$, we define

$$Q(t) = \{x : (t, x) \in Q\}.$$

We say a set $Q \subset \mathbb{R}^{d+1}$ is a bowl-shaped domain if $Q(t)$ is convex for each t and $Q(t_1) \subset Q(t_2)$ for $t_1 \leq t_2$. Let $Q \subset \mathbb{R}^{d+1}$ be a bowl-shaped domain and denote $t_0 = \inf\{t : (t, x) \in Q\}$. Then the parabolic boundary of a bowl-shaped domain Q is

$$\partial_p Q := \{t_0\} \times \overline{Q(t_0)} \cup \bigcup_{t \in \mathbb{R}} (\{t\} \times \partial Q(t)),$$

where $\overline{Q(t_0)}$ denotes the closure of $Q(t_0)$ and $\partial Q(t)$ denotes the boundary of $Q(t)$. For a parabolic convex function ϕ, the canonical parabolic section at point $z_0 = (t_0, x_0)$ with height h is

$$Q_\phi(z_0, h) = \{z = (t, x) : \phi(z) \leq \phi(z_0) + \nabla\phi(t_0, x_0) \cdot (x - x_0) + h,\ t \leq t_0\}.$$

Moreover if it is clear, we omit ϕ in the definition of the parabolic sections as well. Throughout this paper, we use $C(\cdot)$ to denote constants and their dependences. For example, $C(\alpha, \beta)$ is a constant depending on α and β. We also use abbreviations as follows: if a constant C depends on β and the structure conditions, we simply denote $C = C(\beta)$. Furthermore, the constant may vary from line to line. If $z = (0, 0)$, we denote $\tilde{Q}(h) := \tilde{Q}_{c_0}((0, 0), h)$.

3. Critical density argument

As hinted in the introduction, in this section we prove that the level sets of u satisfy the *critical density argument*, which is important in our proof of the Harnack inequity.

First we state Lemma 4.2 in [10] about the engulfing property of sections in the x variable at different time.

Lemma 3.1. *Let ϕ satisfy* (1.2). *Suppose that $(t_1, x_1), (t_2, x_2) \in Q(z_0, h)$. Then there exists $\theta > 1$ depending only on d, λ, Λ, C_1, and C_2 such that*

$$S(x_1, h|t_1) \subset S(x_2, \theta h|t_2), \quad S(x_2, h|t_2) \subset S(x_1, \theta h|t_1).$$

We use Lemma 3.1 frequently in our proofs especially for the case $x_1 = x_2$. Note that under the condition (1.2), there exists $\hat{c}_0 = \hat{c}_0(C_2)$ sufficiently small such that for any $h > 0$ and $c_0 \leq \hat{c}_0$

$$(t_0 - c_0 h, x_0) \in Q((t_0, x_0), \frac{h}{2\theta}), \tag{3.6}$$

which, by Lemma 3.1, implies

$$S(x_0, \frac{h}{2\theta}|t_0 - c_0 h) \subset S(x_0, \frac{h}{2}|t_0) \subset S(x_0, \frac{h\theta}{2}|t_0 - c_0 h). \tag{3.7}$$

In the rest of the paper, we restrict $c_0 < \hat{c}_0/2$. We note that dividing $\hat{c}_0$ by 2 is a technical requirement in Lemma 4.1. We introduce the following notation. Let $z = (t, x) \in \mathbb{R}^{d+1}$, $t_0 = t$, and $t_k = t + \sum_{j=1}^{k} c_0 \rho \theta^j$ for $k \geq 1$, where θ is the constant in Lemma 3.1. Denote $K_0(z, \rho) = \tilde{Q}_{c_0}(z, \rho)$ and

$$K_i(z, \rho) = \tilde{Q}_{c_0}((t_i, x), \rho\theta^i). \tag{3.8}$$

Notice that by (3.7), it is easy to see for any $\rho > 0, j \geq 0$

$$S(x, \frac{\rho\theta^j}{2}|t_j) \subset S(x, \frac{\rho\theta^{j+1}}{2}|t_{j+1}). \tag{3.9}$$

Then we have the following observation.

Lemma 3.2. *Let $T_0 = \theta\hat{c}_0$, and $z = (t, x) \in \tilde{Q}(1)$. Then there exist universal constants $\mathfrak{c}_0 < \hat{c}_0$ and r_0 such that for $c_0 < \mathfrak{c}_0$ and $\rho > 0$*

$$\tilde{Q}_{c_0}((T_0, 0), r_0) \subset (\cup_{j=1}^{\infty} K_j(z, \rho)) \cap \{t \in (0, T_0]\}.$$

Proof. Let k be the integer such that $t_k \leq T_0 < t_{k+1}$, which implies that

$$\theta^k \leq \frac{(T_0 - t)(\theta - 1)}{c_0 \rho \theta} + 1 < \theta^{k+1}. \tag{3.10}$$

Since $c_0 \leq \hat{c}_0$ and $t_{k+1} \leq T_0 + c_0 \rho \theta^{k+1}$, by (3.7) we have

$$S(x, \frac{\rho\theta^k}{2}|T_0) \subset S(x, \frac{\rho\theta^{k+1}}{2}|t_{k+1}),$$

which by (3.10) implies

$$S(x, \frac{(T_0 - t)(\theta - 1)}{2c_0\theta^2} + \frac{\rho}{2\theta}|T_0) \subset S(x, \frac{1}{2}\rho\theta^{k+1}|t_{k+1}).$$

Since $t \leq 0$ and $\rho > 0$,

$$S(x, \frac{T_0(\theta - 1)}{2c_0\theta^2}|T_0) \subset S(x, \frac{1}{2}\rho\theta^{k+1}|t_{k+1}). \tag{3.11}$$

Because $x \in S(0, \frac{1}{2}|0)$, by the engulfing property $0 \in S(x, \frac{1}{2}\theta|0)$. Since $T_0 = \hat{c}_0\theta$, from (3.7),

$$S(x, \frac{1}{2}\theta|0) \subset S(x, \frac{1}{2}\theta^2|T_0). \tag{3.12}$$

Now let us choose $\mathfrak{c}_0$ so small that

$$\frac{\theta^2}{2} \leq \frac{T_0(\theta - 1)}{8\mathfrak{c}_0\theta^2}, \tag{3.13}$$

i.e.,

$$\frac{\hat{c}_0}{\mathfrak{c}_0} \geq \frac{4\theta^3}{\theta - 1}.$$

For $c_0 \le \mathfrak{c}_0$, from (3.12) and (3.13),

$$0 \in S(x, \frac{T_0(\theta-1)}{4c_0\theta^2}|T_0).$$

We repeat the same argument above with time $T_0 - b$ instead of T_0, where b is a small constant depending on $\theta, \mathfrak{c}_0$, and $\hat{c}_0$ so that

$$\frac{1}{2}\theta^2 \le \frac{(T_0-b)(\theta-1)}{4\mathfrak{c}_0\theta^2}. \tag{3.14}$$

This yields that for $c_0 \le \mathfrak{c}_0$,

$$0 \in S(x, \frac{(T_0-b)(\theta-1)}{4c_0\theta^2}|T_0-b).$$

By property (iii), we know that there exists a constant r_1 such that

$$S(0, r_1|T_0) \subset S(x, \frac{T_0(\theta-1)}{2c_0\theta^2}|T_0), \tag{3.15}$$

$$S(0, r_1|T_0-b) \subset S(x, \frac{(T_0-b)(\theta-1)}{2c_0\theta^2}|T_0-b). \tag{3.16}$$

It is easy to see that (3.15) combining with (3.11) yields

$$S(0, r_1|T_0) \subset S(x, \frac{1}{2}\rho\theta^{k+1}|t_{k+1}).$$

Furthermore, from (3.14) r_1 does not depend on b. Hence we may restrict $b < r_1/(\theta C_2)$ if necessary so that

$$(T_0-b, 0) \in Q((T_0, 0), \frac{r_1}{\theta}).$$

From Lemma 3.1, it yields

$$S(0, \frac{r_1}{\theta}|T_0) \subset S(0, r_1|T_0-b). \tag{3.17}$$

We claim that

$$(T_0-b, T_0] \times S(0, \frac{r_1}{\theta}|T_0) \subset (\cup_{j=0}^{\infty} K_j(z,\rho)) \cap \{t \in (0, T_0]\}. \tag{3.18}$$

Indeed, let $\tilde{k}$ be the integer such that $t_{\tilde{k}} \le T - b < t_{\tilde{k}+1}$. Therefore, by (3.17) and (3.16) we have

$$\{T_0 - b\} \times S(0, \frac{r_1}{\theta}|T_0) \subset K_{\tilde{k}+1}(z),$$

which combining with (3.9) and (3.17) yields (3.18). Now we pick a section included in the cylindrical domain $(T_0-b, T_0] \times S(0, \frac{r_1}{\theta}|T_0)$ and centered at $(T_0, 0)$ with height $r_0 = \min\{r_1/\theta, b/c_0\}$:

$$\tilde{Q}_{c_0}((T_0, 0), r_0) \subset (T_0 - c_0, T_0] \times S(0, r_0|T_0).$$

Therefore, the proof is completed. $\square$

Next we discuss the scaling property of the linearized parabolic Monge-Ampère equation. Let T be an affine transformation which normalizes $S(x_0, h|t_0)$ and define $\tilde{\phi}$ as follows

$$\tilde{\phi}(s,y) = \frac{1}{h}(\phi(t_0 + sh, T^{-1}y) - \nabla\phi(t_0, x_0)\cdot(T^{-1}y - x_0) - \phi(x_0)). \quad (3.19)$$

With a simple calculation, one can see that

$$-\tilde{\phi}_t \det D^2\tilde{\phi}(s,y) = -\frac{1}{h^d(\det T)^2}\phi_t \det D^2\phi(t,x).$$

From (1.2),

$$\frac{\lambda}{C_2} \le \det D^2\phi(t,x) \le \frac{\Lambda}{C_1}.$$

Combining the fact that $\det T \approx |S(x_0, h|t_0)|^{-1}$ with Corollary 2.2, we know that

$$\tilde{\lambda} \le -\tilde{\phi}_t \det D^2\tilde{\phi}(s,y) \le \tilde{\Lambda}, \quad (3.20)$$

where $\tilde{\lambda}, \tilde{\Lambda}$ are universal constants. Define

$$\tilde{u}(s,y) = u(t_0 + sh, T^{-1}y). \quad (3.21)$$

By a simple calculation, it is easy to check that

$$L_{\tilde{\phi}}\tilde{u} = -\frac{\tilde{u}_s}{\tilde{\phi}_s} - \mathrm{tr}((D^2\tilde{\phi})^{-1}D^2\tilde{u}) = 0.$$

We state the following version of the Alexandroff-Bakelman-Pucci-Krylov-Tso (ABP) estimate of the parabolic type, the proof of which can be found in [19] and [12].

Lemma 3.3. *Assume that u is smooth in a bowl-shape domain Q and $u \ge 0$ on $\partial_p Q$. Then*

$$\sup_Q(u^-) \le C\Big(\iint_{\Gamma(u)} |u_t \det D^2u|\, dx\, dt\Big)^{\frac{1}{d+1}},$$

where $u^- = -\min\{u, 0\}$, C is a universal constant, and $\Gamma(u) = \{(t,x) \in Q : u \le 0, u_t \le 0, D^2u \ge 0\}$.

Now we are ready to prove the following lemma.

Lemma 3.4. *Let $z_0 \in \mathbb{R}^{d+1}$ and $h > 0$. There exists a universal constant $\hat{c}_1$ such that the following statement holds: For any $c_0 \le \hat{c}_1$, if u nonnegative satisfies (1.1)–(1.2) in $\tilde{Q}_{c_0}(z_0, 2h)$ and $\inf_{\tilde{Q}_{c_0}(z_0,h)} u \le 1$, then there exists a constant $\varepsilon_0 = \varepsilon_0(d, \lambda, \Lambda, C_1, C_2, c_0)$ so that*

$$|\{u \le 8\} \cap \tilde{Q}_{c_0}(z_0, 2h)| \ge \varepsilon_0 |\tilde{Q}_{c_0}(z_0, 2h)|.$$

Proof. Let T be an affine transformation that normalizes $S(x_0, h|t_0)$ and define $\tilde{u}$ and $\tilde{\phi}$ as follows,

$$\tilde{u}(s, y) = u(t_0 + sh, T^{-1}y),$$
$$\tilde{\phi}(s, y) = \frac{1}{h}(\phi(t_0 + sh, T^{-1}y) - \nabla\phi(t_0, x_0) \cdot (T^{-1}y - x_0) - \phi(t_0, x_0)) - 1.$$

Define $T_p : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ to be

$$T_p(t, x) = (\frac{t - t_0}{h}, Tx). \tag{3.22}$$

It is sufficient to prove

$$|T_p(\{u \leq 8\} \cap \tilde{Q}_{c_0}(z_0, 2h))| \geq \varepsilon_0 |T_p(\tilde{Q}_{c_0}(z_0, 2h))|,$$

i.e.,

$$|\{\tilde{u} \leq 8\} \cap \{(-2c_0, 0) \times T(S(x_0, h|t_0))\}| \geq \varepsilon_0 |\{(-2c_0, 0) \times T(S(x_0, h|t_0))\}|.$$

By the definition of $\tilde{u}$ and $\inf_{\tilde{Q}_{c_0}(z_0,h)} u \leq 1$, it follows

$$\inf_{(-c_0,0)\times T(S(x_0,\frac{1}{2}h|t_0))} \tilde{u} \leq 1.$$

From the definition of $\tilde{\phi}$,

$$\tilde{\phi} \leq -\frac{1}{2} \quad \text{on} \quad \{0\} \times T(S(x_0, \frac{1}{2}h|t_0)).$$

Since

$$C_1 \leq -\tilde{\phi}_t \leq C_2,$$

we can choose $\hat{c}_1$ small, which depends on C_2, such that for $c_0 \leq \hat{c}_1$

$$\tilde{\phi} \leq -\frac{1}{4} \quad \text{on} \quad (-c_0, 0] \times T(S(x_0, \frac{1}{2}h|t_0)).$$

It is easy to see that $\tilde{\phi} \geq 0$ on $(-2c_0, 0] \times \partial T(S(x_0, h|t_0))$ and $\tilde{\phi} \geq -1$ on $\{-2c_0\} \times T(S(x_0, h|t_0))$. We choose a decreasing smooth function $\psi : [-2c_0, 0] \to \mathbb{R}$ such that $\psi(-2c_0) = 1$ and $\psi(t) = 0$ for $t \in [-c_0, 0]$. There exists $N_1 = N_1(c_0)$ satisfying $|\psi_t| \leq N_1$.

Let us consider $w := \tilde{u}(z) + 8(\tilde{\phi}(z) + \psi(t))$ in $T_p(\tilde{Q}_{c_0}(z_0, 2h))$, which satisfies $w \geq 0$ on $\partial_p(T_p(\tilde{Q}_{c_0}(z_0, 2h)))$. By Lemma 3.3 and noticing that

$$\inf_{(-c_0,0)\times T(S(x_0,\frac{1}{2}h|t_0))} w \leq -1,$$

we have

$$1 \leq \sup(w^-) \leq C(\iint_{\Gamma(w)} |w_t \det D^2 w| \, dx \, dt)^{\frac{1}{d+1}}. \tag{3.23}$$

By the inequality of arithmetic and geometric means, the definition of $\Gamma(w)$, and (3.20),

$$\begin{aligned}
&\iint_{\Gamma(w)} |w_t \det D^2 w|\,dx\,dt = \iint_{\Gamma(w)} |\frac{w_t \det D^2 w}{\tilde{\phi}_t \det D^2\tilde{\phi}}||\tilde{\phi}_t \det D^2\tilde{\phi}|\,dx\,dt \\
&\leq \tilde{\Lambda} \iint_{\Gamma(w)} |\frac{w_t \det D^2 w}{\tilde{\phi}_t \det D^2\tilde{\phi}}| dxdt \\
&\leq \frac{\tilde{\Lambda}}{(d+1)^{d+1}} \iint_{\Gamma(w)} |-\frac{w_t}{\tilde{\phi}_t} - \operatorname{tr}((D^2\tilde{\phi})^{-1}D^2 w)|^{d+1}\,dx\,dt. \qquad (3.24)
\end{aligned}$$

Since $w = \tilde{u} + 8(\tilde{\phi} + \psi)$, we get

$$|-\frac{w_t}{\tilde{\phi}_t} - \operatorname{tr}((D^2\tilde{\phi})^{-1}D^2 w)| = |L_{\tilde{\phi}}\tilde{u} - 8(d+1+\frac{\psi_t}{\tilde{\phi}_t})| \leq 8(d+1+\frac{N_1}{C_1}),$$

which, combining with (3.24) implies that

$$\iint_{\Gamma(w)} |w_t \det D^2 w|\,dx\,dt \leq \frac{\tilde{\Lambda}}{(d+1)^{d+1}} 8^{d+1}(d+1+\frac{N_1}{C_1})^{d+1}|\Gamma(w)|. \quad (3.25)$$

On the other hand, $\Gamma(w) \subset \{w \leq 0\}$, which implies $\Gamma(w) \subset \{\tilde{u} \leq -8(\tilde{\phi}+\psi)\}$. Moreover, it is easily seen that

$$\min_{T_p(\tilde{Q}_{c_0}(z_0,2h))} \tilde{\phi}(t_0, x) \geq -1.$$

Combining the fact that $\psi \geq 0$, we have $\Gamma(w) \subset \{\tilde{u} \leq 8\}$. From (3.23) and (3.25), we prove that

$$|\{\tilde{u} \leq 8\} \cap T_p(\tilde{Q}_{c_0}(z_0, 2h))| \geq \frac{1}{C} = \frac{|T_p(\tilde{Q}_{c_0}(z_0, 2h))|}{C|T_p(\tilde{Q}_{c_0}(z_0, 2h))|},$$

where C is universal. Because $T(S(x_0, h|t_0))$ is normalized and $T_p(\tilde{Q}_{c_0}(z_0, 2h)) = (-2c_0, 0) \times T(S(x_0, h|t_0))$, it is easy to obtain that ε_0 depending on c_0 and the structure conditions. Therefore, we prove the lemma. $\square$

Now we fix the parameter $c_0 = \min\{\mathfrak{c}_0, \hat{c}_1, \hat{c}_0/2\}$, where $\mathfrak{c}_0$ and $\hat{c}_1$ are in Lemma 3.2 and Lemma 3.4 respectively, and $\hat{c}_0$ is defined in the beginning of this section. Since $\mathfrak{c}_0$, $\hat{c}_1$, and $\hat{c}_0$ are all universal constants, so is c_0. It follows that ε_0 is universal as well. With the help of Lemma 3.1, we are ready to prove the following corollary of Lemma 3.4.

Corollary 3.5. *Assume that u is a nonnegative solution of* (1.1) *and* (1.2). *Then there exists a universal constant τ such that the following property holds: For any $z_0 = (t_0, x_0) \in \mathbb{R}^{d+1}$, $h > 0$, and $\hat{\tau} \leq \tau$, denote*

$$Z = (t_0, t_0 + \hat{\tau}h] \times S(x_0, \frac{h}{2\theta}|t_0 + \hat{\tau}h),$$

where θ is the constant in Lemma 3.1. If $\inf_Z u \leq 1$, then there exists a universal constant ε_1 such that

$$|\{u \leq 8\} \cap \tilde{Q}_{c_0}(z_0, 2h)| \geq \varepsilon_1 |\tilde{Q}_{c_0}(z_0, 2h)|.$$

Proof. First note that for $\tau < 1/(C_2\theta^2)$, we have $(t_0, x_0) \in Q((t_0+\tau h, x_0), h/\theta^2)$. Then from Lemma 3.1, we get

$$S(x_0, \frac{h}{\theta}|t_0 + \tau h) \subset S(x_0, h|t_0), \quad S(x_0, \frac{h}{\theta^2}|t_0) \subset S(x_0, \frac{h}{\theta}|t_0 + \tau h). \tag{3.26}$$

Denote $z_\tau = (t_0+\tau h, x_0)$. Second, if $\tau < c_0/\theta$, $\inf_Z u \leq 1$ implies $\inf_{\tilde{Q}(z_\tau, \frac{h}{\theta})} u \leq 1$. Hence, from Lemma 3.4, we know that

$$|\{u \leq 8\} \cap \tilde{Q}(z_\tau, \frac{2h}{\theta})| \geq \varepsilon_0 |\tilde{Q}(z_\tau, \frac{2h}{\theta})|,$$

which can be written as

$$\begin{aligned} &|\{u \leq 8\} \cap ((t_0, t_0 + \tau h] \times S(x_0, \frac{h}{\theta}|t_0 + \tau h))| \\ &+ |\{u \leq 8\} \cap ((t_0 + \tau h - c_0\frac{2h}{\theta}, t_0] \times S(x_0, \frac{h}{\theta}|t_0 + \tau h))| \\ &\geq \frac{2c_0\varepsilon_0 h}{\theta}|S(x_0, \frac{h}{\theta}|t_0 + \tau h)|. \end{aligned}$$

This implies that

$$\begin{aligned} &|\{u \leq 8\} \cap ((t_0 + \tau h - c_0\frac{2h}{\theta}, t_0] \times S(x_0, \frac{h}{\theta}|t_0 + \tau h))| \\ &\geq (\frac{2c_0\varepsilon_0 h}{\theta} - \tau h)|S(x_0, \frac{h}{\theta}|t_0 + \tau h)|. \end{aligned}$$

From (3.26), we obtain that

$$|\{u \leq 8\} \cap ((t_0 + \tau h - c_0\frac{2h}{\theta}, t_0] \times S(x_0, h|t_0))| \geq (\frac{2c_0\varepsilon_0 h}{\theta} - \tau h)|S(x_0, \frac{h}{\theta^2}|t_0)|. \tag{3.27}$$

Since $\tau > 0$ and $\theta > 1$, it follows from (3.27)

$$|\{u \leq 8\} \cap \tilde{Q}(z_0, 2h)| \geq (\frac{2c_0\varepsilon_0 h}{\theta} - \tau h)|S(x_0, \frac{h}{\theta^2}|t_0)|.$$

From property (v), we know that

$$|S(x_0, \frac{h}{\theta^2}|t_0)| \geq C(\theta)|S(x_0, h|t_0)|,$$

where $C(\theta)$ is a universal constant. Therefore, we get

$$|\{u \leq 8\} \cap \tilde{Q}(z_0, 2h)| \geq (\frac{2c_0\varepsilon_0}{\theta} - \tau)\frac{C(\theta)}{2c_0}|S(x_0, h|t_0)|2c_0 h.$$

After taking $\tau = \min\{c_0\varepsilon_0/\theta, c_0/\theta, 1/(C_2\theta^2)\}$, we can choose

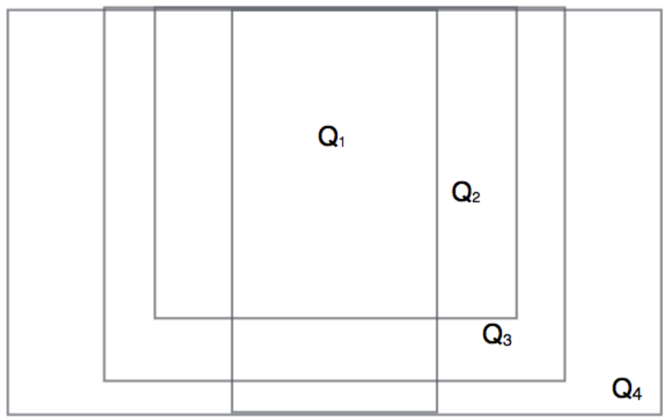
$$\varepsilon_1 = \frac{C(\theta)\varepsilon_0}{2\theta}$$

to prove the lemma. $\square$

Now we take

$$\varepsilon = \min\{\varepsilon_0, \varepsilon_1\}, \tag{3.28}$$

and Lemma 3.4 and Corollary 3.5 hold with ε in place of ε_0 and ε_1, respectively. Next we define the following four domains:



$$\begin{aligned}
Q_4 &= (t_0 - 4Kh, t_0] \times S(x_0, 2\theta h|t_0),\\
Q_3 &= (t_0 - 4(K-\beta)h, t_0] \times S(x_0, 2\theta(1-\beta)h|t_0),\\
Q_2 &= (t_0 - 4(K-\sigma)h, t_0] \times S(x_0, \theta h|t_0),\\
Q_1 &= (t_0 - 4Kh, t_0] \times S(x_0, \frac{1}{2}h|t_0).
\end{aligned}$$

where $K > \sigma > \beta > 0, \beta < \frac{1}{2}$.

Lemma 3.6. *Let K, σ, β, and $Q_i, i = 1, 2, 3, 4$, be as above and $K \leq c_0\theta$. Suppose that u is a nonnegative solution of* (1.1) *and* (1.2) *in Q_4. Then there exists a small universal constant $\gamma_0 > 0$ such that if $K - \beta \leq \gamma_0$, then the following property holds: If $\inf_{Q_1} u \geq 1$, then $\inf_{Q_2} u \geq \frac{1}{L}$, where L depends on K, β, and the structure conditions.*

Proof. Let T be an affine transformation that normalizes $S(x_0, 2\theta h|t_0)$ and $T_p : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ be as follows,

$$T_p(s, y) = (\frac{s - t_0}{h}, Ty).$$

Define $\tilde{\phi}$ and $\tilde{u}$ as in (3.19) and (3.21) respectively and $Z_i = T_p(Q_i)$ for $i = 1, 2, 3, 4$. Denote $y_0 = Tx_0$ and for $l, h > 0$, $S^*(y_0, l|0) = T(S(x_0, lh|t_0))$, for instance $S^*(y_0, 2\theta|0) = T(S(x_0, 2\theta h)|t_0)$. Moreover, it is obvious that S^* is the section of $\tilde{\phi}$ in the x variable.

Since $S^*(y_0, 2\theta|0)$ is normalized, from [9, Corollary 3.3.6],

$$\operatorname{dist}(\partial S^*(y_0, 2\theta|0), S^*(y_0, 2\theta(1-\beta)|0)) \geq C(\beta), \tag{3.29}$$

where $C(\beta)$ is a constant depending on β.

Consider $H_\eta = \{z \in Z_3 : \Delta\tilde{\phi} \geq \frac{1}{\eta}\}$ and it is easy to show that

$$\begin{aligned} |H_\eta| &\leq \eta \int_{-4(K-\beta)}^{0} \int_{S^*(y_0, 2\theta(1-\beta)|0)} \Delta\tilde{\phi}\, dx\, dt \\ &\leq \eta \int_{-4(K-\beta)}^{0} \int_{\partial S^*(y_0, 2\theta(1-\beta)|0)} |\nabla\tilde{\phi}|\, ds\, dt, \end{aligned} \tag{3.30}$$

where $|H_\eta|$ is the Lebesgue measure of H_η. Since we restrict $K \leq \theta c_0$, by (3.6) we have

$$Z_4 \subset Q_{\tilde{\phi}}((0, y_0), 4\theta).$$

From (3.29), for each $t \in (-4(K-\beta), 0)$,

$$\operatorname{dist}(S^*(y_0, 2\theta(1-\beta)|0), \partial Q_{\tilde{\phi}}((0, y_0), 4\theta)(t)) \geq C(\beta). \tag{3.31}$$

After subtracting 4θ, we obtain $(\tilde{\phi} - 4\theta)|_{\partial_P(Q_{\tilde{\phi}}((0,y_0),4\theta))} \leq 0$. By Lemma 2.1 and (3.31), we find that there exists a constant $C(\beta)$ depending on β such that

$$|\nabla\tilde{\phi}| \leq C(\beta) \quad \text{on} \quad (-4(K-\beta), 0) \times \partial S^*(y_0, 2\theta(1-\beta)|0).$$

Combining (3.30) with the fact that $S^*(y_0, 2\theta|0)$ is normalized, we obtain

$$|H_\eta| \leq C(\beta)(K-\beta)\eta.$$

Let

$$g = -\tilde{\phi}_t \det D^2\tilde{\phi}.$$

Set $\tilde{H}_\eta$ be an open subset in Z_4 such that $\overline{H}_\eta \subset \tilde{H}_\eta$ and $|\tilde{H}_\eta \setminus H_\eta| \leq \eta$. Take $f_\eta(x)$ to be a smooth function such that $f_\eta \in C_0^\infty(\tilde{H}_\eta)$, $0 \leq f_\eta \leq 1$, and $f_\eta = 1$ on H_η. Consider the boundary value problem for the following parabolic Monge-Ampère equation:

$$\begin{aligned} -w_t \det D^2 w = f_\eta g/(\alpha)^{d+1} \quad &\text{in} \quad (-4K, 0) \times S^*(y_0, 2\theta|0), \\ w = \zeta(x) - \varepsilon(t + 4K) \quad &\text{on} \quad \partial_p((-4K, 0) \times S^*(y_0, 2\theta|0)). \end{aligned}$$

Here $\varepsilon > 0$ is a small constant, α is a large constant to be determined later, and ζ is a strictly convex function in x with $|\zeta(x)| < \varepsilon, \zeta|_{\partial S^*(y_0, 2\theta|0)=0}$. From the existence result in [21], we know that the equation above has a unique classical solution $w \in C^{1,2}(Z_4) \cap C(\overline{Z_4})$, which is parabolic convex. By applying Lemma 3.3 to $w - \inf_{\partial_p Z_4} w$, we obtain that

$$\begin{aligned} \sup_{Z_4}(w^-) \leq& \sup_{\partial_p Z_4}(w^-) + C(\alpha, d)(\iint_{Z_4} f_\eta g\, dx\, dt)^{\frac{1}{d+1}} \\ \leq& 4K\varepsilon + C(\tilde{\Lambda}, \alpha, d)|\tilde{H}_\eta| \leq 4K\varepsilon + C(\tilde{\Lambda}, \alpha, d)\eta, \end{aligned} \tag{3.32}$$

where we use the fact that $\tilde{\lambda} \leq g \leq \tilde{\Lambda}$ and $C(\tilde{\Lambda}, \alpha, d)$ is a constant depending on Λ, α, and d.

Now let us recall some properties of $\tilde{\phi}$. Since $0 \leq \tilde{\phi} \leq 1/2$ in $\{0\} \times S^*(y_0, \frac{1}{2}|0)$ and $K \leq c_0\theta$, we can choose a large constant $C_3 = C_3(K, C_1, C_2)$

such that $0 \le \tilde{\phi}/C_3 \le 1/2$ in $(-4K, 0) \times S^*(y_0, \frac{1}{2}|0)$. Then we take $\alpha = C_3$ and consider

$$\min_{Z_4}\{\tilde{u}^\delta - (w - \frac{\tilde{\phi}}{\alpha})\},$$

where $\delta > 0$ to be determined later. Assume that the minimum is attained at $P = (t_p, x_p)$.

First, suppose that $P \in Z_1$ and it is obvious that

$$\tilde{u}^\delta(t,x) - (w(t,x) - \frac{\tilde{\phi}(t,x)}{\alpha}) \ge \tilde{u}^\delta(P) - (w(P) - \frac{\tilde{\phi}(P)}{\alpha}).$$

Since $\inf_{Z_1} u \ge 1$, from (3.32) the inequality above yields

$$\begin{aligned}\tilde{u}^\delta(t,x) &\ge \tilde{u}^\delta(P) + (w(t,x) - w(P)) + \frac{\tilde{\phi}(P) - \tilde{\phi}(t,x)}{\alpha} \\ &\ge 1 - \frac{1}{2} - 2(4K\varepsilon + C\eta) = \frac{1}{2} - C(K)(\varepsilon + \eta),\end{aligned}$$

where $C(K)$ depends on K and the structure conditions. For ε and η sufficiently small, we get $\tilde{u}(t,x) \ge (\frac{1}{4})^{1/\delta}$.

Second, assume $P \in Z_4 \setminus Z_3$. In this case, we know that

$$\tilde{u}^\delta(t,x) - (w(t,x) - \frac{\tilde{\phi}(t,x)}{\alpha}) \ge -(w(P) - \frac{\tilde{\phi}(P)}{\alpha}).$$

It follows from (3.32) that

$$\tilde{u}^\delta(t,x) \ge \frac{\tilde{\phi}(P) - \tilde{\phi}(t,x)}{\alpha} - C(K)(\varepsilon + \eta). \tag{3.33}$$

We restrict $(t,x) \in Z_2$. Because $C_1 \le -\tilde{\phi}_t \le C_2$,

$$\max_{z \in Z_2} \tilde{\phi} = \tilde{\phi}(-4(K-\sigma), \tilde{x}) \le \theta + 4C_2(K - \sigma),$$

where $\tilde{x} \in \partial S^*(y_0, \theta|t_0)$, and

$$\min_{z \in Z_4 \setminus Z_3} \tilde{\phi} = \tilde{\phi}(0, \hat{x}) = 2\theta(1-\beta),$$

where $\hat{x} \in S^*(y_0, 2\theta(1-\beta)|0)$. Therefore,

$$\tilde{\phi}(P) - \tilde{\phi}(t,x) \ge 2\theta(1-\beta) - (\theta + 4C_2(K-\sigma)) = (1-2\beta)\theta - 4C_2(K-\sigma). \tag{3.34}$$

Then from (3.33) and (3.34), it yields that

$$u^\delta(t,x) \ge \frac{(1-2\beta)\theta}{\alpha} - \frac{4C_2(K-\sigma)}{\alpha} - C(K)(\varepsilon + \eta).$$

Since $K - \sigma < K - \beta \le \gamma_0$, by taking γ_0, ε and η sufficiently small and $\beta < \frac{1}{4}$, we prove that $u(t,x) \ge 1/L$ for the second case, where L depends on β, K, and the structure conditions.

Finally, let us consider the last case that $P \in Z_3 \setminus Z_1$. Notice that at P

$$D(\tilde{u}^\delta)(P) = Dw(P) - \frac{D\tilde{\phi}(P)}{\alpha}, \quad D^2(\tilde{u}^\delta)(P) \geq D^2 w(P) - \frac{D^2\tilde{\phi}(P)}{\alpha},$$

$$\partial_t \tilde{u}^\delta(P) \leq \partial_t w(P) - \frac{\partial_t \tilde{\phi}(P)}{\alpha}.$$

Therefore,

$$\begin{aligned} L_{\tilde{\phi}}\tilde{u}^\delta(P) &\leq \frac{\partial_t w(P) - \partial_t\tilde{\phi}(P)/\alpha}{-\partial_t \tilde{\phi}(P)} - \operatorname{tr}((D^2\tilde{\phi})^{-1} D^2(w - \frac{\tilde{\phi}}{\alpha}))(P) \\ &= \frac{d+1}{\alpha} + L_{\tilde{\phi}} w(P). \end{aligned} \tag{3.35}$$

Moreover, since $-w_t \det D^2 w = f_\eta g/(\alpha)^{d+1}$, by the inequality of arithmetic and geometric means

$$L_{\tilde{\phi}} w \leq \frac{-(d+1) f_\eta^{1/(d+1)}}{\alpha}. \tag{3.36}$$

On the other hand, following the proof of [5, Lemma 2.1], we can show that

$$L_{\tilde{\phi}}(\tilde{u}^\delta) \geq \frac{1-\delta}{\delta \tilde{u}^\delta} \frac{|D(\tilde{u}^\delta)|^2}{\Delta\tilde{\phi}},$$

in particular,

$$L_{\tilde{\phi}}(\tilde{u}^\delta)(P) \geq \frac{1-\delta}{\delta \tilde{u}^\delta(P)} \frac{|D(\tilde{u}^\delta)(P)|^2}{\Delta\tilde{\phi}(P)},$$

which, with (3.35) and (3.36), implies that

$$\frac{d+1}{\alpha}(1 - f_\eta^{1/(d+1)}(P)) \geq \frac{1-\delta}{\delta \tilde{u}(P)^\delta} \frac{|Dw(P) - D\tilde{\phi}(P)/C_3|^2}{\Delta\tilde{\phi}(P)}. \tag{3.37}$$

For the right-hand side of the inequality above, applying Lemma 2.1 to w, we find that

$$|Dw(P)| \leq \frac{-w(P)}{\operatorname{dist}(P, \partial Z_4(t_P))}.$$

Because $P \in Z_3$, $\operatorname{dist}(P, \partial Z_4(t_P)) \geq C(\beta)$. Combining with the fact that $|w(P)| \leq C(K)(\varepsilon + \eta)$, we find

$$|\nabla w(P)| \leq C(K, \beta)(\varepsilon + \eta).$$

Next we show that $|D\tilde{\phi}(P)|$ has a lower bound for $P \in Z_3 \setminus Z_1$. Recall that $\nabla\tilde{\phi}(0, y_0) = 0$ and

$$Q_{\tilde{\phi}}((0, y_0), h) = \{z : \tilde{\phi}(z) \leq \tilde{\phi}(0, y_0) + h,\, t \leq 0\}.$$

We choose γ_0 so small depending on C_2 that $(-4(K-\beta), 0) \times \{y_0\} \subset Q_{\tilde{\phi}}((0, y_0), 1/8)$ and certainly $(-4(K-\beta), 0) \times \{y_0\} \subset Q_{\tilde{\phi}}((0, y_0), 1/4)$. Then for each $s \in (-4(K-\beta), 0)$, let us consider

$$\min_{Q_{\tilde{\phi}}((0,y_0),1/4) \cap \{t=s\}} \tilde{\phi}. \tag{3.38}$$

One can easily show that the point q_s, where (3.38) is attainted, must be in $Q_{\tilde{\phi}}((0, y_0), 1/8) \cap \{t = s\}$. Since $\nabla\tilde{\phi}(s, q_s) = 0$, it follows immediately that

$$S^*(q_s, \frac{1}{8}|s) \subset Q_{\tilde{\phi}}((0, y_0), \frac{1}{4}) \cap \{t = s\}.$$

Because

$$Q_{\tilde{\phi}}((0, y_0), \frac{1}{4}) \cap \{t \in (-4(K-\beta), 0)\} \subset (-4(K-\beta), 0) \times S^*(y_0, \frac{1}{2}|0),$$

for $(t, x) \in Z_3 \setminus Z_1$, it follows that $x \notin S^*(q_t, 1/8|t)$. This indicates that $|D\tilde{\phi}(t, x)| \geq C$, where C is universal, i.e., $|D\tilde{\phi}(P)|$ has a lower bound. From (3.37), we get

$$\frac{d+1}{\alpha}(1 - f_\eta^{1/(d+1)}(P)) \geq \frac{1-\delta}{\delta\tilde{u}^\delta(P)} \frac{|C - C(\beta, K)(\varepsilon + \eta)|^2}{\Delta\tilde{\phi}(P)}.$$

Hence, for ε, η sufficiently small, the right-hand side of the inequality above is strictly positive, which means that f_η cannot be 1, i.e., $P \notin H_\eta$. In other words, $\Delta\tilde{\phi}(P) \leq 1/\eta$. We modify the inequality above to obtain

$$\tilde{u}^\delta(P) \geq \frac{1-\delta}{\delta} C(K, \beta).$$

Since P is the minimum point of $\tilde{u}^\delta - (w - \tilde{\phi}/\alpha)$, we get

$$\begin{aligned} \tilde{u}^\delta(t, x) &\geq \tilde{u}^\delta(P) + (w(t, x) - \frac{\tilde{\phi}(t, x)}{\alpha}) - (w - \frac{\tilde{\phi}}{\alpha})(P) \\ &\geq \frac{1-\delta}{\delta} C(K, \beta) - 2 \sup_{z \in Z_3}(|w| + \frac{|\tilde{\phi}|}{\alpha}). \end{aligned}$$

Note that we have the upper bounds for $|w|$ and $|\tilde{\phi}|$. By taking δ sufficiently small, we obtain the lower bound for $\tilde{u}$ in Z_2. Hence, the last case is proved and so is the lemma. □

Denote $\tau_0 = \min\{\tau, \gamma_0/2\}$, where τ and γ_0 are the constants in Corollary 3.5 and Lemma 3.6, respectively. Combining Lemma 3.4, Corollary 3.5, and Lemma 3.6, we get the following theorem.

Theorem 3.7. *Assume that u is a nonnegative solution of* (1.1) *and* (1.2). *Let $h > 0, z_0 = (t_0, x_0)$, and $z' = (t_0 + c_0\theta h, x_0)$. Suppose that*

$$\inf_{\tilde{Q}_{c_0}(z', \theta h)} u \leq 1,$$

then there exists a universal constant M_0 such that

$$|\{z \in \tilde{Q}_{c_0}(z_0, h) : u(z) \leq M_0\}| \geq \varepsilon|\tilde{Q}_{c_0}(z_0, h)|, \tag{3.39}$$

where ε is the constant in (3.28).

Proof. By a scaling and translation of the coordinates, we assume $h = 1$ and $z_0 = (0, 0)$. We prove the theorem by contradiction. If (3.39) does not hold, then applying Lemma 3.4 to $8u/M_0$, we get

$$\inf_{\tilde{Q}(\frac{1}{2})} u \geq \frac{M_0}{8}. \tag{3.40}$$

By Corollary 3.5 with $h = 1/4$, it follows that

$$\inf\{u(z) : z \in (0, \frac{\tau_0}{4}] \times S(0, \frac{1}{8\theta}|\frac{\tau_0}{4})\} \geq \frac{M_0}{8^2}. \tag{3.41}$$

Since $\tau_0 \leq \tau$, where τ is the constant in Corollary 3.5, and notice that in the proof of Corollary 3.5 we restrict $\tau < c_0/\theta$, by Lemma 3.1 we have

$$S(0, \frac{1}{8\theta}|\frac{\tau_0}{4}) \subset S(0, \frac{1}{8}|0) \subset S(0, \frac{1}{4}|0). \tag{3.42}$$

From (3.40), (3.41), and (3.42) we get

$$\inf\{u(z) : z \in (-\frac{1}{2}c_0, \frac{\tau_0}{4}] \times S(0, \frac{1}{8\theta}|\frac{\tau_0}{4})\} \geq \frac{M_0}{8^2}.$$

Then applying Lemma 3.6 with

$$Q_1 = (-\frac{\tau_0}{4}, \frac{\tau_0}{4}] \times S(0, \frac{1}{8\theta}|\frac{\tau_0}{4}),$$
$$Q_2 = (0, \frac{\tau_0}{4}] \times S(0, \frac{1}{4}|\frac{\tau_0}{4}),$$

and $\sigma = 3\beta/4 = K/2 = \tau_0/16$, we obtain

$$\inf\{u(z) : z \in (0, \frac{\tau_0}{4}] \times S(0, \frac{1}{4}|\frac{\tau_0}{4})\} \geq \frac{M_0}{8^2 L_1},$$

where L_1 is universal. Next, we claim that there exists a universal constant $L \geq L_1$ so that

$$\inf_{\tilde{Q}_{c_0}((\frac{\tau_0}{4}, 0), \frac{1}{2})} u \geq \frac{M_0}{8^2 L^2}. \tag{3.43}$$

It is sufficient to prove

$$\inf\{u(z) : z \in (\frac{\tau_0}{4} - \frac{1}{2}c_0, 0] \times S(0, \frac{1}{4}|\frac{\tau_0}{4})\} \geq \frac{M_0}{8^2 L^2}.$$

By Lemma 3.1, for any $t \in (\frac{\tau_0}{4} - \frac{1}{2}c_0, 0]$

$$S(0, \frac{1}{4}|\frac{\tau_0}{4}) \subset S(0, \frac{\theta}{4}|t). \tag{3.44}$$

On the other hand, since $\{t\} \times S(0, \frac{1}{4\theta}|t) \subset \tilde{Q}(\frac{1}{2})$ and from (3.40)

$$\inf\{u(z) : z \in \{t\} \times S(0, \frac{1}{4\theta}|t)\} \geq \frac{M_0}{8}.$$

Denote $\hat{Q}_1 = (t - \frac{\tau_0}{4}, t] \times S(0, \frac{1}{4\theta}|t)$ and $\hat{Q}_2 = (t - \frac{\tau_0}{8}, t] \times S(0, \frac{1}{2}|t)$. We apply Lemma 3.6 to $\hat{Q}_1$ and $\hat{Q}_2$, and obtain

$$\inf\{u(z) : z \in (t - \frac{\tau_0}{8}, t] \times S(0, \frac{1}{2}|t)\} \geq \frac{M_0}{8L_2}.$$

Denote $\hat{Q}_3 = (t - \frac{\tau_0}{16}, t] \times S(0, \theta|t)$. Then we apply Lemma 3.6 again to $\hat{Q}_2$ and $\hat{Q}_3$

$$\inf\{u(z) : z \in (t - \frac{\tau_0}{16}, t] \times S(0, \theta|t)\} \geq \frac{M_0}{8L_2L_3}.$$

It follows from the inequality above and (3.44) that

$$\inf\{u(z) : z \in \{t\} \times S(0, \frac{1}{4}|\frac{\tau_0}{4})\} \geq \frac{M_0}{8L_2L_3}.$$

Therefore, we prove the claim by taking $L = \max\{L_1, L_2, L_3\}$.

For convenience, we denote $M_1 = 8^2L^2$ and $t_k = k\tau_0/4$ for $k \geq 0$. Then it is easy to apply an induction argument to show that

$$\inf\{u(z) : z \in \tilde{Q}_{c_0}((t_k, 0), \frac{1}{2})\} \geq \frac{M_0}{M_1^k}. \tag{3.45}$$

Let k_0 be an integer such that $t_{k_0-1} \leq c_0\theta < t_{k_0}$. Next we claim that there exists a universal constant $\hat{L}$ so that

$$\inf_{\tilde{Q}_{c_0}((c_0\theta,0),\theta)} u \geq M_2^{k_0}\hat{L}.$$

For any $t \in (0, c_0\theta]$, by Lemma 3.1

$$S(0, \theta|c_0\theta) \subset S(0, \theta^2|t).$$

There exists a $j \leq k_0$ such that $t_{j-1} \leq t < t_j$. We first consider $t_{j-1} < t < t_j$. From (3.45) and Lemma 3.1,

$$\inf\{u(z) : z \in (t_j - \frac{c_0}{2}, t] \times S(0, \frac{1}{4\theta}|t)\} \geq \frac{M_0}{M_1^j}. \tag{3.46}$$

By our choice of τ_0 ($\tau_0 < c_0$), we have $t - (t_j - c_0/2) \geq \tau_0/4$. Similar to the proof of (3.43), we apply Lemma 3.6 repeatedly. Denote $\mathcal{Q}_1 = (t - \tau_0/4, t] \times S(0, 1/(4\theta)|t)$ and $\mathcal{Q}_2 = (t - \tau_0/8, t] \times S(0, 1/2|t)$ and apply Lemma 3.6 to $\mathcal{Q}_1$ and $\mathcal{Q}_2$ to get

$$\inf\{u(z) : z \in (t - \frac{\tau_0}{8}, t] \times S(0, \frac{1}{2}|t)\} \geq \frac{M_0}{M_1^j\hat{L}_1}.$$

By induction, it is easy to see that

$$\inf\{u(z) : z \in (t - \frac{\tau_0}{2^i}, t] \times S(0, \frac{(2\theta)^{i-1}}{2}|t)\} \geq \frac{M_0}{M_1^j\hat{L}_1 \cdots \hat{L}_i}$$

for any $i \geq 1$, where each $\hat{L}_j$ is universal. Then for i_0 so that $(2\theta)^{i_0-1}/2 \geq \theta^2$, we have

$$\inf\{u(z) : z \in \{t\} \times S(0, c_0\theta^2|t)\} \geq \frac{M_0}{M_1^j\hat{L}_1 \cdots \hat{L}_{i_0}}.$$

Therefore, for any $j \leq k_0$ and $t \in (t_{j-1}, t_j)$

$$\inf\{u(z) : z \in \{t\} \times S(0, c_0\theta^2|t)\} \geq \frac{M_0}{M_1^{k_0}\hat{L}_1 \cdots \hat{L}_{i_0}}. \tag{3.47}$$

For the case when $t = t_{j-1}$, instead of (3.46), we have

$$\inf\{u(z) : z \in \tilde{Q}_{c_0}((t_{j-1}, 0), \frac{1}{2})\} \geq \frac{M_0}{M_1^{j-1}}.$$

The rest of the proof is the same. We combine (3.47) and (3.46) to get that

$$\inf_{\tilde{Q}_{c_0}((c_0\theta,0),\theta)} u \geq \frac{M_0}{M_1^{k_0}\hat{L}_1 \cdots \hat{L}_{i_0}},$$

which proves the claim. Moreover, this gives us an contradiction with

$$\inf_{\tilde{Q}_{c_0}((c_0\theta,0),\theta)} u \leq 1$$

if $M_0 > M_1^{k_0}\hat{L}_1 \cdots \hat{L}_{i_0}$. Therefore, we prove the lemma.

□

With the help of Theorem 3.7, we obtain the following lemma.

Lemma 3.8. *Assume all the conditions in Lemma 3.2 with* $z_0 = (0, 0)$ *and*

$$\inf_{\tilde{Q}_{c_0}((T_0,0),r_0)} u \leq 1.$$

For any $z \in \tilde{Q}(1)$, $\rho > 0$, *and* $M > 0$ *such that*

$$|\tilde{Q}_{c_0}(z, \rho) \cap \{u \geq M\}| \geq (1 - \varepsilon)|\tilde{Q}_{c_0}(z, \rho)|,$$

where ε *is defined in* (3.28)*, then*

$$\rho \leq CM^{-\sigma},$$

where C *and* σ *are universal constants.*

Proof. Let $t_k = t + \sum_{j=1}^{k} c_0\rho\theta^j$ and k^* be the integer such that $t_{k^*} \geq T_0 > t_{k^*-1}$, which implies that $\tilde{Q}_{c_0}((t_k, x), \theta^k\rho) \cap \{t = T_0\} \neq \emptyset$. This yields that

$$\log_\theta(1 + \frac{(T_0 - t)(\theta - 1)}{c_0\theta\rho}) \leq k^* \leq 1 + \log_\theta(1 + \frac{(T_0 - t)(\theta - 1)}{c_0\theta\rho}). \tag{3.48}$$

Applying Theorem 3.7 to uM_0/M, we obtain $u \geq M/M_0$ in $\tilde{Q}_{c_0}((t_1, x_0), \theta\rho)$. Implementing Theorem 3.7 repeatedly, we get

$$u \geq \frac{M}{M_0^{k^*}} \quad \text{in} \quad \cup_{j=1}^{k^*} K_j(z), \tag{3.49}$$

where $K_j(z)$ is defined in (3.8). From Lemma 3.2, we know that

$$\tilde{Q}_{c_0}((T_0, 0), r_0) \subset (\cup_{j=0}^{\infty} K_j(z)) \cap \{t \in (0, T_0]\}. \tag{3.50}$$

Since

$$\inf_{\tilde{Q}_{c_0}((T_0,0),r_0)} u \leq 1,$$

from (3.49) and (3.50),

$$M \leq M_0^{k^*}.$$

From the estimate of k^* in (3.48), the fact that $-c_0 \le t \le 0$, and the inequality above, by a simple calculation, we get

$$\rho \le \frac{(c_0 + \hat{c}_{c_0})(\theta - 1)(eM_0)^{\log\theta}}{c_0 M^{\log\theta}} = CM^{-\sigma},$$

where C and σ are universal constants. □

4. Calderón-Zygmund Decomposition

In this section, we obtain a Calderón-Zygmund type decomposition with respect to our parabolic sections, which is a necessary tool in establishing a power decay of the distribution function. As noted in [7], based on the strong geometrical properties provided in Section 2, for fixed t, $s(x, h|t)$ satisfies the list of axioms in [18, Section 1.1], so several theorems in real analysis hold using $S(x, h|t)$ in place of Euclidean balls. In this section, we show our parabolic sections satisfy similar properties so that these theorems hold for the parabolic sections as well. For the Besides our parabolic section $\tilde{Q}_{c_0}(z, r)$, we use the following domains in our proof. Let $m \ge 1$ be an integer. Set

$$\begin{aligned}
\tilde{Q}^{(m)}_{c_0}(z, h) &= (t + (m-2)c_0 h, t + (m-1)c_0 h] \times S(x_0, \frac{1}{2}h|t),\\
\overline{\tilde{Q}^{m}_{c_0}}(z, h) &= \cup_{i=1}^{m+1} \tilde{Q}^{(i)}_{c_0}(z, h) = (t - c_0 h, t + mc_0 h] \times S(x_0, \frac{1}{2}h|t),\\
\tilde{Q}^{m}_{c_0}(z, h) &= \cup_{i=2}^{m+1} \tilde{Q}^{(i)}_{c_0}(z, h) = (t, t + mc_0 h] \times S(x_0, \frac{1}{2}h|t).
\end{aligned}$$

We denote $\tilde{Q}^{\delta}_{c_0}(z, h) = (t - c_0 h, t + \delta h) \times S(x, h/2|t)$, where $\delta \le c_0$ is a small parameter to be determined later. Then we prove the following lemma which is an important ingredient in proving the Calderón-Zygmund decomposition.

Lemma 4.1. *There exist positive universal constants K_1, K_2, K_3, and ε_2 with the following property: Given two sections $\tilde{Q}_{c_0}(z_0, r_0), \tilde{Q}_{c_0}(z, r)$ with $r \le r_0$, T an affine transformation that normalizes $S(x_0, r_0/2|t_0)$, and T_p is defined in* (3.22)*, if*

$$\tilde{Q}^{\delta}_{c_0}(z_0, r_0) \cap \tilde{Q}^{\delta}_{c_0}(z, r) \ne \emptyset,$$

then

$$\begin{aligned}
(t' - c_0\frac{r}{r_0}, t' + \delta\frac{r}{r_0}) \times B(x', K_1(\frac{r}{r_0})) &\subset T_p(\tilde{Q}^{\delta}_{c_0}(z, r))\\
&\subset (t' - c_0\frac{r}{r_0}, t' + \delta\frac{r}{r_0}) \times B(x', K_2(\frac{r}{r_0})^{\varepsilon_2}),
\end{aligned}$$

where

$$T_p(z) = (t', x'),$$

and $T_p(z) \in (-c_0 - \delta, c_0 + \delta) \times B(0, K_3)$.

Proof. Upon taking a translation of the coordinates, without loss of generality, we assume $z_0 = (0,0)$ and hence $T_p(s,y) = (s/r_0, Ty)$. First we consider the case when $t < 0$. Since

$$\tilde{Q}^\delta_{c_0}(0,r_0) \cap \tilde{Q}^\delta_{c_0}(z,r) \neq \emptyset,$$

obviously $t \in (-c_0 r_0 - \delta r, 0)$ and

$$S(x, \frac{1}{2}r|t) \cap S(0, \frac{1}{2}r_0|0) \neq \emptyset.$$

By Lemma 3.1, $\delta \leq c_0$, and as we hinted in the beginning of Section 3 $c_0 \leq \hat{c}_0/2$ so that $c_0 + \delta \leq \hat{c}$ we have

$$S(0, \frac{r_0}{2}|0) \subset S(0, \frac{\theta r_0}{2}|t), \tag{4.51}$$

which implies

$$S(x, \frac{1}{2}r|t) \cap S(0, \frac{\theta}{2}r_0|t) \neq \emptyset. \tag{4.52}$$

Because $T(S(0, \frac{1}{2}r_0|0))$ is normalized, i.e., $\exists\, a_d > 0$ such that

$$B(0, a_d) \subset T(S(0, \frac{1}{2}r_0|0)) \subset B(0,1).$$

Then there exist $0 < b_d < c_d$ such that

$$B(0, b_d) \subset T(S(0, \frac{\theta}{2}r_0|t)) \subset B(0, c_d). \tag{4.53}$$

In fact, by Lemma 3.1

$$S(0, \frac{\theta}{2}r_0|t) \subset S(0, \frac{\theta^2}{2}r_0|0),$$

and from property (v) there exists a constant λ_0 large such that

$$T(S(0, \frac{\theta^2}{2}r_0|0)) \subset T(\lambda_0 S(0, \frac{1}{2}r_0|0)) \subset B(0, \lambda_0).$$

Hence,

$$T(S(0, \frac{\theta}{2}r_0|t)) \subset T(S(0, \frac{\theta^2}{2}r_0|0)) \subset B(0, \lambda_0).$$

In this way, we find $c_d = \lambda_0$. On the other hand, since $S(0, r_0/2|0) \subset S(0, \theta r_0/2|t)$, we take $b_d = a_d$. From (4.52), (4.53), and [9, Theorem 3.3.8], we know that

$$B(Tx, K_1\frac{r}{\theta r_0}) \subset T(S(x, \frac{1}{2}r|t)) \subset B(Tx, K_2(\frac{r}{\theta r_0})^{\varepsilon_2}),$$

and $Tx \in B(0, K_3)$. Therefore,

$$\begin{aligned}(\frac{t - c_0 r}{r_0}, \frac{t+\delta r}{r_0}) \times B(Tx, K_1\frac{r}{\theta r_0}) &\subset T_p(\overline{\tilde{Q}^\delta}(z,r)) \\ &\subset (\frac{t - c_0 r}{r_0}, \frac{t+\delta r}{r_0}) \times B(Tx, K_2(\frac{r}{\theta r_0})^{\varepsilon_2}).\end{aligned}$$

Since $t \in (-c_0 r_0 - \delta r, 0)$, let $x' = Tx$, $t' = t/r_0 \in (-c_0 - \delta, 0)$. We prove this case.

Next, we consider the case when $t \geq 0$. Obviously, $t \in [0, \delta r_0 + c_0 r]$. Combining the fact that $r \leq r_0$ and $\delta \leq c_0$, we find that (4.51) still holds in this case. The rest of the proof follows exactly the previous case. Therefore, we prove the lemma. □

Denote $K^\delta(z, r) = (t - c_0 r, t + \delta r) \times B(x, r/2)$ and we have the following Lemma.

Lemma 4.2. *There exists a small constant $c > 0$ depending on δ and the structure conditions such that the following holds: let $z \notin \tilde{Q}^\delta_{c_0}(z_0, r)$ and T_p is an affine transformation that normalizes $\tilde{Q}_{c_0}(z_0, r)$, then*

$$K^\delta(T_p(z), c\varepsilon^d) \cap T_p(\tilde{Q}^\delta_{c_0}(z_0, (1-\varepsilon)r)) = \emptyset, \quad \textit{for } 0 < \varepsilon < 1.$$

Proof. First if $t \notin (t_0 - c_0 r, t_0 + \delta r)$, we only need to consider that the intersection is empty in the t variable. Indeed,

$$T_p(\tilde{Q}^\delta_{c_0}(z_0, (1-\varepsilon)r)) = (-c_0(1-\varepsilon), \delta(1-\varepsilon)) \times T(S(x_0, \frac{1-\varepsilon}{2} r | t_0)).$$

It is easy to find $c < \delta/c_0$ such that

$$K^\delta(T_p(z), c\varepsilon^d) \cap T_p(\tilde{Q}^\delta_{c_0}(z_0, (1-\varepsilon)r)) = \emptyset.$$

If $t \in (t_0 - c_0 r, t_0 + \delta r)$, then $x \notin S(x_0, r/2|t_0)$. By [9, Corollary 3.3.6], there exists $c > 0$ such that

$$B(T(x), c\varepsilon^d) \cap T(S(x_0, \frac{1-\varepsilon}{2} r | t_0)) = \emptyset.$$

Hence, we prove the lemma. □

Now we are ready to prove a Besicovitch's type covering lemma with respect to $\tilde{Q}^\delta_{c_0}(z, h)$. For the covering lemma in a metric setting, please see [8].

Lemma 4.3. *Let O be a bounded set. Suppose that for each $z \in O$ a section $\tilde{Q}^\delta_{c_0}(z, h)$ is given such that $h \leq M$, where M is fixed. Denote by F this family of parabolic sections. Then there exists a countable subfamily of F, $\{\tilde{Q}^\delta_{c_0}(z_k, h_k)\}_{k=1}^\infty$, with the following properties:*

(i) $O \subset \cup_{k=1}^\infty \tilde{Q}^\delta_{c_0}(z_k, h_k)$.

(ii) $z_k \notin \cup_{k<j} \tilde{Q}^\delta_{c_0}(z_j, h_j)$, $\forall k \geq 2$.

(iii) For $\varepsilon > 0$ small and universal, we have that the family $F_\varepsilon = \{\tilde{Q}^\delta_{c_0}(z_k, (1-\varepsilon)h_k)\}_{k=1}^\infty$ has bounded overlap. More precisely

$$\sum_{k=1}^\infty \chi_{\tilde{Q}^\delta_{c_0}(z_k, (1-\varepsilon)h_k)}(z) \leq C_0 \log(1/\varepsilon),$$

where C_0 depends on δ, and the structure conditions; χ_E denotes the characteristic function of E.

Proof. We follow the lines of the proof of [11, Lemma 2.1] and [4, Lemma 1], though some details are different. Following the same process as in [11] and [4], we construct a sequence of the families of the parabolic sections $\{\mathcal{F}_i'\}$ for $i \geq 0$. Precisely, assume $N = \sup\{r : \tilde{Q}^\delta_{c_0}(z,r) \in F\}$. Let

$$\mathcal{F}_0 = \{\tilde{Q}^\delta_{c_0}(z,r) : \frac{N}{2} < r \leq N, \tilde{Q}^\delta_{c_0}(z,r) \in F\},$$

and

$$O_0 = \{z : \tilde{Q}^\delta_{c_0}(z,r) \in \mathcal{F}_0\}.$$

Pick $\tilde{Q}^\delta_{c_0}(z_1,r_1) \in \mathcal{F}_0$. If $O_0 \subset \tilde{Q}^\delta_{c_0}(z_1,r_1)$, then we stop. Otherwise, we pick $\tilde{Q}^\delta_{c_0}(z_2,r_2) \in \mathcal{F}_0$ with $z_2 \in O_0 \setminus \tilde{Q}^\delta_{c_0}(z_1,r_1)$. If $O_0 \subset \tilde{Q}^\delta_{c_0}(z_1,r_1) \cup \tilde{Q}^\delta_{c_0}(z_2,r_2)$, we stop. Otherwise we continue the process. In this way, we construct a subfamily $\mathcal{F}_0' = \{\tilde{Q}^\delta_{c_0}(z_i^0,r_i^0)\}_{i=1}^\infty$. Next we consider

$$\mathcal{F}_1 = \{\tilde{Q}^\delta_{c_0}(z,r) \in F : \frac{N}{4} < r \leq \frac{N}{2}\},$$

and

$$O_1 = \{z : \tilde{Q}^\delta_{c_0}(z,r) \in \mathcal{F}_1 \text{ and } z \notin \cup_{i=1}^\infty \tilde{Q}^\delta_{c_0}(z_i^0,r_i^0)\}.$$

We repeat the construction above for the set O_1 and obtain a family of sections denoted by $\mathcal{F}_1' = \{\tilde{Q}^\delta_{c_0}(z_i^1,r_i^1)\}_{i=1}^\infty$. In the same way, at kth step, we obtain $\mathcal{F}_k' = \{\tilde{Q}^\delta_{c_0}(z_i^k,r_i^k)\}_{i=1}^\infty$.

With the help of Lemma 4.1, we are able to show that each $\mathcal{F}_i'$ has bounded overlapping. Indeed let us assume that $\tilde{Q}^\delta_{c_0}(z_j,r_j) \in \mathcal{F}_i'$ for $1 \leq j \leq K$ and

$$z \in \tilde{Q}^\delta_{c_0}(z_1,r_1) \cap \cdots \cap \tilde{Q}^\delta_{c_0}(z_K,r_K).$$

For simplicity, we suppose that $\tilde{Q}^\delta_{c_0}(z_0,r_0)$ is a section in $\{\tilde{Q}^\delta_{c_0}(z_j,r_j)\}_{j=1}^K$ with $r_0 = \max\{r_j : 1 \leq j \leq K\}$ and from the construction of $\mathcal{F}_i'$ we know $z_l \notin \tilde{Q}^\delta_{c_0}(z_k,r_k)$ for $l > k$. Let T_p be an affine transformation that normalizes $\tilde{Q}_{c_0}(z_0,r_0)$. By Lemma 4.1, it is obvious that for $l > k$

$$T_p(z_l) \notin (t_k' - c_0\frac{r_k}{r_0}, t_k' + \delta\frac{r_k}{r_0}) \times B(x_k', K_1(\frac{r_k}{r_0})),$$

where $T_p(z_k) = (t_k', x_k')$. This, together with the fact that $\frac{1}{2} \leq r_j/r_0 \leq 2$, which is guaranteed by the construction, implies that

$$|T_p(z_l) - T_p(z_k)| > C \quad \text{for} \quad l > k,$$

where C depends on K_1, c_0, and δ. Therefore, an argument similar to that of Lemma 1 in [4] shows that overlapping in each $\mathcal{F}_i'$ is at most α depending on C and the structure conditions but not on i.

Next since O is bounded, combining the fact that each $\mathcal{F}_i'$ has finite overlapping with Lemma 4.1, we show that $\mathcal{F}_i'$ is finite for each i. Consider $\mathcal{F}_i' = \{\tilde{Q}^\delta_{c_0}(z_j,r_j)\}_{j=1}^\infty$. Since O is bounded and by construction $N2^{-(i+1)} \leq r_j \leq N2^{-i}$, there is a constant $\mathcal{C} \geq 2$ depending on $\operatorname{diam}(O), N, c_0$, and δ, such that $O \subset \tilde{Q}^\delta_{c_0}(z_1, \mathcal{C}r_1)$ and $\mathcal{C}r_1 \geq N2^{-i}$. Let T_p be an affine transformation that normalizes $\tilde{Q}_{c_0}(z_1, \mathcal{C}r_1)$. From Lemma 4.1, it follows that

$$\begin{aligned}(t'_j - c_0\frac{r_j}{\mathcal{C}r_1}, t' + \delta\frac{r_j}{\mathcal{C}r_1}) \times B(x'_j, K_1\frac{r_j}{\mathcal{C}r_1}) &\subset T_p(\tilde{Q}^\delta_{c_0}(z_j, r_j)) \\ &\subset (t'_j - c_0\frac{r_j}{\mathcal{C}r_1}, t'_j + \delta\frac{r_j}{\mathcal{C}r_1}) \times B(x'_j, K_2(\frac{r_j}{\mathcal{C}r_1})^{\varepsilon_2}),\end{aligned}$$

where $z'_j := (t'_j, x'_j) \in K^\delta(0, K_3)$ and K_3 is a large constant. Hence

$$\begin{aligned}(t'_j - \frac{c_0}{2\mathcal{C}}, t'_j + \frac{\delta}{2\mathcal{C}}) \times B(x'_j, \frac{K_1}{2\mathcal{C}}) &\subset T_p(\tilde{Q}^\delta_{c_0}(z_j, r_j)) \\ &\subset K^\delta(0, K_4), \end{aligned} \tag{4.54}$$

where K_4 is a large constant depending on $K_1, K_2, K_3, \varepsilon_2$, and d. Since $\mathcal{F}'_i$ has overlapping bounded by α, then

$$\sum_j \chi_{T_p(\tilde{Q}^\delta_{c_0}(z_j, r_j))} \leq \alpha.$$

By (4.54)

$$\sum_j \chi_{K^\delta(z'_j, d_1)} \leq \alpha \chi_{K^\delta(0, K_4)},$$

where d_1 depends on $K_1, K_2, K_3, \mathcal{C}$, and d. Then it follows immediately that there are only finitely many sections in $\mathcal{F}'_i$.

Finally, after shrinking the sections to $\tilde{Q}^\delta_{c_0}(z, (1-\varepsilon)r)$, we are able to prove that the overlappings between different $\mathcal{F}'_i$s are bounded. In fact, let $\varepsilon \in (0,1)$ and

$$z_0 \in \cap_i \overline{\tilde{Q}^\delta_{c_0}}(z^{e_i}_{j_i}, (1-\varepsilon) r^{e_i}_{j_i}), \tag{4.55}$$

where $e_1 < e_2 < \cdots < e_i < \cdots, N2^{-(e_i+1)} < r^{e_i}_{j_i} \leq N2^{-e_i}$. We denote $z_i = z^{e_i}_{j_i}$ and $r_i = r^{e_i}_{j_i}$. Fixing i and $l > i$, we estimate the gap between e_i and e_l. Let T_p be an affine transformation that normalizes $\tilde{Q}_{c_0}(z_i, r_i)$. By our construction, $r_i > r_l$. From Lemma 4.1,

$$\begin{aligned}&(t' - \frac{c_0(1-\varepsilon)r_l}{r_i}, t' + \frac{\delta(1-\varepsilon)r_l}{r_i}) \times B(x', K_1\frac{(1-\varepsilon)r_l}{r_i}) \subset T_p(\tilde{Q}^\delta_{c_0}(z_l, (1-\varepsilon)r_l)) \\ &\subset (t' - \frac{c_0(1-\varepsilon)r_l}{r_i}, t' + \frac{\delta(1-\varepsilon)r_l}{r_i}) \times B(x', K_2(\frac{(1-\varepsilon)r_l}{r_i})^{\varepsilon_2}),\end{aligned}$$

where $T_p(z_l) = (t', x')$. Since $z_l \notin \overline{\tilde{Q}^\delta_{c_0}(z_i, r_i)}$, by Lemma 4.2

$$K^\delta(T_p(z_l), c\varepsilon^d) \cap T_p(\overline{\tilde{Q}^\delta}_{c_0}(z_i, (1-\varepsilon)r_i)) = \emptyset,$$

which implies that

$$\begin{aligned}&\frac{1}{2}c\varepsilon^d < |T_p(z_l) - T_p(z_0)| \leq K_2((\frac{(1-\varepsilon)r_l}{r_i})^{\varepsilon_2}) + \frac{\delta(1-\varepsilon)r_l}{r_i} \\ &\leq K_2(\frac{r_l}{r_i})^{\varepsilon_2} + \delta\frac{r_l}{r_i} \leq K_2 2^{(e_i - e_l + 1)\varepsilon_2} + \delta 2^{e_i - e_l} \leq K_2(\delta)\max\{2^{(e_i-e_l+1)\varepsilon_2}, 2^{e_i - e_l}\},\end{aligned}$$

which implies that $e_l - e_i \le \overline{C}\log(1/\varepsilon)$, where $\overline{C}$ depends on δ, c, ε_2, and K_2. Hence, the sections in (4.55) are at most $\overline{C}\log(1/\varepsilon)$.

Denote $\mathcal{F}' = \cup_i \mathcal{F}_i'$ and we claim $\mathcal{F}'$ satisfies properties (i)–(iii). Since each $\mathcal{F}_i'$ has finitely many sections, $\mathcal{F}'$ has countably many sections. Moreover, from our construction, $\mathcal{F}_i'$ covers each O_i, hence $\mathcal{F}'$ covers O. The second property holds automatically by our construction and relabeling the sections. Combining the fact that each $\mathcal{F}_i'$ has finitely many sections and the overlapping between different families of sections are bounded, we prove the third property. Therefore, the proof is completed. □

With Lemma 4.3, the following version of Calderón-Zygmund decomposition theorem follows Theorem 2.1 and Lemma 2.3 in [11] with measure $\mathcal{M}$ replaced by the Lebesgue measure exactly.

Theorem 4.4. *Assume that $\lambda \in (0,1), z_0 \in \mathbb{R}^{d+1}, h_0 > 0$, and ϕ satisfies (1.1) and (1.2). Given a bounded open set $O \subset \tilde{Q}_{c_0}(z_0, h_0)$, there exists a family of parabolic sections $F = \{\tilde{Q}_{c_0}(z_k, h_k)\}$ with the following properties:*

1. *$z_k \in O, \forall k$.*
2. *$O \subset \cup_{k=1}^{\infty} \tilde{Q}^{\delta}_{c_0}(z_k, h_k)$.*
3. *$\frac{|O \cap \tilde{Q}^{\delta}_{c_0}(z_k,h_k)|}{|\tilde{Q}^{\delta}_{c_0}(z_k,h_k)|} = \lambda$.*
4. *$|O| \le c(\lambda)\frac{m+1}{m}|\cup_{k=1}^{\infty} \tilde{Q}^{m^*}_{c_0}(z_k, h_k)|$, for any $m \ge 1$, where $c(\lambda) \in (0,1)$ depends on λ but not on O and F,*

where $\delta > 0$, and m^* is the smallest integer such that $m^* \ge m + \delta m$.

5. Harnack Inequality

Before proving our main result, we have the following lemma about the shrinking property at different time.

Lemma 5.1. *There exist universal constants $C_0 > 0$ and $p_1 \ge 1$ such that for $0 < r < s \le 1, h \ge 0$ and $z = (t,x) \in (t_0 - rc_0h, t_0 + rc_0h) \times S(x_0, rh/2|t_0)$ we have*

$$S(x, \frac{C_0(s-r)^{p_1}h}{2}|t) \subset S(x_0, \frac{sh}{2}|t_0).$$

Proof. By a translation of the coordinates, we may assume that $z_0 = (0,0)$. Upon applying the transformation T_p, let us assume that $h = 1$ and $S(0, 1/2|0)$ is normalized. First we consider $t \in (-c_0 r, 0)$. By Lemma 3.1, we have

$$x \in S(0, \frac{r}{2}|0) \subset S(0, \frac{\theta r}{2}|t) \subset S(0, \frac{\theta}{2}|t).$$

Following the same argument as in Lemma 4.1, there exist $0 < b_d < c_d$ such that

$$B(0, b_d) \subset S(0, \frac{\theta}{2}|t) \subset B(0, c_d).$$

This implies that $S(0,\theta/2|t)$ is almost normalized and so is $S(0,1/2|t)$. Since $x \in S(0,\theta/2|t)$, by property (ii) with $T = \mathrm{id}$, there exist two universal constants K_1, ε such that

$$S(x,\rho|t) \subset B(x, K\rho^{\varepsilon}) \tag{5.56}$$

for $\rho < 1$. Because $x \in S(0,r/2|0)$, by property (iii) we have

$$S(x, C(\frac{s-r}{2})^p|0) \subset S(0,\frac{s}{2}|0),$$

where C, p are universal. Since $S(0,1/2|0)$ is normalized, by property (ii) with $T = id$, we obtain

$$B(x, K_2 C(\frac{s-r}{2})^p|0) \subset S(x, C(\frac{s-r}{2})^p|0) \tag{5.57}$$

Combining (5.56) with (5.57), we choose

$$\rho \le 2(\frac{CK_2}{K_1 2^p})^{1/\varepsilon}(s-r)^{p/\varepsilon},$$

which indicates that

$$C_0 = 2(\frac{CK_2}{K_1 2^p})^{1/\varepsilon}, \quad p_1 = p/\varepsilon.$$

Therefore, the proof of the case when $t \in (-c_0 r, 0)$ is completed.

For $t \in (0, c_0 r)$, one can check easily that the whole argument above holds as well. Therefore we prove the lemma. □

We then show a weak Harnack inequality by proving a power decay of the distribution function of u.

Lemma 5.2. *Let $z_0 = (0,0), T_0 = \hat{c}_0\theta$, and r_0 is the constant in Lemma 3.2. Assume that*

$$\inf_{\tilde{Q}_{c_0}((T_0,0),r_0)} u \le 1.$$

Then there exists a cylindrical domain

$$Q := (-\frac{5c_0}{8}, -\frac{3c_0}{8}) \times S(0,\frac{1}{8}|-\frac{c_0}{2}) \subset \tilde{Q}_{c_0}(z_0,1)$$

and universal constants M large, $0<\gamma<1$, $C>0$ and integer m such that

$$|\{u \ge M^{km}\} \cap Q| \le C\gamma^k |Q| \quad \forall k \in \mathbb{N}. \tag{5.58}$$

Proof. We are going to construct a sequence of decreasing domains Q_k converging to Q and satisfying

$$|\{u \ge M^{km}\} \cap Q_k| \le C\gamma^k |Q| \quad \forall k \in \mathbb{N}, \tag{5.59}$$

from which (5.58) follows immediately. First we choose

$$Q_k = (-\frac{c_0}{2} - t_k, -\frac{c_0}{2} + t_k) \times S(0, \alpha_k | \frac{c_0}{2}),$$

where $t_k \le c_0/4$ and $\alpha_k \le 1/4$ to be determined later. For $k = k_0$, where k_0 large to be determined later, we pick C depending on $d, \lambda, \Lambda, C_1, C_2$, and k_0 to be sufficiently large so that

$$|(-\frac{3c_0}{4}, -\frac{c_0}{4}) \times S(0, \frac{1}{4}| - \frac{c_0}{2})| \le C\gamma^{k_0}|Q|.$$

Now assume that (5.59) is valid for k and let us consider $k+1$. By applying Theorem 4.4 to $O = \{u \ge M^{(k+1)m}\} \cap Q_{k+1}$ with $\lambda > 1 - \varepsilon$, where ε is the constant in Theorem 3.7, we can find $z_j \in \{u \ge M^{(k+1)m}\} \cap Q_{k+1}$ and ρ_j such that

$$|\tilde{Q}^{\delta}_{c_0}(z_j, \rho_j) \cap \{u \ge M^{(k+1)m}\} \cap Q_{k+1}| = \lambda|\tilde{Q}^{\delta}_{c_0}(z_j, \rho_j)|$$

for $j \ge 1$, which implies that

$$|\tilde{Q}^{\delta}_{c_0}(z_j, \rho_j) \cap \{u \ge M^{(k+1)m}\}| \ge \lambda|\tilde{Q}^{\delta}_{c_0}(z_j, \rho_j)|.$$

Furthermore, for δ small, we have

$$|\tilde{Q}_{c_0}(z_j, \rho_j) \cap \{u \ge M^{(k+1)m}\}| \ge (\lambda - \frac{\delta}{c_0})|\tilde{Q}_{c_0}(z_j, \rho_j)|.$$

Let us choose δ sufficiently small such that $\lambda - \delta/c_0 \ge 1 - \varepsilon$ and fix this δ. By Theorem 4.4,

$$|\{u \ge M^{(k+1)m}\} \cap Q_{k+1}| \le c(\lambda)\frac{m+1}{m}| \cup_{j=1}^{\infty} \tilde{Q}^{m^*}_{c_0}(z_j, \rho_j)|.$$

We take $\gamma \in (c(\lambda), 1)$ and choose m sufficiently large so that $c(\lambda)(m+1)/m < \gamma$. Then it follows that

$$|\{u \ge M^{(k+1)m}\} \cap Q_{k+1}| \le \gamma| \cup_{j=1}^{\infty} \tilde{Q}^{m^*}_{c_0}(z_j, \rho_j)|.$$

Next after selecting proper Q_k, i.e., α_k and t_k, we want to show that for any j

$$\tilde{Q}^{m^*}_{c_0}(z_j, \rho_j) \subset \{u \ge M^{km}\} \cap Q_k. \tag{5.60}$$

By Lemma 3.1 and (3.7), one can easily check that

$$\tilde{Q}^{m^*}_{c_0}(z, \rho) \subset \cup_{i=1}^{m^*} K_i(z),$$

where $K_i(z)$ is defined in (3.8). Upon applying Theorem 3.7 m^* times, we find that

$$u \ge \frac{M^{(k+1)m}}{M_0^{m^*}} \quad \text{in} \quad \tilde{Q}^{m^*}_{c_0}(z_j, \rho_j). \tag{5.61}$$

If $M > M_0^{1+\delta}$, then

$$\frac{M^{(k+1)m}}{M_0^{m^*}} \ge M^{km},$$

which by (5.61) yields

$$\tilde{Q}^{m^*}_{c_0}(z_j, \rho_j) \subset \{u \ge M^{km}\}.$$

It only remains to prove $\tilde{Q}_{c_0}^{m^*}(z_j, \rho_j) \subset Q_k$. We treat the x variable and the t variable separately. From Lemma 3.8, we know that

$$\rho_j \leq CM^{-\delta(k+1)m}. \tag{5.62}$$

For the x variable by Lemma 5.1,

$$S(x_j, C_0(\alpha_k - \alpha_{k+1})^{p_1}|t_j) \subset S(0, \alpha_k| - \frac{c_0}{2}). \tag{5.63}$$

Let us take $\alpha_0 = 1/4$ and

$$\alpha_k = \frac{1}{4} - \sum_{j=0}^{k-1} (\frac{C}{C_0})^{1/p_1} M^{-\delta(j+1)m/p_1},$$

where C is the constant in (5.62). Then $\{\alpha_k\}$ satisfies

$$\alpha_k - \alpha_{k+1} = (\frac{C}{C_0})^{1/p_1} M^{-\delta(k+1)m/p_1}.$$

Therefore, by (5.62)

$$\rho_j \leq CM^{-\delta(k+1)m} = C_0(\alpha_k - \alpha_{k+1})^{p_1},$$

which from (5.63) implies

$$S(x_j, \rho_j|t_j) \subset S(0, \alpha_k| - \frac{c_0}{2}).$$

On the other hand, for the t variable we need

$$t_{k+1} + m^* c_0 \rho_j \leq t_k. \tag{5.64}$$

So we choose $t_0 = c_0/4$ and

$$t_k = \frac{c_0}{4} - m^* c_0 C \sum_{j=1}^{k-1} M^{-\delta(j+1)m}.$$

In this way

$$t_k - t_{k+1} = m^* c_0 C M^{-\delta(k+1)m},$$

which combining with (5.62) shows (5.64).

Finally, we need that the summations in the formulas of α_k and t_k to converge, which are guaranteed by taking M sufficiently large, for instance,

$$m^* C \sum_{j=1}^{\infty} M^{-\delta(j+1)m} < \frac{1}{8},$$

and

$$(\frac{C}{C_0})^{1/p_1} \sum_{j=1}^{\infty} M^{-\delta(j+1)m/p_1} < \frac{1}{8}.$$

Now we finish the proof of (5.60). Therefore

$$|\{u \geq M^{(k+1)m}\} \cap Q_{k+1}| \leq \gamma |\{u \geq M^{km}\} \cap Q_k| \leq C\gamma^{k+1}|Q|.$$

It remains to pick k_0 large such that $Q_{k_0} \subset \tilde{Q}(1)$. Hence, the proof is completed. $\square$

For the domain Q in Lemma 5.2, following a standard argument we deduce that there exists a constant $\mathfrak{p} > 0$ such that for any $h > 0$

$$|\{u \geq h\} \cap Q| \leq Ch^{-\mathfrak{p}},$$

which implies

$$\int_Q u^{\tilde{\mathfrak{p}}}\, dx\, dt \leq C, \tag{5.65}$$

where $\tilde{\mathfrak{p}} < \mathfrak{p}$. For simplicity of the notation, we still use $\mathfrak{p}$ to denote the constant in (5.65). In the rest of the paper, we denote $z = (-c_0/2, 0)$ and $r_1 = 1/8$. Let Q_- denote $\tilde{Q}_{c_0}(z, r_1)$ so that $Q_- \subset Q$, and $Q_+ = \tilde{Q}_{c_0}((T_0, 0), r_0)$, where T_0 and r_0 are as in Lemma 5.2. Therefore, we prove the following weak Harnack inequality

Theorem 5.3. *Assume that $u \geq 0$ satisfies* (1.1) *and* (1.2)*. Then there exist universal constants $C, \mathfrak{p}$ such that*

$$\int_{Q_-} u^{\mathfrak{p}}\, dx\, dt \leq C \inf_{Q_+} u.$$

Next we follow the idea in [11] to estimate $\sup_{Q_-} u$. For convenience, we denote $\nu = 16/15$.

Lemma 5.4. *Let $u \geq 0$ satisfy* (1.1) *and* (1.2)*. Suppose that $Q_- = \tilde{Q}_{c_0}(z, r_1)$, and*

$$\int_{Q_-} u^{\mathfrak{p}} dx\, dt \leq C|Q_-| \tag{5.66}$$

for some $C > 0$. Then there exist constants C_3, δ_1, and j_0 depending on C, $\mathfrak{p}$, and the structure conditions such that for $z' = (t', x') \in \tilde{Q}_{c_0}(z, 3r_1/4)$, and $j \geq j_0$, if $u(z') \geq 8\nu^{j-1}$, then

$$\sup_{\tilde{Q}(z',\rho)} u \geq 8\nu^j,$$

where $\rho = C_3 \nu^{-\delta_1 j} r_1$ and $\tilde{Q}_{c_0}(z', \rho) \subset Q_-$.

Proof. We prove the lemma by contradiction. Suppose that $\sup_{\tilde{Q}_{c_0}(z',\rho)} u < 8\nu^j$ and consider

$$w(z) = \frac{8\nu^j - u(z)}{8\nu^{j-1}(\nu - 1)}.$$

It follows that $w > 0$ and w satisfies (1.1). One can easily check that $w(z') \leq 1$. Applying Lemma 3.4 to w, we obtain

$$|\{w \leq 8\} \cap \tilde{Q}_{c_0}(z', \rho)| \geq \varepsilon_0 |\tilde{Q}_{c_0}(z', \rho)|,$$

which is

$$|\{u \geq 4\nu^j\} \cap \tilde{Q}_{c_0}(z', \rho)| \geq \varepsilon_0 |\tilde{Q}_{c_0}(z', \rho)|.$$

Since $z' \in \tilde{Q}_{c_0}(z, \frac{3r_1}{4})$ and by Lemma 5.1, it is easily seen that there exists a large constant j_0 depending on C_3 and δ_1 so that for any $j \geq j_0$, $\tilde{Q}_{c_0}(z', \rho) \subset Q_-$. By the Chebyshev inequality and (5.66)

$$|\{u \geq 4\nu^j\} \cap \tilde{Q}_{c_0}(z', \rho)| \leq (4\nu^j)^{-\mathfrak{p}} \int_{Q_-} u^{\mathfrak{p}}\, dx\, dt \leq C(4\nu^j)^{-\mathfrak{p}}|Q_-|.$$

Combining the two inequalities above, we obtain

$$C(4\nu^j)^{-\mathfrak{p}}|Q_-| \geq \varepsilon_0|\tilde{Q}_{c_0}(z', \rho)|,$$

i.e.,

$$C(4\nu^j)^{-\mathfrak{p}}c_0 r_1|S(x, r_1|t)| \geq \varepsilon_0 c_0 \rho|S(x', \rho|t')|. \tag{5.67}$$

Since $z' \in \tilde{Q}(z, \frac{3}{4}r_1)$, by Lemma 3.1 we know that

$$|S(x, r_1|t)| \leq |S(x, \theta r_1|t')|.$$

Then by property (iv) of sections, it follows that

$$|S(x, \theta r_1|t')| \leq \hat{C}(\frac{r_1}{\rho})^d|S(x', \rho|t')|,$$

where $\hat{C}$ is universal. Combining the two inequalities above with (5.67), we get

$$C\hat{C}\nu^{-\mathfrak{p}j}r_1^{d+1} \geq \rho^{1+d}.$$

This implies that if we set $C_3 = (\hat{C}C)^{1/(d+1)} + 1$ and $\delta_1 = \mathfrak{p}/(d+1)$, the inequality above contradicts with $\rho = C_3\nu^{-\delta_1 j}r_1$. Hence we prove the lemma. $\square$

The following theorem is about the estimate of $\sup_{Q_-} u$.

Theorem 5.5. *Let $u \geq 0$ satisfy* (1.1) *and* (1.2). *Suppose that u satisfies* (5.66). *Then there exists j_1 such that $\sup_{\tilde{Q}_{c_0}(z, r_1/2)} u \leq 8\nu^{j_1-1}$, where j_1 only depends on $C, \mathfrak{p}$, and the structure conditions.*

Proof. Choose $j_1 \geq j_0$ such that

$$\sum_{j \geq j_1}(\frac{C_3\theta\nu^{-\delta_1(j_1+k)}}{C_0})^{1/p_1} < \frac{1}{4}, \tag{5.68}$$

where j_0, C_3, and δ_1 are the constants in Lemma 5.4, p_1 and C_0 are the constants in Lemma 5.1. We claim that $\sup_{\tilde{Q}(z, r_1/2)} u \leq 8\nu^{j_1-1}$. Otherwise, suppose that there exists $z' \in \tilde{Q}_{c_0}(z, r_1/2)$ such that $u(z') \geq 8\nu^{j_1-1}$. By applying Lemma 5.4 we are going to find a sequence of points $\{z_k\}$ and parabolic sections $\tilde{Q}_{c_0}(z, h_k)$ such that $z_k \in \tilde{Q}_{c_0}(z, h_k)$, $\tilde{Q}_{c_0}(z, h_k) \subset \tilde{Q}_{c_0}(z, \frac{3}{4}r_1)$, and $u(z_k) \geq 8\nu^{k-1+j_1}$. Denote $z_0 = z'$ and $\tilde{Q}_{c_0}(z, h_0) = \tilde{Q}_{c_0}(z, \frac{r_1}{2})$. Suppose that we find $\{z_j\}_{j=1}^k$ and $\{h_j\}_{j=1}^k$ satisfy our conditions. Then for $k+1$, since $u(z_k) \geq 8\nu^{j_1+k-1}$, by Lemma 5.4, we have $z_{k+1} \in \tilde{Q}_{c_0}(z_k, \rho_k)$ such that $u(z_{k+1}) \geq 8\nu^{j_1+k}$, where $\rho_k = C_3\nu^{-\delta_1(j_1+k)}r_1$. It suffices to choose h_{k+1} such that

$$\tilde{Q}_{c_0}(z_k, \rho_k) \subset \tilde{Q}_{c_0}(z, h_{k+1}).$$

We consider the spatial variables x and time variable t separately. Since $z_k \in \tilde{Q}_{c_0}(z, h_k)$, for the t variable we only need

$$h_{k+1} - h_k \geq \rho_k = C_3 \nu^{-\delta_1(j_1+k)}. \tag{5.69}$$

For the spatial variables x, by Lemma 3.1, it is sufficient to consider

$$S(x_k, \frac{\theta \rho_k}{2}|t) \subset S(x, \frac{h_{k+1}}{2}|t).$$

Because $x_k \in S(x, \frac{h_k}{2}|t)$, by Lemma 5.1, we need

$$\theta \rho_k \leq C_0 (h_{k+1} - h_k)^{p_1},$$

which can be rewritten as

$$h_{k+1} - h_k \geq (\frac{\theta \rho_k}{C_0})^{1/p_1} = (\frac{C_3 \theta \nu^{-\delta_1(j_1+k)}}{C_0})^{1/p_1}. \tag{5.70}$$

Since $p_1, \nu \geq 1$, we have $\nu^{-\delta_1} \leq \nu^{-\delta_1/p_1}$. Then for j_1 sufficiently large depending on C_0, C_3, θ, and δ_1, (5.70) implies (5.69), which indicates that we can pick

$$h_{k+1} = \frac{1}{2} r_1 + \sum_{j=1}^{k} (\frac{C_3 \theta \nu^{-\delta_1(j_1+k)}}{C_0})^{1/p_1} r_1.$$

By (5.68), we know that $\tilde{Q}_{c_0}(z, h_k) \subset \tilde{Q}_{c_0}(z, \frac{3}{4} r_1)$ for any $k \in \mathbb{N}$. On the other hand, since for each k, $u(z_{k+1}) \geq \nu^{j_1+k}$, this contradicts with the assumption that u is continuous in $\tilde{Q}_{c_0}(z, \frac{3}{4} r_1)$. Therefore, we prove the lemma. □

Proof of Theorem 1.1. First we prove

$$\sup_{\tilde{Q}_{c_0}(z, \frac{r_1}{2})} u \leq C \inf_{\tilde{Q}_{c_0}((T_0,0), r_0)} u, \tag{5.71}$$

where $z = (-c_0/2, 0)$, $r_1 = 1/8$ and $\tilde{Q}_{c_0}((T_0, 0), r_0) = Q^+$ as in Theorem 5.3. Without loss of generality, we assume that $\inf_{\tilde{Q}_{c_0}((T_0,0),r_0)} u \leq 1$. Hence, we only need to prove $\sup_{\tilde{Q}_{c_0}(z, \frac{r_1}{2})} u \leq C$, which can be easily shown by combining Theorem 5.5 and Theorem 5.3. Therefore we prove (5.71). Then Theorem 1.1 follows easily by scaling and translation of the coordinates. □

Proof of Corollary 1.3. By scaling and translation of the coordinates, we may assume that $z_0 = (0, 0)$ and $R = 1$. Let $M = \sup_{\tilde{Q}(1)} u$ and $m = \inf_{\tilde{Q}(1)} u$. We consider $M - u$ and $u - m$ which are both nonnegative solution of (1.1). By Theorem 1.1, Remark 1.2, and the arguments of scaling and translation of the coordinates, we can find parabolic sections Q_1 and Q_2 such that for any nonnegative solution v of (1.1) and (1.2),

$$\sup_{Q_1} v \leq C \inf_{Q_2} v,$$

where

$$Q_1 = \tilde{Q}_{c_0}((t_1, x_1), r_1), \quad Q_2 = \tilde{Q}(r_0),$$

and $t_1 < -c_0 r_0$. Then it is sufficient to prove the corollary for $\rho \le r_0$. We apply Theorem 1.1 to $u - m$ and $M - u$ in Q_1 and Q_2 to get,

$$\sup_{Q_1}(u - m) \le C \inf_{Q_2}(u - m),$$
$$\sup_{Q_1}(M - u) \le C \inf_{Q_2}(M - u).$$

We add the two inequalities above to obtain

$$M - m + \sup_{Q_1} u - \inf_{Q_1} u \le C(M - m - (\sup_{Q_2} u - \inf_{Q_2} u)).$$

This implies that

$$osc_{Q_2} u \le \frac{C-1}{C} osc_{\tilde{Q}(1)} u,$$

which is

$$osc_{\tilde{Q}(r_0)} u \le \frac{C-1}{C} osc_{\tilde{Q}(1)} u.$$

Then an elementary iteration proves the corollary, for instance see [15]. □

Remark 5.6. We can obtain the Hölder continuity of u from Corollary 1.3, but the Hölder constant depends on the norm of the affine transformation which normalizes the section under consideration. The detail can be found in [11] and [5].

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References

[1] L. A. Caffarelli. A localization property of viscosity solutions to the Monge-Ampère equation and their strict convexity. *Ann. of Math. (2)*, 131(1):129–134, 1990.

[2] Luis A. Caffarelli. Interior $W^{2,p}$ estimates for solutions of the Monge-Ampère equation. *Ann. of Math. (2)*, 131(1):135–150, 1990.

[3] Luis A. Caffarelli. Some regularity properties of solutions of Monge Ampère equation. *Comm. Pure Appl. Math.*, 44(8-9):965–969, 1991.

[4] Luis A. Caffarelli and Cristian E. Gutiérrez. Real analysis related to the Monge-Ampère equation. *Trans. Amer. Math. Soc.*, 348(3):1075–1092, 1996.

[5] Luis A. Caffarelli and Cristian E. Gutiérrez. Properties of the solutions of the linearized Monge-Ampère equation. *Amer. J. Math.*, 119(2):423–465, 1997.

[6] Panagiota Daskalopoulos and Ovidiu Savin. $C^{1,\alpha}$ regularity of solutions to parabolic Monge-Ampére equations. *Amer. J. Math.*, 134(4):1051–1087, 2012.

[7] Guido De Philippis and Alessio Figalli. $W^{2,1}$ regularity for solutions of the Monge-Ampère equation. *Invent. Math.*, 192(1):55–69, 2013.

[8] Giuseppe Di Fazio, Cristian E. Gutiérrez, and Ermanno Lanconelli. Covering theorems, inequalities on metric spaces and applications to PDE's. *Math. Ann.*, 341(2):255–291, 2008.

[9] Cristian E. Gutiérrez. *The Monge-Ampère equation.* Progress in Nonlinear Differential Equations and their Applications, 44. Birkhäuser Boston Inc., Boston, MA, 2001.

[10] Cristian E. Gutiérrez and Qingbo Huang. $W^{2,p}$ estimates for the parabolic Monge-Ampère equation. *Arch. Ration. Mech. Anal.*, 159(2):137–177, 2001.

[11] Qingbo Huang. Harnack inequality for the linearized parabolic Monge-Ampère equation. *Trans. Amer. Math. Soc.*, 351(5):2025–2054, 1999.

[12] N. V. Krylov. Sequences of convex functions, and estimates of the maximum of the solution of a parabolic equation. *Sibirsk. Mat. Ž.*, 17(2):290–303, 478, 1976.

[13] N. Q. Le and O. Savin. Boundary regularity for solutions to the linearized Monge-Ampère equations. *Arch. Ration. Mech. Anal.*, 210(3):813–836, 2013.

[14] N. Q. Le and O. Savin. On boundary Hölder gradient estimates for solutions to the linearized Monge-Ampère equations. *ArXiv*, March 2013.

[15] Gary M. Lieberman. *Second order parabolic differential equations*. World Scientific Publishing Co. Inc., River Edge, NJ, 1996.

[16] O. Savin. A localization property at the boundary for the Monge-Ampère equation. *ArXiv*, October 2010.

[17] O. Savin. Pointwise $C^{2,\alpha}$ estimates at the boundary for the Monge-Ampère equation. *J. Amer. Math. Soc.*, 26(1):63–99, 2013.

[18] Elias M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[19] Kaising Tso. On an Aleksandrov-Bakel′man type maximum principle for second-order parabolic equations. *Comm. Partial Differential Equations*, 10(5):543–553, 1985.

[20] Lihe Wang. On the regularity theory of fully nonlinear parabolic equations. I. *Comm. Pure Appl. Math.*, 45(1):27–76, 1992.

[21] Rou Huai Wang and Guang Lie Wang. On existence, uniqueness and regularity of viscosity solutions for the first initial-boundary value problems to parabolic Monge-Ampère equation. *Northeast. Math. J.*, 8(4):417–446, 1992.

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