CONTROL WITH PARTIAL OBSERVATIONS AND AN EXPLICIT SOLUTION OF MORTENSEN'S EQUATION

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Abstract

We formulate a stochastic control problem with a general information structure, and show that an optimal law exists and is characterized as the unique solution of a recursive stochastic equation. For a special information structure of the "signal-plus-noise" type and with quadratic cost-functions, this recursive equation is solved for the value function of the control problem. This value function is then shown to satisfy the Mortensen equation of Dynamic Programming in function-space.

Key words: Stochastic control, partial observations, filtering, recursive stochastic equations, Komlós theorem, Mortensen equation.

1 INTRODUCTION

This paper discusses a feed-forward stochastic control problem inspired by the work of Beneš (1991), with general partial observation structure. For a class of convex cost-functions on the control and on the terminal state, it is shown in Section 2 that an optimal control process exists and is characterized as the unique solution of a certain *Recursive* (or "backwards") *Stochastic Equation*. The methodology of this section is based on the theorem of Komlós (1967) and on straightforward variational arguments.

In the special case of quadratic cost-functions, simple stochastic analysis shows in Section 3 that this equation can be solved explicitly, and that the solution leads to a general formulation of the Certainty-Equivalence Principle of stochastic control (Remark 3.1).

Section 4 specializes these results to the case of an observation-filtration of the "signal-plus-noise" type, generated by a Brownian motion with independent, random drift B with known probability distribution μ. In this context, filtering theory leads to explicit computations for the value function of the problem and for its time-derivative (subsections 4.1 and 4.2, respectively).

Once such computation has been achieved, it is natural to try and connect the results to the dynamic programming equation of Mortensen (1966). His approach, reviewed briefly and formally in Section 5, was designed to trade off the finite-dimensional, partially-observed control problem for a completely-observed but infinite-dimensional one, in which the role of "state" is played by the conditional distribution of the unobservable random drift B shifted by the cumulative action of control, given the observations; see equation (4.7). This conditional distribution satisfies the Kushner-Stratonovich stochastic partial differential equation (5.3), and the formal dynamic programming equation for the corresponding fully-observed control problem is the Mortensen equation (5.6).

Giving rigorous meaning to the first- and second-order functional derivatives appearing in this equation turns out to be a challenging task. (The difficulty stems from the fact that, even in this relatively simple context, there is no clear choice for the space of variations in which directional derivatives such as those in (6.2), (6.3) can be interpreted in a rigorous manner.) Formal arguments, leading to the explicit computations (6.4) and (6.5) for these derivatives, are carried out in Section 6. These computations indeed justify the validity of the Mortensen equation, and lead to a feed-back expression for the optimal control law.

The formal arguments of Section 6 are justified rigorously by a finite-dimensional analysis carried out in Section 7. Such finite-dimensional analysis is possible in this case because of the simple and explicit formula (4.7), that maps the "prior" distribution μ into its "posterior" version μ_t^u, given the observations, and the control process $u(\cdot)$ used, up to time t. This analysis bypasses the above difficulties by singling out and analyzing only a special, finite-dimensional space of variations. In this space, ordinary derivatives can be computed explicitly, and are then shown to "factor" correctly through the functional derivatives that appear in the Mortensen equation. Such computations turn out to be sufficient for our purpose, and provide a rigorous derivation of Itô's rule in our context.

To our knowledge, the results of Sections 6, 7 constitute the first instance of an explicit solution to Mortensen's equation.

2 THE CONTROL PROBLEM

We shall place ourselves on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, endowed with a fitration $\mathbb{F} = \{\mathcal{F}_t\}_{0\le t\le T}$ that satisfies $\mathcal{F}_0 = \{\emptyset, \Omega\}$, mod. $\mathbb{P}$, as well as the "usual conditions" of right-continuity and augmentation by $\mathbb{P}$-null sets. On this space, we are given a random variable $B : \Omega \to \mathbb{R}$ with known distribution $\mu(A) \stackrel{\triangle}{=} \mathbb{P}[B \in A],\ A \in \mathcal{B}(\mathbb{R})$, and consider the class of *control processes*

(2.1)
$$\mathcal{U} = \left\{ u : [0,T] \times \Omega \to R \ / \ u(\cdot) \text{ is } \mathbb{F}\text{−progressively measurable and } \|u\| \stackrel{\triangle}{=} \mathbb{E} \int_0^T \big|u(t)\big|\, dt < \infty \right\}.$$

Let us consider also two "cost-functions" $\varphi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$; both of them are convex and bounded from below, $\psi(\cdot)$ is strictly convex with $\psi(\pm\infty) = \infty$, and $\mathbb{E}[\varphi(B)] < \infty$. With these ingredients, we pose the *Stochastic Control Problem* of minimizing the expected cost

$$J(u) = \mathbb{E}\left[\lambda \int_0^T \psi\big(u(t)\big)\, dt + \varphi\left(B + \int_0^T u(t)\, dt\right)\right] \tag{2.2}$$

over $u(\cdot) \in \mathcal{U}$. Here $\lambda > 0$ is a given real constant, which weighs the "cost-of-control" $\int_0^T \psi\big(u(t)\big)\, dt$ relative to the cost $\varphi\left(B + \int_0^T u(t)\, dt\right)$ of "missing the random target $-B$". A typical situation is the quadratic $\varphi(x) = \psi(x) = x^2$, to be studied in detail in Section 4 below.

We have the following general result.

THEOREM 2.1. *There exists a unique (up to equivalence a.e. on* $[0,T]$*) control process* $u^*(\cdot) \in \mathcal{U}$ *which is optimal for the problem of* (2.2)*, i.e.,*

$$V \stackrel{\triangle}{=} \inf_{u\in\mathcal{U}} J(u) = J(u^*). \tag{2.3}$$

Proof: Consider a sequence $\{u_n(\cdot)\}_{n\in\mathbb{N}} \subseteq \mathcal{U}$ which is "minimizing" for the control problem, i.e. $\lim_{n\to\infty} \downarrow J(u_n) = V$. Because

$$-\infty < \lambda T \cdot \inf_{\mathbb{R}} \psi(\cdot) + \inf_{\mathbb{R}} \varphi(\cdot) \le V \le \lambda T \cdot \psi(0) + \mathbb{E}[\varphi(B)] < \infty,$$

it is clear that the sequence $\{J(u_n)\}_{n\in\mathbb{N}}$ is bounded; and because $\varphi(\cdot)$ is bounded from below, it follows that

$$\text{the sequence } \left\{\mathbb{E}\int_0^T \psi\big(u_n(t)\big)\, dt\right\}_{n\in\mathbb{N}} \text{ is bounded.} \tag{2.4}$$

However, our assumptions on the function $\psi(\cdot)$ imply a lower-bound of the type

$$\psi(x) \geq k_1 + k_2|x|, \qquad \forall\, x \in \mathbb{R} \tag{2.5}$$

for some $k_1 \in \mathbb{R}$, $k_2 > 0$; and from (2.4), (2.5) we see that

$$\text{the sequence} \quad \left\{ \mathbb{E} \int_0^T \big|u_n(t)\big|\, dt \right\}_{n\in\mathbb{N}} \quad \text{is bounded.}$$

Thus, from a theorem of Komlós (1967) (see also Schwartz (1985)), there exists a measurable process $u^* : [0,T] \times \Omega \to \mathbb{R}$ and a subsequence $\{u'_n(\cdot)\}_{n\in\mathbb{N}}$ of $\{u_n(\cdot)\}_{n\in\mathbb{N}}$, such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} u'_j(\cdot) = u^*(\cdot), \qquad (\ell \otimes \mathbb{P}) - \text{a.e. on} \;\; [0,T] \times \Omega\,,$$

where ℓ stands for Lebesgue measure. Thanks to the conditions imposed on the filtration $\mathbb{F}$, this process $u^*(\cdot)$ can be considered in its $\mathbb{F}$-adapted, (thus also in its $\mathbb{F}$-progressively measurable) modification; recall Proposition 1.1.12 in Karatzas & Shreve (1991). On the other hand, Fatou's lemma gives

$$\mathbb{E} \int_0^T \big|u^*(t)\big|\, dt \;\leq\; \liminf_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \mathbb{E} \int_0^T \big|u'_j(t)\big|\, dt \;\leq\; \sup_{n\in\mathbb{N}} \mathbb{E} \int_0^T \big|u_n(t)\big|\, dt \;<\; \infty\,,$$

so that $u^*(\cdot) \in \mathcal{U}$; and the *convexity* of $\varphi(\cdot)$, $\psi(\cdot)$ implies

$$J(u^*) \;\leq\; \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} J(u'_j) \;=\; \inf_{u\in\mathcal{U}} J(u),$$

which means that $u^*(\cdot)$ attains this last infimum. The *strict* convexity of $\psi(\cdot)$ guarantees that $u^*(\cdot)$ is the only (modulo a.e.-equivalence) process in $\mathcal{U}$ with this property. □

To proceed further, let us suppose that the convex functions $\varphi(\cdot)$ and $\psi(\cdot)$ satisfy the condition

$$f(x \pm 1) \leq c_1 + c_2 \cdot f(x), \qquad \forall\, x \in \mathbb{R} \tag{2.6}$$

for some suitable constants $c_1 > 0$, $c_2 > 0$.

Remark 2.1. For a convex function $f : \mathbb{R} \to \mathbb{R}$, condition (2.6) guarantees that $f(\cdot)$ "does not grow too fast"; e.g. $f(x) = |x|^p$ for $p \geq 1$, or $f(x) = e^{\alpha|x|}$ for $\alpha > 0$, both satisfy this condition (but $f(x) = e^{\alpha x^2}$ for $\alpha > 0$ does not). The derivatives

$$D^{\pm} f(x) \;\triangleq\; \lim_{\varepsilon\downarrow 0} \frac{f(x \pm \varepsilon) - f(x)}{\varepsilon}\,, \qquad x \in \mathbb{R}$$

exist, satisfy $D^{-}f(\cdot) \leq D^{+}f(\cdot)$ everywhere on $\mathbb{R}$, and may differ on a set which is at most countable. It can be checked that every convex function $f(\cdot)$ satisfying (2.6), also satisfies the condition

$$\big|D^{\pm} f(x)\big| \leq d_1 + d_2 \cdot f(x), \qquad \forall\, x \in \mathbb{R} \tag{2.7}$$

for suitable real constants $d_1 > 0, d_2 > 0$; furthermore, for any real $h > 0$, there exist $\beta_j = \beta_j(h)$, $j = 1, 2$ such that

$$f(x + y) \leq \beta_1 + \beta_2 \cdot f(x); \qquad \forall\, x \in \mathbb{R},\; |y| \leq h. \tag{2.8}$$

We shall assume, from now on, that

$$\varphi(\cdot) \quad \text{is continuously differentiable,} \tag{2.9}$$

and introduce the family of *Recursive Stochastic Operators*

$$L^{\pm}(t;u) \triangleq \lambda \cdot D^{\pm}\psi\big(u(t)\big) + \mathbb{E}\left[\varphi'\left(B + \int_0^T u(s)\,ds\right) \middle|\, \mathcal{F}_t \right], \qquad 0 \le t \le T \tag{2.10}$$

for each $u(\cdot) \in \mathcal{U}$. Clearly, if $\psi(\cdot)$ is also continuously differentiable, we have

$$L^{\pm}(t;u) = L(t;u) \triangleq \lambda \cdot \psi'\big(u(t)\big) + \mathbb{E}\left[\varphi'\left(B + \int_0^T u(s)\,ds\right) \middle|\, \mathcal{F}_t \right]. \tag{2.10$'$}$$

THEOREM 2.2. *If the process $\tilde{u} \in \mathcal{U}$ satisfies*

$$L^{-}(\cdot\,;\,u) \le 0 \le L^{+}(\cdot\,;\,u), \qquad (\ell \otimes \mathbb{P}) - a.e. \ on \ [0,T] \times \Omega, \tag{2.11}$$

then $\tilde{u}(\cdot)$ is optimal for the control problem (i.e., attains the infimum of (2.3)). Conversely, if both $\varphi(\cdot)$ and $\psi(\cdot)$ satisfy the condition (2.6) *and the process $u^*(\cdot) \in \mathcal{U}$ is optimal for the control problem, then $u^*(\cdot)$ satisfies the* Recursive Stochastic Inequalities *of (2.11).*

COROLLARY 2.1. Suppose that both $\varphi(\cdot)$ and $\psi(\cdot)$ are continuously differentiable, and satisfy the condition (2.6). Then the *Recursive Stochastic Equation*

$$\lambda \cdot \psi'\big(u(t)\big) + \mathbb{E}\left[\varphi'\left(B + \int_0^T u(s)\,ds\right) \middle|\, \mathcal{F}_t \right] = 0\,, \qquad \forall\ 0 \le t \le T \tag{2.12}$$

admits a unique solution $u^*(\cdot) \in \mathcal{U}$, and this process $u^*(\cdot)$ is optimal for the control problem.

Proof of Sufficiency in Theorem 2.2: Assume that $\tilde{u}(\cdot) \in \mathcal{U}$ satisfies (2.11); let $u(\cdot)$ be an arbitrary but fixed element of $\mathcal{U}$, and set $v(\cdot) = u(\cdot) - \tilde{u}(\cdot) \in \mathcal{U}$. Then the convexity of $\varphi(\cdot), \psi(\cdot)$ gives

$$\psi\big(u(t)\big) - \psi\big(\tilde{u}(t)\big) \ge v(t) \cdot \left[D^{+}\psi\big(u(t)\big) \cdot 1_{\{v(t)\ge 0\}} + D^{-}\psi\big(\tilde{u}(t)\big) \cdot 1_{\{v(t)<0\}} \right], \qquad 0 \le t \le T,$$

as well as

$$\varphi\left(B + \int_0^T u(t)\,dt\right) - \varphi\left(B + \int_0^T \tilde{u}(t)\,dt\right) \ge \int_0^T v(t)\,dt \cdot \varphi'\left(B + \int_0^T \tilde{u}(s)\,ds\right)$$

almost surely, and thus

$$\mathbb{E}\left[\varphi\left(B + \int_0^T u(t)\,dt\right) - \varphi\left(B + \int_0^T \tilde{u}(t)\,dt\right)\right] \ge \mathbb{E}\int_0^T v(t) \cdot \mathbb{E}\left[\varphi'\left(B + \int_0^T \tilde{u}(s)\,ds\right) \middle|\, \mathcal{F}_t\right] dt$$

$$\mathbb{E}\left[\int_0^T \psi\big(u(t)\big)\,dt - \int_0^T \psi\big(\tilde{u}(t)\big)\,dt\right] \ge \mathbb{E}\int_0^T v(t) \cdot \left\{ D^{+}\psi\big(\tilde{u}(t)\big) 1_{\{v(t)\ge 0\}} + D^{-}\psi\big(\tilde{u}(t)\big) 1_{\{v(t)<0\}} \right\} dt.$$

Therefore,

$$J(u) - J(\tilde{u}) \ge \mathbb{E}\int_0^T v(t)\left(L^{+}(t;\tilde{u}) \cdot 1_{\{v(t)\ge 0\}} + L^{-}(t;\tilde{u}) \cdot 1_{\{v(t)<0\}} \right) dt \ge 0$$

holds for every $u(\cdot) \in \mathcal{U}$, thanks to (2.11), and the optimality of $\tilde{u}(\cdot) \in \mathcal{U}$ follows.

Proof of Necessity in Theorem 2.2: Now suppose that $u^*(\cdot) \in \mathcal{U}$ is optimal for the control problem, and take $v(\cdot) \in \mathcal{U}$ with values in $[0,1]$ but otherwise arbitrary. Then $J(u^* + \varepsilon v) \geq J(u^*)$ holds for each $0 < \varepsilon < 1$, and

$$\liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \left[J(u^* + \varepsilon v) - J(u^*) \right] \geq 0. \tag{2.13}$$

Clearly, the inequalities

$$\psi\big(u^*(t) + \varepsilon v(t)\big) - \psi\big(u^*(t)\big) \leq \varepsilon v(t) \cdot D^+\psi\big(u^*(t) + \varepsilon v(t)\big), \quad 0 \leq t \leq T$$

and

$$\varphi\left(B + \int_0^T \big(u^*(t) + \varepsilon v(t)\big)\, dt\right) - \varphi\left(B + \int_0^T u^*(t)\, dt\right) \leq \varepsilon \int_0^T v(t)\, dt \cdot \varphi'\left(B + \int_0^T \big(u^*(t) + \varepsilon v(t)\big)\, dt\right)$$

hold almost surely, and in the light of (2.13) they lead to

(2.14)
$$\liminf_{\varepsilon \downarrow 0}\ \mathbb{E}\left[\lambda \int_0^T v(t) \cdot D^+\psi\big(u^*(t) + \varepsilon v(t)\big)\, dt + \int_0^T v(t) \cdot \varphi'\left(B + \int_0^T \big(u^*(s) + \varepsilon v(s)\big)\, ds\right)\, dt\right] \geq 0.$$

In this last expression, we would like to interchange the limit and the expectation. This can be justified as follows: From the assumption (2.6) and Remark 2.1 following it, we deduce the a.s. bounds

$$\left|D^+\psi\big(u^*(t) + \varepsilon v(t)\big)\right| \leq d_1 + d_2 \cdot \psi\big(u^*(t) + \varepsilon v(t)\big) \leq \gamma_1 + \gamma_2 \cdot \psi\big(u^*(t)\big),$$

for suitable positive constants d_j, γ_j $(j = 1, 2)$; similarly,

$$\begin{aligned} \left|\varphi'\left(B + \int_0^T u^*(t)\, dt + \varepsilon \int_0^T v(t)\, dt\right)\right| &\leq d_1' + d_2' \cdot \varphi\left(B + \int_0^T u^*(t)\, dt + \varepsilon \int_0^T v(t)\, dt\right) \\ &\leq \gamma_1' + \gamma_2' \cdot \varphi\left(B + \int_0^T u^*(t)\, dt\right), \qquad \text{a.s.} \end{aligned}$$

But

$$\mathbb{E}\left[\lambda \int_0^T \psi\big(u^*(t)\big)\, dt + \varphi\left(B + \int_0^T u^*(t)\, dt\right)\right] = J(u^*) < \infty,$$

and so, by the right-continuity of $D^+\psi(\cdot), \varphi(\cdot)$ and the Dominated Convergence Theorem, we obtain

$$\begin{aligned} 0 &\leq \mathbb{E}\left[\lim_{\varepsilon \downarrow 0} \int_0^T v(t) \left\{\lambda \cdot D^+\psi\big(u^*(t) + \varepsilon v(t)\big) + \varphi'\left(B + \int_0^T u^*(s)\, ds + \varepsilon \int_0^T v(s)\, ds\right)\right\} dt\right] \\ &= \mathbb{E}\left[\int_0^T v(t) \left\{\lambda \cdot D^+\psi\big(u^*(t)\big) + \mathbb{E}\left[\varphi'\left(B + \int_0^T u^*(s)\, ds\right) \,\middle|\, \mathcal{F}_t\right]\right\} dt\right]. \end{aligned}$$

Since $v(\cdot)$ is arbitrary, this implies $L^+(\cdot\,; u^*) \geq 0$, $(\ell \otimes \mathbb{P})$-a.e. on $[0,T] \times \Omega$. It can be shown similarly that $L^-(\cdot\,; u^*) \leq 0$, $(\ell \otimes \mathbb{P})$-a.e. on $[0,T] \times \Omega$. $\square$

3 THE QUADRATIC CASE

In the special case $\varphi(x) = \psi(x) = x^2$, the Recursive Stochastic Equation of (2.12) for the optimal control process $u^*(\cdot) \in \mathcal{U}$, becomes

$$\lambda u^*(t) + \mathbb{E}\left[B + \int_0^T u^*(s)\, ds \,\middle|\, \mathcal{F}_t \right] = 0, \qquad 0 \le t \le T. \tag{3.1}$$

In particular, $u^*(\cdot)$ is a martingale in this case. Thus, the equation (3.1) can be written in terms of another martingale, namely, the conditional expectation

$$\hat{B}(t) \stackrel{\triangle}{=} \mathbb{E}\big[B \,\big|\, \mathcal{F}_t\big], \qquad 0 \le t \le T \tag{3.2}$$

of the random variable B given the observations, in the simpler form

$$(\lambda + T - t) \cdot u^*(t) + \int_0^t u^*(s)\, ds + \hat{B}(t) = 0, \qquad 0 \le t \le T. \tag{3.1$'$}$$

This last equation can be solved readily:

$$u^*(t) = -\vartheta_T(t)\hat{B}(t) + \int_0^t \vartheta_T^2(s)\hat{B}(s)\, ds = u^*(0) - \int_0^t \vartheta_T(s)\, d\hat{B}(s)\ , \quad 0 \le t \le T \tag{3.3}$$

with

$$\vartheta_T(t) \stackrel{\triangle}{=} \frac{1}{\lambda + T - t}\ , \qquad u^*(0) = -\frac{\mathbb{E}(B)}{\lambda + T}\ . \tag{3.4}$$

Furthermore, the value of the stochastic control problem can be expressed in the form

$$V \;=\; J(u^*) \;=\; \inf_{u \in \mathcal{U}} J(u) \;=\; \mathbb{E}(B^2) - \lambda \int_0^T \vartheta_T^2(t) \cdot \mathbb{E}\left[\hat{B}^2(t)\right] dt. \tag{3.5}$$

Remark 3.1. The formula (3.3) shows that *the certainty-equivalence principle holds for the problem of (2.2)*, when $\psi(\cdot)$ and $\varphi(\cdot)$ are quadratic but the filtration $\mathbb{F}$ is quite general. To see this, think of the process

$$X^u(t) = B + \int_0^t u(s)\, ds, \qquad 0 \le t \le T \tag{3.6}$$

as the "state-process". Because $X^u(\cdot)$ has linear dynamics, in a degenerate sense, for each fixed control law $u(\cdot)$, problem (2.2) with quadratic $\psi(\cdot)$ and $\varphi(\cdot)$ is just a partially-observed *Linear Quadratic (LQ) control problem*, differing from the standard LQ problem only in that the "observation" filtration, with respect to which the control is adapted, is general and fixed (does not depend on the control process $u(\cdot)$). Now the equation (3.1)$'$ may be re-written in the form

$$u^*(t) \;=\; -\vartheta_T(t) \cdot \hat{X}^{u^*}(t) \stackrel{\triangle}{=} \vartheta_T(t) \cdot \mathbb{E}\left[X^{u^*}(t) \,\middle|\, \mathcal{F}_t\right]\ , \quad 0 \le t \le T\,, \tag{3.7}$$

which identifies the function $\vartheta_T(\cdot)$ of (3.4) as a "gain". But it is easily checked that if B is observable (i.e., $\mathcal{F}_0-$measurable), then we have

$$\bar{u}^*(t) \;=\; \vartheta_T(t)\cdot\left[B+\int_0^t \bar{u}^*(s)\,ds\right] \;=\; -\vartheta_T(t)\cdot X^{\bar{u}^*}(t)\,,\quad 0\le t\le T \tag{3.8}$$

for the optimal control process $\bar{u}^*(\cdot)$. Comparison of (3.7) and (3.8) gives the certainty-equivalence principle. Notice that no assumption of normality was placed upon the initial (prior) distribution of $X^u(0)=B$.

We are not aware of such a general statement of the certainty-equivalence principle, although the generality seems to be well-recognized. Whittle (1990) develops LQ stochastic control in discrete time, assuming that $\mathbb{F}$ is generated by an observation process whose dependence on the state-process noise is linear. Haussmann (1987) analyzes the standard case of linear observations in additive white noise, with initial law not necessarily normal.

Proof of (3.3)*:* With the notation of (3.2), (3.4) and using the martingale property of $u^*(\cdot)$, the equation (3.1) takes the form

$$\hat{B}(t)+\int_0^t u^*(s)\,ds = -(\lambda+T-t)\cdot u^*(t),\qquad 0\le t\le T$$

of (3.1)$'$. In particular, $u^*(0)=-\mathbb{E}(B)/(\lambda+T)$. This expression can be written in the equivalent, differential form

$$d\hat{B}(t) = -(\lambda+T-t)\,du^*(t),$$

which leads directly to (3.3) after integrating by parts. □

Proof of (3.5)*:* The value of the control problem is

$$V=\lambda\cdot\mathbb{E}\int_0^T \left(u^*(t)\right)^2 dt+\mathbb{E}\left(B+\int_0^T u^*(s)\,ds\right)^2. \tag{3.9}$$

Denoting by

$$X(t)\;\stackrel{\triangle}{=}\;\hat{B}(t)+\int_0^t u^*(s)\,ds,\qquad 0\le t\le T \tag{3.10}$$

the conditional expectation $\mathbb{E}\left[X^{u^*}(t)\,\middle|\,\mathcal{F}(t)\right]$ of the expression of (3.6) with $u(\cdot)=u^*(\cdot)$, given the observations up to time t, we see that

$$\begin{aligned}\mathbb{E}\left(B+\int_0^T u^*(s)\,ds\right)^2 \;&=\; \mathbb{E}\left(B+X(T)-\hat{B}(T)\right)^2 = \mathbb{E}\left(B^2\right)+\mathbb{E}\left(X^2(T)-\hat{B}^2(T)\right)\\ &=\; \mathbb{E}\left(B^2\right)+\mathbb{E}\left[\int_0^T u^*(t)\,dt\left(2\hat{B}(T)+\int_0^T u^*(s)\,ds\right)\right].\end{aligned}$$

Thanks to the martingale property of the processes $u^*(\cdot)$ and $\hat{B}(\cdot)$, this last expectation is

$$\begin{aligned}&\mathbb{E}\int_0^T u^*(t)\left[2\hat{B}(T)+\int_0^t u^*(s)\,ds+\int_t^T u^*(s)\,ds\right]dt\\ &\qquad=\mathbb{E}\int_0^T u^*(t)\left[2\hat{B}(t)+\left(X(t)-\hat{B}(t)\right)+(T-t)u^*(t)\right]dt.\end{aligned}$$

Therefore, (3.9) becomes

$$
\begin{aligned}
(3.11) \qquad V &= \mathbb{E}\left(B^2\right) + \mathbb{E}\int_0^T u^*(t)\left[X(t) + \hat{B}(t) + (\lambda + T - t)u^*(t)\right] dt \\
&= \mathbb{E}\left(B^2\right) + \mathbb{E}\int_0^T u^*(t)\hat{B}(t)\, dt,
\end{aligned}
$$

thanks to (3.1)$'$ and (3.10). Substituting from (3.3), we can write this last expectation as

$$
\begin{aligned}
\mathbb{E}\int_0^T \hat{B}(t)u^*(t)\, dt &= \mathbb{E}\int_0^T \hat{B}(t)\left[-\vartheta_T(t)\hat{B}(t) + \int_0^t \hat{B}(s)\vartheta_T^2(s)\, ds\right] dt \\
&= -\,\mathbb{E}\int_0^T \vartheta_T(t)\hat{B}^2(t)\, dt \;+\; \mathbb{E}\int_0^T \hat{B}(s)\vartheta_T^2(s)\left(\int_s^T \hat{B}(t)\, dt\right) ds \\
&= -\int_0^T \vartheta_T(t)\cdot\mathbb{E}\left(\hat{B}^2(t)\right) dt \;+\; \mathbb{E}\int_0^T \hat{B}^2(s)\vartheta_T^2(s)\cdot(T-s)\, ds \\
&= -\int_0^T \mathbb{E}\left(\hat{B}^2(t)\right)\cdot\vartheta_T(t)\left[1-(T-t)\vartheta_T(t)\right] dt \\
&= -\,\lambda\int_0^T \vartheta_T^2(t)\cdot\mathbb{E}\left(\hat{B}^2(t)\right) dt.
\end{aligned}
$$

Finally, substituting back into (3.11), we arrive at the expression of (3.5). □

4 A SPECIFIC FILTRATION

Let us specialize now to the case where the filtration $\mathbb{F}$ is the augmentation $\mathcal{F}(t)$ of $\mathcal{F}^Y(t) \triangleq \sigma\{Y(s),\ 0 \le s \le t\}$, namely, the filtration generated by the observation process

$$(4.1) \qquad Y(t) \triangleq Bt + W(t), \qquad 0 \le t \le T.$$

Here $W(\cdot)$ is a standard Brownian motion, independent of the random variable B. It will be assumed throughout that

$$(4.2) \qquad \mathbb{E}\left(B^2\right) < \infty.$$

The distribution of B will be denoted by μ, and, if μ is absolutely continuous with respect to Lebesgue measure, its density $\frac{d\mu}{dx}$ will be denoted by $p(\cdot)$. Given a function f, we adopt the notation

$$\mu(f) \triangleq \int f(x)d\mu(x), \qquad \langle f, p\rangle \triangleq \int f(x)p(x)\, dx$$

whenever the integrals exist. For convenience, we shall denote by $h(\cdot)$ the identity mapping $h(x) = x$, $x \in \mathbb{R}$; thus, $\mu(h)$ or $\langle h, p\rangle$ is the expectation of the random variable B.

As in Section 3, the *state process* is

$$X^u(t) \triangleq B + \int_0^t u(s)\, ds, \qquad 0 \le t \le T.$$

The stochastic control problem (2.2), using the filtration generated by $Y(\cdot)$, is a variant (special case) of the class of problems with linear state and observation dynamics but non-Gaussian initial distributions, studied by Haussmann (1987). In our formulation, observation and control are already separated. Theorem 2.1 and formula (2.12) lead to a direct construction of the optimal control, by a method different from Haussmann's. For our problem and method, Haussmann's strong moment assumption $\mathbb{E}(e^{\varepsilon B^2}) < \infty$ for some $\varepsilon > 0$, can be relaxed to (4.2).

Let μ_t^u (respectively, $p_t^u(\cdot)$) denote the "posterior distribution" (respectively, the "posterior density"), of the state $X^u(t)$ given $\mathcal{F}(t)$, for a fixed $u \in \mathcal{U}$. Explicit formulae and recursive equations for μ_t^u (or $p_t^u(\cdot)$) are well-known from filtering theory. We state these next. Our purpose is to derive an explicit expression for the dependence of the value function on the law of B in the quadratic case, and to provide background for the Mortensen dynamic programming equation in function-space. We take up these tasks in the subsequent section.

Direct calculation shows that the posterior distribution of B, given the observations up to time t, is given as

$$\eta_t(A) \triangleq \mathbb{P}\big(B \in A \,\big|\, \mathcal{F}_t\big) = \frac{\int_A \exp\big[xY(t) - \frac{t}{2}x^2\big]\,\mu(dx)}{\int_{\mathbb{R}} \exp\big[xY(t) - \frac{t}{2}x^2\big]\,\mu(dx)} \,. \tag{4.3}$$

For later purposes, it is convenient to adopt the notation

$$S(t,y)(x) \;\triangleq\; \exp\big[xy - \frac{t}{2}x^2\big]$$

and to define the multiplication operator $S(t,y)\nu$, which takes the measure ν to the new measure

$$\big[S(t,y)\nu\big](A) \triangleq \int_A \exp\big[xy - \frac{t}{2}x^2\big]\,\nu(dx), \qquad A \in \mathcal{B}(\mathbb{R}). \tag{4.4}$$

Define also the function

$$F(t,y;\mu) \triangleq \big[S(t,y)\mu\big](\mathbb{R}) = \int_{\mathbb{R}} \exp\big[xy - \frac{t}{2}x^2\big]\,\mu(dx)\,, \qquad t>0,\ x \in \mathbb{R}, \tag{4.5}$$

and observe that it satisfies the backward heat equation

$$F_t + \frac{1}{2}F_{yy} \;=\; 0, \qquad \text{on} \;\; (0,\infty)\times\mathbb{R}.$$

Then (4.3) can be written as

$$\eta_t = \frac{S\big(t,Y(t)\big)\mu}{F\big(t,Y(t);\mu\big)}\,. \tag{4.6}$$

Finally, for $\xi \in \mathbb{R}$, let τ_ξ denote the operation of translation by $-\xi$:

$$\big[\tau_\xi \nu\big](A) \triangleq \nu(A-\xi).$$

Then the "posterior" distribution of $X^u(t)$, given the observations (and the control $u(\cdot)$ used) up to time t, is

$$\begin{aligned} \mu_t^u(A) \;&=\; \mathbb{P}\big[X^u(t) \in A \,\big|\, \mathcal{F}(t)\big] = \mathbb{P}\left[B \in \left(A - \int_0^t u(s)\,ds\right) \,\middle|\, \mathcal{F}(t)\right] \\ &=\; \left[\tau_\xi \eta_t\right](A)\Big|_{\xi=U(t)} = \left.\frac{\left[\tau_\xi S(t,y)\mu\right](A)}{F(t,y;\mu)}\right|_{y=Y(t),\ \xi=U(t)} = \frac{\big[S\big(t,Y(t)\big)\mu\big]\big(A-U(t)\big)}{F\big(t,Y(t);\mu\big)}\,, \end{aligned} \tag{4.7}$$

where $U(t) \triangleq \int_0^t u(s)\,ds$. From these identities it follows that

$$\hat{B}(t) = \mathbb{E}\left[B \,\middle|\, \mathcal{F}(t)\right] = G\big(t, Y(t); \mu\big), \qquad t \geq 0, \tag{4.8}$$

where

$$G(t, y; \mu) \triangleq \left(\frac{F_y}{F}\right)(t, y; \mu) = \frac{1}{F(t, y; \mu)} \int_{\mathbb{R}} x S(t, y)(x)\, \mu(dx). \tag{4.9}$$

Similarly, the posterior variance of B is

$$\mathrm{Var}_t(B) \;\triangleq\; \mathbb{E}\left[\big(B - \hat{B}(t)\big)^2 \,\middle|\, \mathcal{F}(t)\right] \;=\; G_y\big(t, Y(t); \mu\big). \tag{4.10}$$

It is well-known from filtering theory that the "innovations process"

$$N(t) \triangleq Y(t) - \int_0^t \hat{B}(s)\, ds = Y(t) - \int_0^t G\big(s, Y(s); \mu\big)\, ds, \qquad 0 \leq t \leq T \tag{4.11}$$

is an $\mathbb{F}$-Brownian motion (cf. Kallianpur (1980), or Liptser & Shiryaev (2000), Chapter 8).

By Itô's rule and the equation

$$G_t + \frac{1}{2} G_{yy} + G G_y = 0, \qquad \text{on} \;\; (0, \infty) \times \mathbb{R}, \tag{4.12}$$

we obtain the equation

$$d\hat{B}(t) = \mathrm{Var}_t(B)\, dN(t), \qquad \hat{B}(0) = \mu(h), \tag{4.13}$$

which is also well-known from filtering theory.

§4.1 The Value Function in the case of Quadratic Cost-Functions

Let us consider the special case $\varphi(x) = \psi(x) = x^2$. Then, in light of (3.5) and (4.8), the value $V \equiv V(T, \mu)$ of the partially-observed stochastic control problem may be written explicitly

$$V(T, \mu) = \mu(h^2) - \lambda \int_0^T \vartheta_T^2(t) \cdot \mathbb{E}_\mu\left[G^2\big(t, Bt + W(t); \mu\big)\right]\, dt\ , \tag{4.14}$$

as a function of the time-to-go T and of the "prior" distribution μ for the random variable B. We are using the notation

$$\mathbb{E}_\mu\left[H\big(t, Bt + W(t)\big)\right] \;\triangleq\; \int_{\mathbb{R}} \mu(db) \left[\int_{\mathbb{R}} H(t, bt + w) \frac{e^{-\frac{w^2}{2t}}}{\sqrt{2\pi t}}\, dw\right]. \tag{4.15}$$

Remark 4.1. A change of variables and an integration show that

$$\mathbb{E}_\mu\left[H\big(t, Bt + W(t)\big)\right] = \mathbb{E}\left[H\big(t, W(t)\big) \cdot F\big(t, W(t); \mu\big)\right] \tag{4.15$'$}$$

in the notation of (4.5) and (4.15). Then, in (4.14), we may write

$$\mathbb{E}_\mu\left[G^2\big(t, Bt + W(t); \mu\big)\right] = \mathbb{E}\left[\left(\frac{F_y^2}{F}\right)\big(t, W(t); \mu\big)\right], \tag{4.16}$$

a simplification which is sometimes useful.

Remark 4.2. We shall assume often that μ has density $p(\cdot) = \frac{d\mu}{dx}$ with respect to Lebesgue measure. In such a case, we shall find it useful to replace μ everywhere in the notation, by $p(\cdot)$, for instance

$$F(t,y;p) = \int_{\mathbb{R}} e^{xy-\frac{t}{2}x^2} p(x)\,dx, \qquad G(t,y;p) = \frac{F_y(t,y;p)}{F(t,y;p)}$$

in (4.5), (4.9), or

$$V(T,p) = \langle h^2, p\rangle - \lambda \int_0^T \vartheta_T^2(t)\cdot \mathbb{E}_p\left[G^2\big(t, Bt+W(t);p\big)\right]\,dt \tag{4.14$'$}$$

in (4.14). This should cause no confusion.

§4.2 The Time-Derivative of $V(t,\mu)$

The following relatively simple calculation will be needed in the sequel.

Lemma 4.1. *The function $V(T,\mu)$ of (4.14) has temporal derivative*

$$\frac{\partial}{\partial T} V(T,\mu) = -\frac{\lambda}{(\lambda+T)^2}\,\mu^2(h) - \lambda\int_0^T \vartheta_T^2(t)\cdot \mathbb{E}_\mu\left[G_y^2\big(t,Bt+W(t);\mu\big)\right]\,dt. \tag{4.17}$$

Proof: Differentiation in (4.14) gives

$$\begin{aligned}
-\frac{1}{\lambda}\frac{\partial}{\partial T}V(T,\mu) &= \vartheta_T^2(T)\cdot\mathbb{E}_\mu\left[G^2\big(T,BT+W(T);\mu\big)\right] - \int_0^T \mathbb{E}_\mu\left[G^2\big(t,Bt+W(t);\mu\big)\right]\cdot\frac{\partial}{\partial t}\big(\vartheta_T^2(t)\big)\,dt \\
&= \vartheta_T^2(0)\mu^2(h) + \int_0^T \vartheta_T^2(t)\cdot\frac{\partial}{\partial t}\mathbb{E}_\mu\left[G^2\big(t,Bt+W(t);\mu\big)\right]\,dt,
\end{aligned}$$

which leads directly to (4.17) in conjunction with the observation

$$\mathbb{E}_\mu\left[G^2\big(t,Bt+W(t);\mu\big)\right] = \mu^2(h) + \int_0^t \mathbb{E}_\mu\left[G_y^2\big(s,Bs+W(s);\mu\big)\right]\,ds, \qquad 0\le t\le T. \tag{4.18}$$

To see that (4.18) holds, re-write it in the form

$$\mathbb{E}\left(\hat{B}^2(t)\right) = \mu^2(h) + \int_0^t \mathbb{E}\left(\mathrm{Var}_s(B)\right)^2\,ds, \qquad 0\le t\le T \tag{4.18$'$}$$

using the notation of (4.8) and (4.10), and observe that

$$d\left(\hat{B}(t)\right)^2 = 2\hat{B}(t)\cdot\mathrm{Var}_t(B)\,dN(t) + \left(\mathrm{Var}_t(B)\right)^2\,dt$$

is a consequence of (4.13) and Itô's rule. In particular,

$$\mathbb{E}\left(\hat{B}^2(t\wedge\tau_n)\right) = \mathbb{E}\left(B^2\right) + \mathbb{E}\int_0^{t\wedge\tau_n}\left(\mathrm{Var}_s(B)\right)^2\,ds, \qquad \forall\, n\in\mathbb{N},$$

where $\tau_n \triangleq \inf\left\{t\ge 0\,/\,|\hat{B}(t)| + \mathrm{Var}_t(B)\ge n\right\}\wedge T$. Then (4.18) follows by letting $n\to\infty$, thanks to the Monotone and the Dominated Convergence Theorems, as well as Doob's inequality

$$\mathbb{E}\left[\sup_{0\le t\le T}\hat{B}^2(t)\right] \le 4\cdot\mathbb{E}\left[\hat{B}^2(T)\right] \le 4\cdot\mathbb{E}\left[B^2\right] < \infty\,,$$

which is valid since $\hat{B}(\cdot)$ is a martingale. $\square$

5 A FORMAL HJB EQUATION OF MORTENSEN

Throughout this section, which is mostly heuristic, it will be assumed that the initial distribution μ of B admits a probability density function $p(\cdot)$.

In the framework of the specific filtration of Section 4, we may express the cost functional $J(u)$ of (2.2) as

$$J(u;T,p) = \mathbb{E}\left[\lambda \int_0^T \psi\big(u(t)\big)\,dt + \varphi\big(X^u(T)\big)\right] = \mathbb{E}\left[\lambda \int_0^T \psi\big(u(t)\big)\,dt + \langle \varphi, p_T^u\rangle\right], \tag{5.1}$$

using the notation of Section 4 for the posterior density $p_T^u(\cdot)$ of the random variable $X^u(T)$ at time T, and indicating explicitly the dependence on time-to-go T and on the prior density $p(\cdot)$. Similarly, we write the value function as

$$V(T,p) \stackrel{\triangle}{=} \inf_{u\in\mathcal{U}} J(u;T,p). \tag{5.2}$$

Now observe that the process $\{p_t^u(\cdot),\ 0\le t\le T\}$, satisfies the *Kushner-Stratonovich filtering equation*

$$\left\{\begin{array}{lcll} dp_t^u(x) &=& -u(t)\cdot \frac{\partial}{\partial x}p_t^u(x)\,dt + \big(x-\langle h,p_t^u\rangle\big)p_t^u(x)\,dN(t) & ; \qquad t>0,\ x\in\mathbb{R} \\ p_0^u(x) &=& p(x) & ; \qquad x\in\mathbb{R} \end{array}\right. \tag{5.3}$$

(cf. Kallianpur (1980), or Liptser & Shiryaev (2000), Chapter 8), where $N(\cdot)$ is the innovation process of (4.11). If the "prior" density $p(\cdot)$ is continuously differentiable, then (5.3) follows by Itô differentiation of the formula

$$p_t^u(x) = \frac{\exp\left[\big(x-U(t)\big)Y(t) - \frac{t}{2}\big(x-U(t)\big)^2\right]}{F\big(t,Y(t);p\big)}\cdot p\big(x-U(t)\big)\ , \tag{5.4}$$

where $U(t)=\int_0^t u(s)\,ds$. The expression of (5.4) is just the density version of (4.7).

More generally, (5.3) is valid in a weak form. If $f:\mathbb{R}\to\mathbb{R}$ is any $\mathcal{C}^1$-function with compact support, then from (4.7) we have

$$d\mu_t^u(f) = u_t\cdot\mu_t^u(f')\,dt + \mu_t^u\big((h-\mu_t(h))f\big)\,dN(t). \tag{5.5}$$

Now the equations (5.2)-(5.3) describe a fully-observed stochastic control problem. Mortensen (1966) suggested a Hamilton-Jacobi-Bellman-type equation for this kind of reformulation of partially observed control, by analogy with ordinary stochastic control. If one has (5.3) as the equation of evolution in an infinite-dimensional space of densities $p(\cdot)$, along with a definition of first- and second-order functional derivatives $D_pV(T,p)[q]$ and $D_{pp}V(T,p)[q_1,q_2]$ on this space (as in (6.2) and (6.3)), Mortensen's equation takes the form

(5.6)
$$\left\{\begin{array}{lcl} \frac{\partial V}{\partial T}(T,p) &=& \frac{1}{2}D_{pp}V(T,p)\,[h-\langle h,p\rangle, h-\langle h,p\rangle] + \min_{a\in\mathbb{R}}\Big(\lambda\psi(a) - a\cdot D_pV(T,p)[p']\Big), \\ V(0,p) &=& \langle\varphi,p\rangle, \end{array}\right.$$

for the problem under consideration. Going further, if for a fixed $T \in (0, \infty)$ we set

$$u^*(t,p) \stackrel{\triangle}{=} \operatorname{argmin}_{a\in\mathbb{R}} \left\{ \lambda\psi(a) - a \cdot D_p V(T-t,p)[p'] \right\}, \tag{5.7}$$

or equivalently,

$$u^*(t,p) = \left(\psi'\right)^{-1} \left(\frac{1}{\lambda} \cdot D_p V(T-t,p)[p'] \right), \tag{5.8}$$

then (5.8) should provide an optimal law in feedback form; that is, if $p^*(\cdot)$ solves the evolution equation

$$\begin{cases} dp_t^*(x) &= -u^*(t,p_t^*) \cdot \frac{\partial}{\partial x} p_t^*(x)\, dt + \big(x - \langle h, p_t^* \rangle\big) p_t^*(x)\, dN(t), \qquad t > 0 \\ p_0^*(x) &= p(x), \end{cases}$$

then $\{u^*(t,p_t^*),\ 0 \le t < T\}$ should be an optimal control law for the partially-observed stochastic control problem under consideration.

As usual, *the connection between (5.6) and stochastic control is forged through Itô's rule.* Suppose one has a candidate solution V to (5.6). Then from (5.3) it follows that

$$\begin{aligned} dV(T-t,p_t^u) = \Bigg\{ &-\frac{\partial}{\partial T} V(T-t,p_t^u) - u_t\, D_p V(T-t,p_t^u) \left[(p_t^u)' \right] \\ &+ \frac{1}{2} D_{pp} V(T-t,p_t^u) \left[\big(h - \langle h, p_t^u \rangle\big) p_t^u, \big(h - \langle h, p_t^u \rangle\big) p_t^u \right] \Bigg\}\, dt \\ &+ D_p V(T-t,p_t^u) \left[\big(h - \langle h, p_t^u \rangle\big) p_t^u \right]\, dN(t). \end{aligned} \tag{5.9}$$

In conjunction with (5.6), the semimartingale decomposition (5.9) implies, again formally, that $\left\{ V(T-t,p_t^u) + \lambda \int_0^t \psi\big(u(s)\big)\, ds,\ 0 \le t \le T \right\}$ is a (local) supermartingale, and hence, with some extra work, that $V(T,p) \le J(u,T,p)$. In other words, $V(T,p)$ is a lower bound on the value of the stochastic control problem. Likewise, if $u^*(\cdot)$ as in (5.7) defines an admissible control, (5.6) and (5.9) should imply $V(T,p) = J(u^*;T,p)$, since then $\left\{ V(T-t,p_t^{u^*}) + \lambda \int_0^t \psi\big(u^*(s)\big)\, ds,\ 0 \le t \le T \right\}$ is a (local) martingale.

To what extent can Mortensen's idea be developed into a rigorous theory that covers the stochastic control problem of (5.2)-(5.3)? This question was posed and discussed in a broad context by Beneš & Karatzas (1983). They gave an analytic sense to (5.6) but offered no examples. In Section 6, we show that the function $V(T,p)$, defined in (4.14) for the quadratic case, does indeed provide such an example, if we calculate $D_p V$ and $D_{pp} V$ formally.

The ideal theory for Mortensen's equation would start with an analytic and general definition of the functional derivatives $D_p V$ and $D_{pp} V$, build spaces of regular functionals based on these derivatives — analogues of $\mathbb{L}^2(\mathbb{R})$ or the Sobolev space $H^2(\mathbb{R})$ — and prove that V belongs to such a class. This we have not attemped, for reasons to be discussed shortly. But we do show in Section 7 that (5.9) is rigorously valid for our formally defined $D_p V(T,p)[p']$ and $D_{pp} V \left[\big(h - \langle h, p \rangle\big) p, \big(h - \langle h, p \rangle\big) p \right]$. From the point of view of stochastic control, (5.9) is the fact one really needs; its validity explains, in a loose sense, why the formal verification of Mortensen's equation is valid.

Fleming & Pardoux (1982) introduced another viewpoint on the idea of regarding partially-observed control as the fully-observed control of an infinite-dimensional stochastic evolution. They re-wrote the problem as one of the controlling the Zakai equation for the unnormalized conditional density. Because the Zakai equation can often be treated as an evolution equation in a nice Hilbert space, for example $\mathbb{L}^2(\mathbb{R})$, the Fleming-Pardoux approach can be made rigorous under fairly general assumptions and does provide a framework for Mortensen's equation. See Lions (1988, 1989) for analytic developments of this theory using viscosity solution ideas. The advantage of the Hilbert space context is that it allows for easy and general definitions of the first- and second-order functional derivatives.

A parallel development of a general theory for partially-observed stochastic control (on an infinite horizon, with discounting) appears in the work of Hijab (1991, 1992). This authos discusses the existence, uniqueness and first-order smoothness of solutions for the analogue of Mortensen equation (5.6) in the space $\mathcal{P}(\mathbb{R}^d)$ of probability measures on $\mathbb{R}^d$, with suitable definitions of first- and second-derivatives. Hijab studies first-order smoothness of the value function of partially-observed control problems as a functional on the space $\mathcal{P}(\mathbb{R}^d)$, using the following approach. Consider, for example, a functional Φ on $\mathcal{P}(\mathbb{R}^d)$; the tangent space $\mathcal{T}(\mathcal{P};\mu)$ to $\mathcal{P} \equiv \mathcal{P}(\mathbb{R}^d)$ at a probability measure μ, is defined to be the set of bounded, signed measures ν, such that $\mu + |t|\nu \in \mathcal{P}(\mathbb{R}^d)$ for all sufficiently small t. (This "tangent space" differs for different μ; the space of probability measures cannot be characterized as a manifold modelled on a Banach space.) The differential of Φ at μ, if it exists, is given by a function $\psi_\mu(\cdot)$ such that $\frac{d}{dt}\Phi(\mu+t\nu)|_{t=0} = \nu(\psi_\mu) \equiv \int \psi_\mu(x)\,dx$, for every $\nu \in \mathcal{T}(\mathcal{P};\mu)$. It can be verified that $V(T,\mu)$ as defined in (4.14) is differentiable in that sense. Hijab gives general conditions under which the value function has such first-order regularity.

In the problem of our paper, *these approaches do not appear helpful when the cost functions* $\varphi(\cdot), \psi(\cdot)$ *are unbounded*. For example, $V(T,p)$ as defined in (4.14) does not extend naturally to a linear space such as $\overline{\mathbb{L}}^1 \triangleq \{\rho \in \mathbb{L}^1 \;/\; \langle h^2,\rho\rangle < \infty\}$, because the term $F(t,y;p)$ that appears in the denominator of G can vanish if $p(\cdot)$ is not positive. *This difficulty goes to the heart of defining* $D_pV(T,p)[p']$ *and* $D_{pp}V(T,p)[q_1,q_2]$ *rigorously*, because the choice of space of the variation in direction $q(\cdot)$ from $p(\cdot)$ is delicate. In our treatment, we stick with the formal definitions and do not try rigorously to formulate a notion of first- and second-order derivatives. The formal directions of differentiation for the expressions in Mortensen equation are not necessarily in the tangent space, anyway. Rather, they are the expressions, to the first order, of a *special, finite-dimensional, space of nonlinear variations* in the space of probability measures. Fortunately, in order to establish (5.9), one need analyze rigorously only these variations, and show that they may be computed in terms of the formal derivatives derived in Section 6. This program is carried out in Section 7.

6 FORMAL VALIDITY OF MORTENSEN'S EQUATION

Let $\varphi(x) = \psi(x) = x^2$. Then Mortensen's equation (5.6) becomes

$$\text{(6.1)} \qquad \left\{ \begin{array}{rcl} \frac{\partial V}{\partial T}(T,p) & = & \frac{1}{2} D_{pp}V(T,p)\,[h-\langle h,p\rangle, h-\langle h,p\rangle] - \frac{1}{4\lambda}\left(D_pV(T,p)[p']\right)^2, \qquad T>0 \\ V(T,p) & = & \langle h^2,p\rangle. \end{array} \right.$$

In this section, we undertake all our formal calculations assuming $p \in \mathbb{L}^1(\mathbb{R}) \cap \mathcal{C}^1(\mathbb{R})$ and $\mathbb{E}(B^4) = \langle h^4,p\rangle < \infty$. Let $V(T,p)$ be defined as in (4.14), Remark 4.2. We shall define the directional derivatives

$$D_p V(T,p)[q] \;\triangleq\; \frac{d}{d\varepsilon} V(T, p+\varepsilon q)\Big|_{\varepsilon=0} \tag{6.2}$$

$$D_{pp} V(T,p)[q,r] \;\triangleq\; \frac{d}{d\varepsilon} D_p V(T, p+\varepsilon r)[q]\Big|_{\varepsilon=0}\,, \tag{6.3}$$

formally assuming that $V(T, p+\varepsilon q)$, $V(T, p+\varepsilon r)$ make sense for all sufficiently small ε, that the derivatives and integrals may be interchanged, and that all final expressions are well-defined.

In the resulting expression for $D_p V(T,p)[p']$, all dependence on $p'(\cdot)$ factors through integrals of the type $\langle g, p'\rangle$. We evaluate $D_p V(T,p)[p']$, assuming the integration by parts formula $\langle g, p'\rangle = -\langle g', p\rangle$ whenever $\langle g, p'\rangle$ appears. With these assumptions, we show formally that

$$D_p V(T,p)[p'] \;=\; -\frac{2\lambda}{\lambda+T} \cdot \langle h, p\rangle\;, \tag{6.4}$$

$$D_{pp} V(T,p)\left[\big(h - \langle h,p\rangle\big)p, \big(h - \langle h,p\rangle\big)p\right] \;=\; -2\lambda \int_0^T \vartheta_T^2(t)\cdot \mathbb{E}_p\left[G_y^2\big(t, Bt+W(t);p\big)\right]\,dt. \tag{6.5}$$

In conjunction with

$$\frac{\partial}{\partial T} V(T,p) \;=\; -\frac{\lambda}{(\lambda+T)^2}\,\langle h,p\rangle^2 - \lambda \int_0^T \vartheta_T^2(t)\cdot \mathbb{E}\left[G_y^2\big(t, Bt+W(t);p\big)\right]\,dt,$$

the expression of (4.17) for the temporal derivatives of $V(T,p)$, *the formal computations (6.4) and (6.5) show that V obeys the Mortensen equation (6.1).*

Recall from (3.7) that the optimal control $u^*(\cdot)$ satisfies

$$-(\lambda+T-t)\cdot u^*(t) = \mathbb{E}\left[X^{u^*}(t)\,\middle|\,\mathcal{F}(t)\right] = \langle h, p_t^{u^*}\rangle\;;$$

thus, in light of (6.4), we have

$$u^*(t) = \frac{1}{2\lambda}\cdot D_p V(T-t, p_t^{u^*})\left[\big(p_t^{u^*}\big)'\right], \qquad\qquad 0 \le t < T,$$

thereby verifying the feedback formula (5.7).

§6.1 Heuristic Computation of (6.2) and (6.3)

Differentiating formally the expression of $(4.14)'$ for $V(T,p)$, Remark 4.2, with respect to its functional argument as in (6.2), yields

$$\begin{aligned} D_p V(T,p)[q] = \langle h^2, q\rangle &- \lambda \int_0^T \vartheta_T^2(t)\cdot \mathbb{E}_q\left[G^2\big(t, Bt+W(t);p\big)\right]\,dt \\ &- \lambda \int_0^T \vartheta_T^2(t)\cdot \mathbb{E}_p\left[2G\big(t, Bt+W(t);p\big)\cdot D_p G\big(t, Bt+W(t);p\big)[q]\right]\,dt\;; \end{aligned} \tag{6.6}$$

here $D_pG(t,y;p)[q]$ is defined formally as in (6.2). Differentiating formally again, as in (6.3), leads to

$$
\begin{aligned}
-\frac{1}{2\lambda}\cdot D_{pp}V(T,p)[q,q] = 2\int_0^T & \vartheta_T^2(t)\cdot \mathbb{E}_q\left[G\big(t,Bt+W(t);p\big)\cdot D_pG\big(t,Bt+W(t);p\big)[q]\right]\,dt \\
& + \int_0^T \vartheta_T^2(t)\cdot \mathbb{E}_p\left[\Big(D_pG\big(t,Bt+W(t);p\big)[q]\Big)^2\right]\,dt \\
& + \int_0^T \vartheta_T^2(t)\cdot \mathbb{E}_p\left[G\big(t,Bt+W(t);p\big)\cdot D_{pp}G\big(t,Bt+W(t);p\big)[q,q]\right]\,dt.
\end{aligned}
\tag{6.7}
$$

§6.2 Formal Derivation of (6.4)

Formal integration by parts leads to the formulae

$$
F(t,y;p') = (tF_y - yF)(t,y;p), \qquad F_y(t,y;p') = (tF_{yy} - yF_y - F)(t,y;p). \tag{6.8}
$$

Hence, by simple calculation using the linearity of F, F_y, etc. in $p(\cdot)$, we get

$$
D_pG(t,y;p)[p'] = \frac{d}{d\varepsilon}G(t,y;p+\varepsilon p')\Big|_{\varepsilon=0} = -\left(1+tG^2 - t\frac{F_{yy}}{F}\right)(t,y;p) = -1+tG_y(t,y;p). \tag{6.9}
$$

On the other hand, integration by parts in (4.15) yields:

$$
\mathbb{E}_{p'}\left[G^2\big(t,Bt+W(t);p\big)\right] = -2t\cdot \mathbb{E}_p\left[(GG_y)\big(t,Bt+W(t);p\big)\right]. \tag{6.10}
$$

By substitution of (6.9) and (6.10) into (6.6) and use of

$$
\mathbb{E}_p\left[G\big(t,Bt+W(t);p\big)\right] = \mathbb{E}[\hat{B}(t)] = \mathbb{E}[B] = \langle h,p\rangle,
$$

we obtain

$$
\begin{aligned}
D_pV(T,p)[p'] &= -2\langle h,p\rangle + 2\lambda\int_0^T \mathbb{E}_p\left[G\big(t,Bt+W(t);p\big)\right]\cdot\frac{\partial}{\partial t}\vartheta_T(t)\,dt \\
&= 2\langle h,p\rangle\left[\lambda\big(\vartheta_T(T)-\vartheta_T(0)\big)-1\right] \\
&= -\frac{2\lambda}{\lambda+T}\cdot\langle h,p\rangle,
\end{aligned}
$$

proving (6.4).

§6.3 Formal Derivation of (6.5)

Let $r = \big(h - \langle h,p\rangle\big)p$. Because $\frac{\partial^k}{\partial y^k}F(t,y;hp) = \frac{\partial^{k+1}}{\partial y^{k+1}}F(t,y;p)$ and $F(t,y;\alpha p) = \alpha F(t,y;p)$ for $\alpha\in\mathbb{R}$, we have

$$
D_pG(t,y;p)[r] = \frac{d}{d\varepsilon}G\big(t,y;p+\varepsilon r\big)\Big|_{\varepsilon=0} = G_y(t,y;p) \tag{6.11}
$$

and

$$
D_{pp}G(t,y;p)[r,r] = \frac{d}{d\varepsilon}D_pG\big(t,y;p+\varepsilon r)\big)[r]\Big|_{\varepsilon=0} = -2G_y(t,y;p)\big[G(t,y;p)-\langle h,p\rangle\big]. \tag{6.12}
$$

Substitution into (6.8), along with the observation that $\mathbb{E}_r[H] = \mathbb{E}_p[BH] - \langle h,p\rangle \mathbb{E}_p[H]$, leads to

$$-\frac{1}{2\lambda}\cdot D_{pp}V(T,p)[r,r] = 2\int_0^T \vartheta_T^2(t)\cdot \mathbb{E}_p\left[G_y^2\big(t,Bt+W(t);p\big) + \xi(t)\left(B-\hat{B}(t)\right)\right]\,dt,$$

where $\xi(t) \stackrel{\triangle}{=} (GG_y)\big(t,Bt+W(t);p\big)$. But $\xi(t)$ is $\mathcal{F}(t)$-measurable for each $0<t\le T$, and so

$$\mathbb{E}_p\left[\xi(t)\left(B-\hat{B}(t)\right)\right] = \mathbb{E}_p\left[\xi(t)\left(\hat{B}(t)-\hat{B}(t)\right)\right] = 0.$$

This last step uses the assumption $\langle h^4,p\rangle<\infty$ since, by the methodology of Lemma 7.1 below (in particular, the inequalities (7.13)-(7.16) with $\varepsilon=0$, $t>0$, $k=1,2$), we have then $|\xi(t)|\le K_t\left[1+|Bt+W(t)|^3\right]$ and $|\hat{B}(t)|\le K_t\left[1+|Bt+W(t)|\right]$ for some real constant K_t; thus $\xi(t)(B-\hat{B}(t))$ is integrable. The result (6.5) follows.

Remark 6.1. Let ν be a positive, finite measure on $\mathbb{R}$, and denote by ν' the distributional derivative of ν. Formally replacing $p(\cdot)$ by ν, and $p'(\cdot)$ by ν', in (6.4) and (6.5), we make the identifications

$$D_pV(T,\nu)[\nu'] \equiv -\frac{2\lambda}{\lambda+T}\ \langle h,\nu\rangle \tag{6.13}$$

and

$$D_{pp}V(T,\nu)\left[\big(h-\langle h,\nu\rangle\big)\nu,\big(h-\langle h,\nu\rangle\big)\nu\right] \equiv -2\lambda\int_0^T \vartheta_T^2(t)\cdot\mathbb{E}_\nu\left[G_y^2\big(t,Bt+W(t);\nu\big)\right]\,dt. \tag{6.14}$$

Here and below, $h\nu$ is the measure $[h\nu](A)\stackrel{\triangle}{=}\int_A x\nu(dx)$.

7 THE FORMAL DERIVATIVE AND ITÔ RULE

In this section we return to considering the value function $V(T,\mu)$ of (4.14) for general probability measures μ such that $\mu(h^2)=\int x^2\mu(dx)<\infty$. Our aim is to establish, with the identifications of $D_pV(T,\mu)[\mu']$ and $D_{pp}V(T,\mu)\left[\big(h-\langle h,\mu\rangle\big)\mu,\big(h-\langle h,\mu\rangle\big)\mu\right]$ as in (6.13) and (6.14), the Itô formula (5.9).

Theorem 7.1. *For $0\le t<T$, with the identifications of (6.13), (6.14), and with μ_t^u defined as in* (4.7)*, we have the Itô rule*

$$\begin{aligned} dV(T-t,\mu_t^u) = \Big\{ &-\frac{\partial}{\partial T}V(T-t,\mu_t^u) - u_t\ D_pV(T-t,\mu_t^u)\left[(\mu_t^u)'\right] \\ &+\frac{1}{2}D_{pp}V(T-t,\mu_t^u)\left[\big(h-\langle h,\mu_t^u\rangle\big)\mu_t^u,\big(h-\langle h,\mu_t^u\rangle\big)\mu_t^u\right]\Big\}\ dt \\ &+D_pV(T-t,\mu_t^u)\left[\big(h-\langle h,\mu_t^u\rangle\big)\mu_t^u\right]\ dN(t). \end{aligned} \tag{7.1}$$

Proof: Fix a probability measure μ on $\mathcal{B}(\mathbb{R})$, and let

$$M(t,y,\xi) \ \triangleq\ \frac{\tau_\xi S(t,y)\mu}{F(t,y;\mu)} \ ,$$

in the notation of (4.4)-(4.7). Define

$$\frac{\partial M}{\partial y}(t,y,\xi) \ \triangleq\ \frac{\tau_\xi S_y(t,y)\mu}{F(t,y;\mu)} - G(t,y;\mu)M(t,y,\xi), \tag{7.2}$$

where

$$[\tau_\xi S_y(t,y)\mu]\,(A) \triangleq \int_A \frac{\partial}{\partial y} S(t,y)(x-\xi)\mu(dx-\xi) = [h\tau_\xi S(t,y)\mu]\,(A) - \xi\cdot[\tau_\xi S(t,y)\mu]\,(A).$$

By substituting this expression in (7.2) and observing that

$$\langle h, M(t,y,\xi)\rangle = \xi + G(t,y;\mu),$$

we get

$$\frac{\partial M}{\partial y}(t,y,\xi) = \big(h - \langle h, M(t,y,\xi)\rangle\big)M(t,y,\xi). \tag{7.3}$$

Now, from (4.7) and with $U(t)=\int_0^t u(s)\,ds$, we have $\mu_t^u = M\big(t,Y(t),U(t)\big)$, thus also

$$V(T-t,\mu_t^u) \ =\ V\big(T-t, M(t,y,\xi)\big)\Big|_{y=Y(t),\ \xi=U(t)} \ . \tag{7.4}$$

We study the function

$$(T,r,y,\xi) \longmapsto V\big(T, M(r,y,\xi)\big)$$

on $\{T>0,\ r>0,\ y\in\mathbb{R},\ \xi\in\mathbb{R}\}$, and show that it is continuously differentiable in (T,r,y,ξ), that it is twice continuously differentiable in y, and that we have

(7.5)

$$\frac{\partial V}{\partial T}\big(T, M(r,y,\xi)\big) = \left(-\frac{\lambda}{(\lambda+T)^2}\ \nu^2(h) - \lambda\int_0^T \vartheta_T^2(t)\cdot\mathbb{E}_\nu\left[G_y^2\big(t, Bt+W(t);\nu\big)\right]\,dt\right)\Bigg|_{\nu=M(r,y,\xi)}$$

by analogy with (4.17), as well as

$$\begin{aligned}
&(7.6)\quad & \frac{\partial V}{\partial r}\big(T,M(r,y,\xi)\big) &= D_pV\big(T,M(r,y,\xi)\big)\left[\frac{\partial M}{\partial r}(r,y,\xi)\right];\\
&(7.7)\quad & \frac{\partial V}{\partial y}\big(T,M(r,y,\xi)\big) &= D_pV\big(T,M(r,y,\xi)\big)\left[\frac{\partial M}{\partial y}(r,y,\xi)\right];\\
&(7.8)\quad & \frac{\partial V}{\partial \xi}\big(T,M(r,y,\xi)\big) &= \frac{2\lambda}{\lambda+T}\,\langle h, M(r,y,\xi)\rangle = -D_pV\big(T,M(r,y,\xi)\big)\left[M(r,y,\xi)'\right];\\
&(7.9)\quad & \frac{\partial^2 V}{\partial y^2}\big(T,M(r,y,\xi)\big) &= D_{pp}V\big(T,M(r,y,\xi)\big)\left[\frac{\partial M}{\partial y}(r,y,\xi), \frac{\partial M}{\partial y}(r,y,\xi)\right]\\
&&&\quad + D_pV\big(T,M(r,y,\xi)\big)\left[\frac{\partial^2}{\partial y^2}M(r,y,\xi)\right],
\end{aligned}$$

with the indentifications of (6.13), (6.14) and (7.3). In (7.6), we interpret

$$\frac{\partial M}{\partial r}(r,y,\xi) \triangleq \frac{\tau_\xi S_r(r,y)\mu}{F(r,y;\mu)} - \frac{F_r(r,y;\mu)}{F(r,y;\mu)} M(r,y,\xi),$$

while in (7.9) we interpret

$$\frac{\partial^2 M}{\partial y^2}(r,y,\xi) \triangleq \frac{\tau_\xi S_{yy}(r,y)\mu}{F(r,y;\mu)} - G(r,y;\mu)\left[\frac{\tau_\xi S_y(r,y)\mu}{F(r,y;\mu)} + \frac{\partial M}{\partial y}(r,y,\xi)\right] - G_y(r,y;\mu)\, M(r,y,\xi)$$

in accordance with (7.2). Since $S_r + \frac{1}{2}S_{yy} = 0$, $F_r + \frac{1}{2}F_{yy} = 0$, and thus $G_y + G^2 = -2F_r/F$, one may check

$$\left[\frac{\partial M}{\partial r} + \frac{1}{2}\frac{\partial^2 M}{\partial y^2}\right](r,y,\xi) \;=\; -\,G(r,y;\mu)\,\frac{\partial M}{\partial y}(r,y,\xi). \tag{7.10}$$

The formulae (7.5)-(7.9) express the fact that *variations in r, y, and ξ factor correctly through the formal functional derivatives.* Part of the assertion of (7.5)-(7.9) is that the right-hand sides are well-defined for $r > 0$, $T > 0$.

The Itô rule of (7.1) now follows easily by application of the ordinary Itô rule to the process $\{V(T-t, M(t,Y(t),\xi(t)));\ 0 < t < T\}$ using (7.2), (7.5)-(7.9), (7.10) and the observation

$$D_pV\big(T, M(r,y,\xi)\big)\left[G(r,y;\mu)\,\frac{\partial M}{\partial y}(r,y,\xi)\right] = G(r,y;\mu)\cdot D_pV\big(T, M(r,y,\xi)\big)\left[\frac{\partial M}{\partial y}(r,y,\xi)\right],$$

which is due to the linearity of $D_pV(T,p)[q]$ in q. Recall $\hat{B}(t) = G\big(t, Y(t);\mu\big)$ in making this calculation.

It remains to establish (7.5)-(7.9). Note first that

$$\int e^{\delta x^2} M(s,y,\xi)(dx) < \infty, \qquad \text{if } 0 < \delta < \frac{s}{2}\,. \tag{7.11}$$

Thus, the measure $M(s,y,\xi)$ has strong moment properties. We already established (7.5) in Lemma 4.1 of Section 4, when M is replaced by a measure μ with weak moment properties, namely, $\mu(h^2) = \int x^2 \mu(dx) < \infty$.

We use the condition (7.11) in a crucial way in the proof of the identities (7.6), (7.7) and (7.9). (Note that Haussmann (1987) imposes (7.11) on the prior distribution μ; namely, he assumes $\int e^{\delta x^2}\,\mu(dx) < \infty$ for some $\delta \in (0,\infty)$. The results of Section 3 show that this is not necessary in our problem in order to obtain certainty-equivalence; and the results of the present section shows that this is not necessary for justifying the Mortensen equation either.)

Proving (7.6), (7.7) and (7.9) is really a matter of interchanging differentiation and integration. If $f(t,x)$ is differentiable in t and if for any compact set $K \subseteq \mathbb{R}$ the set of functions $\{\frac{\partial}{\partial t} f(t,x);\ t \in K\}$ can be dominated by an integrable, then

$$\frac{\partial}{\partial t}\int f(t,x)\,dx = \int \frac{\partial f}{\partial t}(t,x)\,dx\,.$$

We use this principle to establish (7.7)-(7.9). Condition (7.11) is useful for obtaining the dominator.

LEMMA 7.1. *Suppose that* $\int e^{\varepsilon x^2}\nu(dx) < \infty$ *for some* $\varepsilon > 0$. *Then for each* $k \in \mathbb{N}$, *there exists a constant* $K_{\nu,k}$ *such that*

$$\sup_{t\in[0,T]} \left| \frac{\frac{\partial^k}{\partial y^k} F(t,y;\nu)}{F(t,y;\nu)} \right| \le \left(\frac{2|y|}{\varepsilon}\right)^k + K_{\nu,k} \ , \qquad \forall\ y \in \mathbb{R}. \tag{7.12}$$

Proof: Assume that $\nu\big([0,\infty)\big) > 0$. With $y > 0$, integration over the interval $\left[\frac{-2y}{t+\varepsilon}, \frac{2y}{t+\varepsilon}\right]$ and its complement separately, leads to

$$\left|\frac{\partial^k}{\partial y^k} F(t,y;\nu)\right| \ \le \ \left(\frac{2y}{t+\varepsilon}\right)^k F(t,y;\nu) + \int_{\mathbb{R}} |x|^k e^{\frac{1}{2}\varepsilon x^2} \nu(dx) \ , \tag{7.13}$$

for $t \ge 0$, $\varepsilon \ge 0$ with $t+\varepsilon > 0$, because $xy - \frac{t+\varepsilon}{2}x^2 \le 0$ for $|x| \ge \frac{2y}{t+\varepsilon}$. We have also

$$F(t,y;\nu) \ge \int_0^\infty e^{-Tx^2/2}\nu(dx) > 0, \qquad \text{if } \ y > 0,\ 0 < t \le T. \tag{7.14}$$

Thus we obtain (7.12) for $y > 0$ and

$$K_{\nu,k} = \int_{\mathbb{R}} |x|^k e^{\varepsilon x^2/2}\nu(dx) \Big/ \int_0^\infty e^{-Tx^2/2}\nu(dx).$$

A similar bound will hold for $y < 0$, if $\nu\big((-\infty,0)\big) > 0$. On the other hand, if $\nu\big([0,\infty)\big) = 0$ and $y > 0$, we have

$$\left|\frac{\partial^k}{\partial y^k} F(t,y;\nu)\right| \ \le \ \left(\frac{2y}{t+\varepsilon}\right)^k F(t,y;\nu) + e^{-\frac{4}{t+\varepsilon}y^2} \int_{\{x<-2y/(t+\varepsilon)\}} |x|^k e^{\varepsilon x^2/2}\nu(dx) \ , \tag{7.15}$$

for $t \ge 0$, $\varepsilon \ge 0$ with $t+\varepsilon > 0$. Choosing an $a < 0$ such that $\nu\big([a,0]\big) > 0$, we have

$$F(t,y;\nu) \ge e^{ya} \int_a^0 e^{-Tx^2/2}\,\nu(dx), \tag{7.16}$$

and thus (7.12) works for $y > 0$ and

$$K_{\nu,k} = \max_{y>0}\left(e^{-\frac{4}{T+\varepsilon}y^2 - ay}\right) \cdot \int_{\mathbb{R}} |x|^k e^{\varepsilon x^2/2}\nu(dx) \Big/ \int_a^0 e^{-Tx^2/2}\,\nu(dx). \qquad \square$$

COROLLARY 7.1. *If* C *is a compact set in* $(0,\infty)\times\mathbb{R}^2$ *and* $0 < \varepsilon < \inf\{t \,/\, \exists (t,y,\xi) \in C\}$, *then for each* $n \in \mathbb{N}$ *one can choose a constant* K_k *such that* (7.12) *obtains when* $K_{\nu,k}$ *is replaced by* K_k *for all* $\nu \in \{M(t,y,\xi);\ (t,y,\xi) \in C\}$.

Proof: The constants defining $K_{\nu,k}$ in the various cases of the proof of Lemma 7.1 depend continuously on the parameters t, y, ξ. $\square$

Consider (7.7). By direct differentiation, we have

$$\frac{\partial}{\partial y}G^2\big(t,b;M(s,y,\xi)\big) = 2G\big(t,b;M(s,y,\xi)\big)\cdot D_pG\big(t,b;M(s,y,\xi)\big)\left[\frac{\partial M}{\partial y}(s,y,\xi)\right]$$
$$= \quad 2G\big(t,b;M(s,y,\xi)\big)\left\{\frac{F_y\big(t,b;\frac{\partial M}{\partial y}(s,y,\xi)\big)}{F\big(t,b;M(s,y,\xi)\big)} - \frac{F\big(t,b;\frac{\partial M}{\partial y}(s,y,\xi)\big)}{F\big(t,b;M(s,y,\xi)\big)}G\big(t,b;M(s,y,\xi)\big)\right\}.$$

Now, from (7.2), we obtain

$$F\left(t,b;\frac{\partial M}{\partial y}(s,y,\xi)\right) = F_y\big(t,b;M(s,y,\xi)\big) - \langle h, M(s,y,\xi)\rangle\ F\big(t,b;M(s,y,\xi)\big)$$

since $F(t,b;h\mu) = F_y(t,b;\mu)$. Likewise,

$$F_y\left(t,b;\frac{\partial M}{\partial y}(s,y,\xi)\right) = F_{yy}\big(t,b;M(s,y,\xi)\big) - \langle h, M(s,y,\xi)\rangle\ F_y\big(t,b;M(s,y,\xi)\big).$$

Thus

$$\frac{\partial}{\partial y}G^2\big(t,b;M(s,y,\xi)\big) = 2G\big(t,b;M(s,y,\xi)\big)\cdot\left[\frac{F_{yy}}{F} - \left(\frac{F_y}{F}\right)^2\right]\big(t,b;M(s,y,\xi)\big),$$

and Lemma 7.1 gives

$$\sup_{t\in[0,T]}\left|\frac{\partial}{\partial y}G^2\big(t,b;M(s,y,\xi)\big)\right| \le K_C\left(|b|^3+1\right) \tag{7.17}$$

for a constant $K_C\in(0,\infty)$, for (s,y,ξ) varying in any given compact set $C\subseteq(0,\infty)\times\mathbb{R}^2$. Since

$$\frac{\partial}{\partial y}\mathbb{E}\left[G^2\big(t,Bt+W(t);M(s,y,\xi)\big)\right] = \mathbb{E}\left[2G\,D_pG\big(t,Bt+W(t);M(s,y,\xi)\big)\left[\frac{\partial M}{\partial y}(s,y,\xi)\right]\right], \tag{7.18}$$

it is then clear from (7.17) that there is a $dt\times\mu(db)$-integrable function which dominates

$$\frac{\partial}{\partial y}\left[S(t,y)(b-\xi)\cdot\mathbb{E}G^2\big(t,b;M(s,y,\xi)\big)\right] = S_y(t,y)(b-\xi)\cdot\mathbb{E}G^2\big(t,b;M(s,y,\xi)\big)$$
$$+\,S(t,y)(b-\xi)\cdot\mathbb{E}\left[2GD_pG\big(t,b;M(s,y,\xi)\big)\left[\frac{\partial M}{\partial y}(s,y,\xi)\right]\right]$$

for (s,y,ξ) varying in any compact set $C\subseteq(0,\infty)\times\mathbb{R}^2$. Since we have

$$V\big(T,M(s,y,\xi)\big) = \langle h^2, M(s,y,\xi)\rangle - \lambda\int_0^T\left(\int S(t,y)(b-\xi)\cdot\mathbb{E}G^2\big(t,Bt+W(t);M(s,y,\xi)\big)\,\mu(db-\xi)\right)\vartheta_T^2(t)\,dt \tag{7.19}$$

from (4.14), we see that differentiation with respect to y and integration can be integchanged; thus, by the definition of (6.6) of $D_pV(T,p)[q]$, the identity (7.7) is valid. Formulae (7.6) and (7.9) are proved similarly.

Formula (7.8) is handled differently. Observe by direct calculation that, if ν is a measure such that $\langle h^2, \nu\rangle < \infty$, we have

$$G(t, y; \tau_\xi \nu) = \xi + G(t, y - \xi t; \nu)$$

and so

$$\mathbb{E}_{\tau_\xi \nu}\left[G^2\big(t, Bt + W(t); \tau_\xi \nu\big)\right] = \mathbb{E}_\nu\left[G^2\big(t, Bt + \xi t + W(t); \tau_\xi \nu\big)\right] = \mathbb{E}_\nu\left[\xi + G\big(t, Bt + W(t); \nu\big)\right]^2.$$

Therefore, (4.14) gives

$$V(T, \tau_\xi \nu) \;=\; \nu\left((h+\xi)^2\right) - \lambda \int_0^T \left(\mathbb{E}_\nu\left[\xi + G\big(t, Bt + W(t); \nu\big)\right]^2\right) \vartheta_T^2(t)\, dt.$$

It follows that

$$\left.\frac{d}{d\xi} V(T, \tau_\xi \nu)\right|_{\xi=0} \;=\; 2\nu(h) - 2\lambda \int_0^T \left(\mathbb{E}_\nu\left[G\big(t, Bt + W(t); \nu\big)\right]\right) \vartheta_T^2(t)\, dt.$$

But

$$\mathbb{E}_\nu\left[G\big(t, Bt + W(t); \nu\big)\right] = \mathbb{E}[\hat{B}(t)] = \mathbb{E}[B] = \nu(h)$$

so that

$$\left.\frac{d}{d\xi} V(T, \tau_\xi \nu)\right|_{\xi=0} \;=\; \frac{2\lambda}{\lambda + T}\; \nu(h).$$

Formula (7.8) follows immediately. $\square$

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