Dynamic Importance Sampling for Queueing Networks

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Abstract

Importance sampling is the most commonly used technique for speeding up Monte Carlo simulation of rare events. However, little is understood in the design of efficient importance sampling algorithms in the context of queueing networks. The standard approach, which simulates the system using an a priori fixed change of measure suggested by large deviation analysis, has been shown to fail even in the simplest setting such as two-node tandem networks.

Exploiting the connection between importance sampling and differential games, we demonstrate that one can build simple and efficient dynamic importance sampling schemes based on classical subsolutions of the corresponding Isaacs equation. The model we use to illustrate this approach is d-node tandem Jackson networks and rare-events of interest are various types of buffer overflows includes total population overflow and individual buffer overflows.

1 Introduction

For more than two decades, there has been a steady growth of interest in the study of fast simulation techniques for rare events in queueing networks. In particular, importance sampling, a method in which the system is simulated under a different probability distribution (i.e., change of measure), has received much attention [12, 2].

The standard heuristic, which considers a priori fixed static changes of measure suggested by large deviation analysis, has very limited success. It was shown to lead to efficient importance sampling algorithms for simulating large buildups of a single/multiple server queue [1, 15]. However, rather little success has been accomplished in extending this standard heuristic to networks of queues. Even for the simplest setting such as two-node tandem Jackson networks, the change of measure suggested by the standard heuristic, which amounts to exchange the arrival rate and the smallest service rate [14], fails to be asymptotically optimal in general [11] and can lead to importance sampling estimators with infinite variance [3]. The failure was partly due to the discontinuity of the state dynamics on the boundaries of the state space, which is not present in the case of a single queue.

The purpose of the present paper is to build a framework under which one can systematically build simple, efficient *dynamic* (i.e., state-dependent) importance sampling schemes for simulating rare events in queueing networks. Our method heavily exploits the recently discovered connection between importance sampling and deterministic differential games [7, 8] and the role of classical subsolution of the Isaacs equation associated with the game [9, 10]. We demonstrate that one can construct classical subsolutions that lead to simple and efficient importance sampling schemes. As in [10], such subsolutions will be identified as the mollification of the pointwise minimum of affine functions.

To illustrate the main idea, we will specialize to d-node tandem Jackson queueing networks. The rare events of interest are various types of buffer overflows including total population overflow and individual buffer overflows. We will also discuss extensions to more general queueing networks. To the best of our knowledge, the present paper is the first to provide a rigorous theoretical framework from which one can build asymptotically optimal importance sampling algorithms for rare events in networks of queues.

The paper is organized as follows. Section 2 gives a brief review of the basics of importance sampling. In Section 3, we will study in detail the classical problem of total population overflow in two-node tandem Jackson networks. The motivation is to illustrate the key steps in this subsolution

approach with minimal notational inconvenience. Generalization to d-node tandem Jackson networks and general buffer overflow problems will be discussed in Section 4. We will address the application of the subsolution approach to general queueing networks in Section 5. To ease exposition, technical proofs are collected in appendices.

2 Basics of importance sampling

The basic idea of importance sampling is change of measure, that is, the system is simulated under a different probability distribution and the outcomes are multiplied by appropriate likelihood ratios (i.e., Radon-Nikodým derivatives) to form unbiased estimators.

We specialize to the estimation of rare event probabilities and consider a family of events $\{A_n\}$ in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\lim_n -\frac{1}{n} \log \mathbb{P}(A_n) = \gamma$$

for some positive constant γ. In order to estimate $\mathbb{P}(A_n)$, importance sampling generates samples under a different probability measure $\mathbb{Q}$ such that $\mathbb{P}$ is absolutely continuous with respect to $\mathbb{Q}$, and forms an estimator by averaging independent replications of

$$\hat{p}_n \doteq 1_{A_n} \frac{d\mathbb{P}}{d\mathbb{Q}},$$

where $d\mathbb{P}/d\mathbb{Q}$ is the Radon-Nikodým derivative, or the likelihood ratio. It is easy to check that this importance sampling estimator is unbiased since

$$E^{\mathbb{Q}}[\hat{p}_n] = \mathbb{P}(A_n).$$

Its rate of convergence is determined by the variance of $\hat{p}_n$, or equivalently, the second moment, which is

$$[\text{2nd moment of } \hat{p}_n] = E^{\mathbb{Q}}[\hat{p}_n^2] = E^{\mathbb{P}}[\hat{p}_n]. \tag{2.1}$$

The smaller the second moment, the faster the convergence. However, Jensen's inequality implies that

$$\limsup_n -\frac{1}{n} \log E^{\mathbb{Q}}[\hat{p}_n^2] \leq \limsup_n -\frac{2}{n} \log E^{\mathbb{Q}}[\hat{p}_n] = 2\gamma.$$

In other words, the exponential decay rate of the second moment can be at most twice that of the probability. We say the importance sampling estimator is *asymptotically optimal* if the upper bound is achieved, i.e., if

$$\liminf_n -\frac{1}{n} \log E^{\mathbb{P}}[\hat{p}_n] \geq 2\gamma. \tag{2.2}$$

Sometimes 2γ is referred to simply as the "optimal decay rate".

Remark 2.1. The requirement that $\mathbb{P}$ is absolutely continuous with respect to $\mathbb{Q}$ is more stringent than necessary. It is sufficient that $\mathbb{P}$ is absolutely continuous with respect to $\mathbb{Q}$ on a sub-σ-algebra that contains A_n, while the likelihood ratio is defined as the Radon-Nikodým derivative of $\mathbb{P}$ and $\mathbb{Q}$ when they are restricted on this sub-σ-algebra.

3 Two-node tandem Jackson networks

To illustrate the main idea of the game/subsolution approach toward importance sampling, we specialize to two-node Jackson tandem queueing networks, where the arrivals follows a Poisson process with rate λ and the service times are distributed exponentially with rates μ_1 and μ_2 respectively. The system is assume to be stable, that is, $\lambda < \min\{\mu_1, \mu_2\}$.
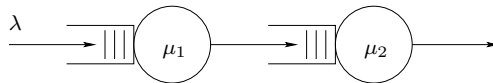
Suppose that the two queues share one buffer with capacity n, and we are interested in buffer overflow probability

$$
\begin{aligned}
p_n \;\doteq\; & \mathbb{P}\{\text{network total population reaches } n \text{ before returning to } 0, \\
& \text{starting from } 0\}.
\end{aligned} \tag{3.1}
$$

Rescaling the time variable will have no effect on this probability, and so without loss of generality we can assume $\lambda + \mu_1 + \mu_2 = 1$. Since exchanging the order of service rates does not affect the value of the probability of interest [17], we further assume that $\mu_2 \leq \mu_1$. Under these conditions, we have the large deviation limit [11]

$$\lim_n -\frac{1}{n} \log p_n = \log \frac{\mu_2}{\lambda} \doteq \gamma. \tag{3.2}$$

3.1 The standard heuristic

Based on a heuristic application of large deviation analysis, [14] proposed a state-independent importance sampling algorithm for estimating p_n, which amounted to interchanging the arrival rate and the smallest service rate. That is, under this change of measure, the system has arrival rate μ_2 and consecutive service rates μ_1 and λ respectively. Even though [14] offered no theoretical justification, it performed some numerical experiments and found that the corresponding importance sampling estimator works well for certain range of parameters.

A rigorous analysis of this importance sampling algorithm first appeared in [11]. The authors showed that the algorithm is asymptotically optimal in certain parameter regions. However, it was also shown that the asymptotic optimality fails in other parameter regions, for example, when the two service rates μ_1 and μ_2 are nearly equal and the arrival rate λ is small. A recent paper [3] extended the results in [11] and asserted that the importance sampling estimator can have infinite variance for certain parameters.

To the best of our knowledge, there have been no known asymptotically optimal (or even provably good) importance sampling algorithms in general even for this simple two-node tandem Jackson network, let alone more general queueing networks. More detailed discussion on importance sampling for queueing networks can be found in the survey paper [12].

3.2 The system dynamics

The system state can be described by the embedded discrete time Markov chain $Z = \{Z(k) : k = 0, 1, 2, \ldots\}$, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, that represents the queue lengths at the transition epochs of the tandem network. More precisely, $Z(k) = (Z_1(k), Z_2(k))$ where $Z_i(k)$ is the length of the queue at node i after the k-th transition. Obviously, Z can only take values at $\mathbb{Z}_+^2$ and p_n equals the probability of $Z_1 + Z_2$ reaching n before returning to 0, given that the system is initially empty.

At times when both queues are non-empty, the increments of the Markov chain Z can take values in the space

$$\mathbb{V} \doteq \{v_0 = (1, 0),\ v_1 = (-1, 1),\ v_2 = (0, -1)\}, \qquad (3.3)$$

with v_0 corresponding to an arrival and v_i to a service at node i for $i = 1, 2$. On the boundary where either queue is empty, the dynamics exhibit different behaviors. For example, suppose that the queue at node i $(i = 1, 2)$ is empty. Then it is impossible for the process Z to have increment v_i since it will lead

to negative queue size. One way to describe this discontinuity in dynamics is to allow Z to make fictitious jumps of size v_i on the boundary, but they have to be accounted for by "pushing back" the state along the direction of constraints

$$d_i = -v_i, \qquad (3.4)$$

so that the state process Z stays non-negative.

To summarize, the evolution of the Markov chain Z can be modeled by equation

$$Z(k+1) = Z(k) + \pi[Z(k), Y(k+1)], \qquad (3.5)$$

where $Y = \{Y(k) : k \geq 1\}$ is a sequence of random variables taking values in the space $\mathbb{V}$, and the mapping π is defined for every $z = (z_1, z_2) \in \mathbb{R}_+^2$ and $y \in \mathbb{V}$ as

$$\pi[z, y] \doteq \begin{cases} 0 & ; \quad \text{if } z_i = 0 \text{ and } y = v_i \text{ for some } i = 1, 2 \\ y & ; \quad \text{otherwise} \end{cases}. \qquad (3.6)$$
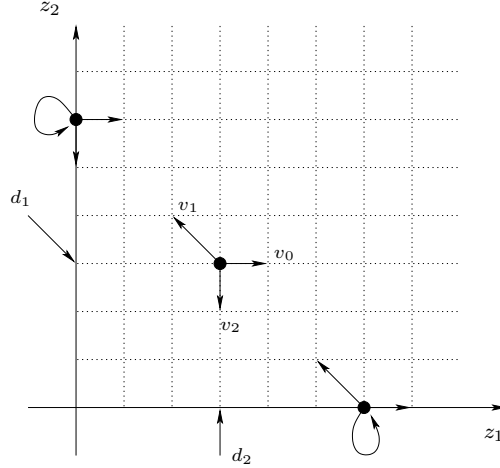

Figure 1: The system dynamics

The distribution of Z is completely determined by that of the sequence $Y = \{Y(k)\}$. Define

$$\mathcal{P}^+(\mathbb{V}) \quad \doteq \quad \{\theta = (\theta_0, \theta_1, \theta_2) : \theta \text{ is a probability measure on } \mathbb{V}$$
$$\text{and } \theta_i = \theta[v_i] > 0 \text{ for every } i = 0, 1, 2\}. \qquad (3.7)$$

Under the (true) probability measure $\mathbb{P}$, Y is a sequence of independent identically distributed (iid) random variables with distribution

$$\Theta \doteq (\lambda, \mu_1, \mu_2) \in \mathcal{P}^+(\mathbb{V}). \tag{3.8}$$

3.3 The dynamic importance sampling algorithms

The importance sampling schemes we consider use state-dependent changes of measure that can be characterized by stochastic kernels $\bar{\Theta}^n[\cdot|\cdot]$ on $\mathbb{V}$ given $\mathbb{R}_+^2$, i.e, $\bar{\Theta}^n[\cdot|x] \in \mathcal{P}^+(\mathbb{V})$ for every $x \in \mathbb{R}_+^2$.

To be more precise, for a given threshold n, define the scaled state process $X^n = Z^n/n$, where Z is defined as in (3.5). Since the definition of π implies $\pi[nx, y] = \pi[x, y]$ for every $x \in \mathbb{R}_+^2$, it is not difficult to see that X^n satisfies the equation

$$X^n(k+1) = X^n(k) + \frac{1}{n}\pi[X^n(k), Y(k+1)], \tag{3.9}$$

with initial condition $X^n(0) = Z^n(0)/n = 0$. The importance sampling generates $\{Y(k)\}$ as follows. The conditional probability of $Y(k+1) = v_i$, given $\{Y(j) : j = 1, 2\ldots, k\}$, is just $\bar{\Theta}^n[v_i|X^n(k)]$ for each $i = 0, 1, 2$.

Define the hitting times

$$\begin{aligned} T_n &\doteq \inf\{k \geq 0 : X_1^n(k) + X_2^n(k) = 1\} & (3.10) \\ T_0 &\doteq \inf\{k \geq 1 : X_1^n(k) = X_2^n(k) = 0\}. & (3.11) \end{aligned}$$

Let A_n be the event of interest, that is,

$$A_n = \{X_1^n + X_2^n \text{ reaches 1 before returning to 0}\} = \{T_n < T_0\}.$$

The importance sampling estimator is just

$$\hat{p}_n = 1_{A_n} \cdot \prod_{k=0}^{T_n-1} \frac{\Theta[Y(k+1)]}{\bar{\Theta}^n[Y(k+1)|X^n(k)]}. \tag{3.12}$$

The second moment of $\hat{p}_n$, thanks to (2.1), equals $E^{\mathbb{P}}[\hat{p}_n]$. The goal is to choose a stochastic kernel $\bar{\Theta}^n$ so that this second moment (whence the variance of $\hat{p}_n$) is as small as possible. Another important consideration is that one would like $\bar{\Theta}^n$ to be simple and easy to implement.

Remark 3.1. The standard heuristic importance sampling algorithm simulates the system using the state independent change of measure under which $\{Y(k)\}$ is iid with distribution $\Theta_{\mathsf{s}} = (\mu_2, \mu_1, \lambda)$. This corresponds to the special choice of stochastic kernel $\bar{\Theta}^n$ with $\bar{\Theta}^n[\cdot|x] \equiv \Theta_{\mathsf{s}}$ for every x.

Remark 3.2. The choice of $\bar{\Theta}^n[\cdot | x]$ is not important for $x \in \mathbb{R}^2_+$ such that $x_1 + x_2 \geq 1$. For example, one can take $\bar{\Theta}^n[\cdot | x] \equiv \Theta$ for such x.

3.4 Notation and terminologies

Before we proceed to construct importance sampling algorithms, we collect in this section some notation and terminologies. Define

$$
\begin{aligned}
\bar{D} &\doteq \{(x_1, x_2) : x_i \geq 0, x_1 + x_2 \leq 1\}, & (3.13)\\
D &\doteq \{(x_1, x_2) : x_i > 0, x_1 + x_2 < 1\}, & (3.14)\\
\partial_1 &\doteq \{(0, x_2) : 0 < x_2 < 1\}, & (3.15)\\
\partial_2 &\doteq \{(x_1, 0) : 0 < x_1 < 1\}, & (3.16)\\
\partial_{\mathsf{e}} &\doteq \{(x_1, x_2) : x_i \geq 0, x_1 + x_2 = 1\}, & (3.17)\\
\bar{D}_n &\doteq \bar{D} \cap \{(z_1, z_2)/n : (z_1, z_2) \in \mathbb{Z}^2_+\}, & (3.18)\\
D_n &\doteq D \cap \{(z_1, z_2)/n : (z_1, z_2) \in \mathbb{Z}^2_+\}. & (3.19)
\end{aligned}
$$

Sometimes we refer to ∂_{e} as the "exit boundary".
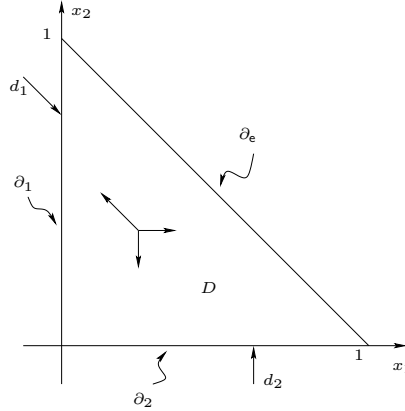

Figure 2: Domains and boundaries

3.5 The Isaacs equation

The main purpose of this section is to derive the Isaacs equation associated with the limit differential game that lies underneath importance sampling algorithms. The derivation will be kept formal. A rigorous argument, though possible, is not necessary for our purpose.

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Recall our goal is to choose a stochastic kernel $\bar{\Theta}^n$ so as to keep the second moment $E^{\mathbb{P}}[\hat{p}_n]$ as small as possible. We can think of this as a stochastic control problem and write down the corresponding Dynamic Programming Equation (DPE). To this end, we extend the dynamics and let, for every $x \in \bar{D}_n$,

$$V_n(x) \doteq \inf_{\bar{\Theta}^n} E^{\mathbb{P}}[\hat{p}_n] = \inf_{\bar{\Theta}^n} E^{\mathbb{P}}\left[1_{A_n} \cdot \prod_{k=0}^{T_n-1} \frac{\Theta[Y(k+1)]}{\bar{\Theta}^n[Y(k+1)|X^n(k)]} \right],$$

where $\hat{p}_n$ is defined in exactly the same fashion as in Section 3.3 except that the initial condition is replaced by $X^n(0) = x$.

For simplicity, we further assume that $x \in D_n$, whence $\pi[x,y] \equiv y$ for every $y \in \mathbb{V}$. Under the original probability measure $\mathbb{P}$, the sequence $\{Y(k)\}$ is iid with distribution Θ. It follows that

$$V_n(x) = \inf_{\bar{\Theta} \in \mathcal{P}^+(\mathbb{V})} \sum_{i=0}^{2} V_n\left(x + \frac{1}{n}v_i \right) \frac{\Theta[v_i]}{\bar{\Theta}[v_i]} \cdot \Theta[v_i].$$

Consider a logarithmic transform of V_n and define

$$W_n(x) \doteq -\frac{1}{n} \log V_n(x).$$

We have

$$W_n(x) = \sup_{\bar{\Theta} \in \mathcal{P}^+(\mathbb{V})} -\frac{1}{n} \log \sum_{i=0}^{2} \exp\left\{ -nW_n\left(x + \frac{1}{n}v_i \right) - \log \frac{\bar{\Theta}[v_i]}{\Theta[v_i]} \right\} \Theta[v_i].$$

A key step in this derivation is to apply the relative entropy representation for exponential integrals to the right-hand-side of the above equation. For completeness, we include the representation in its general form in Remark 3.4. It follows that

$$
\begin{aligned}
W_n(x) \;=\; & \sup_{\bar{\Theta} \in \mathcal{P}^+(\mathbb{V})} \inf_{\theta \in \mathcal{P}^+(\mathbb{V})} \left[\sum_{i=0}^{2} W_n\left(x + \frac{1}{n}v_i \right) \theta[v_i] \right. \\
& \left. + \frac{1}{n}\left(\sum_{i=0}^{2} \theta[v_i] \log \frac{\bar{\Theta}[v_i]}{\Theta[v_i]} + R(\theta\|\Theta) \right) \right].
\end{aligned}
$$

Note that taking infimum over $\theta \in \mathcal{P}^+(\mathbb{V})$ is equivalent to taking infimum over $\theta \in \mathcal{P}(\mathbb{V})$ since by Remark 3.4 the minimizing θ is mutually absolutely continuous to Θ , whence it belongs to $\mathcal{P}^+(\mathbb{V})$.

Suppose for now that $W_n(x)$ converges to $W(x)$. Formally using the approximation

$$W_n\left(x + \frac{1}{n}v_i\right) - W_n(x) \approx \frac{1}{n}\langle DW(x), v_i\rangle,$$

where DW is the gradient of W, and observing $\sum \theta[v_i] = 1$, we arrive at

$$0 = \sup_{\bar{\Theta} \in \mathcal{P}^+(\mathbb{V})} \inf_{\theta \in \mathcal{P}^+(\mathbb{V})} \left[\langle DW(x), \mathbb{F}(\theta)\rangle + \sum_{i=0}^{2} \theta[v_i] \log \frac{\bar{\Theta}[v_i]}{\Theta[v_i]} + R(\theta\|\Theta)\right],$$

$$(3.20)$$

where

$$\mathbb{F}(\theta) \doteq \sum_{i=0}^{2} \theta[v_i] \cdot v_i \qquad (3.21)$$

for each $\theta \in \mathcal{P}^+(\mathbb{V})$. Equation (3.20) is called an *Isaacs equation*.

We now discuss the boundary conditions. For the exit boundary, we have by definition $V_n(x) = 1$ or $W_n(x) = 0$, therefore we impose Dirichlet boundary condition

$$W(x) = 0, \quad \text{for} \quad x \in \partial_{\mathsf{e}}. \qquad (3.22)$$

For ∂_1 and ∂_2, we impose the Neumann boundary condition that is typically associated with constrained dynamics [13]

$$\langle DW(x), d_i\rangle = 0, \quad \text{for} \quad x \in \partial_i. \qquad (3.23)$$

Finally, we would like to make a few remarks on the game interpretation of importance sampling. The Isaacs equation (3.20) indicates that the underlying game has two players. The player who chooses the change of measure (i.e. $\bar{\Theta}$) in order to minimize the second moment becomes the maximizing player in the game due to the negative sign in the logarithmic transform. The minimizing player is artificially introduced, and chooses θ. We will refer to this player as the "large deviation player". The dynamics of the game are completely determined by θ, or the choice of the large deviation player, while the running cost of the game depends on the choices of both players.

Remark 3.3. The original dynamics have initial condition $x = 0$, and $W(0)$ characterizes the asymptotic exponential decay rate of the second moment. We would like $W(0)$ to be large, ideally equal to the optimal decay rate 2γ.

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Remark 3.4. *Relative Entropy Representation for Exponential Integrals*:
Let $(S, \mathcal{F})$ be a measurable space and $f : S \to \mathbb{R}$ a bounded measurable
function. Denote by $\mathcal{P}(S)$ the space of probability measures on $(S, \mathcal{F})$. Then
for any $\gamma \in \mathcal{P}(S)$, we have

$$-\log \int_S e^{-f} d\gamma = \inf_{\theta \in \mathcal{P}(S)} \left[R(\theta \| \gamma) + \int_S f\, d\theta \right].$$

Furthermore, the minimizer to the right-hand-side is mutually absolutely
continuous with respect to γ. Here the relative entropy $R(\cdot \| \cdot)$ is defined as

$$R(\theta \| \gamma) \doteq \begin{cases} \int_S \log \dfrac{d\theta}{d\gamma}\, d\theta & ; \quad \text{if } \theta \ll \gamma \\ \infty & ; \quad \text{otherwise} \end{cases}.$$

We refer the readers to [4, Proposition 1.4.2] for the proof.

3.6   The properties of the Hamiltonian

Our construction of importance sampling algorithms is based on classical
subsolutions to the Isaacs equation. Therefore, it is useful to study the
properties of this equation. To this end, define for each $p \in \mathbb{R}^2$

$$\mathbb{H}(p) \doteq \sup_{\bar{\Theta} \in \mathcal{P}^+(\mathbb{V})} \inf_{\theta \in \mathcal{P}^+(\mathbb{V})} \left[\langle p, \mathbb{F}(\theta) \rangle + \sum_{i=0}^{2} \theta[v_i] \log \frac{\bar{\Theta}[v_i]}{\Theta[v_i]} + R(\theta \| \Theta) \right]. \quad (3.24)$$

The function $\mathbb{H}$ is called the Hamiltonian, and the Isaacs equation (3.20)
can be written as

$$\mathbb{H}(DW) = 0. \quad (3.25)$$

We have the following result, whose proof is deferred to Appendix C.

Proposition 3.5. *Let $\mathbb{H}$ be defined as in (3.24).*

1. *For each $p = (p_1, p_2) \in \mathbb{R}^2$, there exists a saddle point $(\bar{\Theta}^*(p), \theta^*(p)) \in \mathcal{P}^+(\mathbb{V}) \times \mathcal{P}^+(\mathbb{V})$ given by*

$$\bar{\Theta}^*(p) = \theta^*(p) = N(p) \left(\lambda e^{-p_1/2}, \mu_1 e^{(p_1-p_2)/2}, \mu_2 e^{p_2/2} \right),$$

   where
$$N(p) \doteq \left[\lambda e^{-p_1/2} + \mu_1 e^{(p_1-p_2)/2} + \mu_2 e^{p_2/2} \right]^{-1}.$$

   In particular, the order of sup and inf can be exchanged in (3.24).

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2. *We have the representation*

$$\mathbb{H}(p) = \inf_{\theta \in \mathcal{P}^+(\mathbb{V})} [\langle p, \mathbb{F}(\theta) \rangle + 2R(\theta \| \Theta)] = 2 \log N(p).$$

In particular, $\mathbb{H}$ is concave.

Figure 3 is a picture of the zero-level curve of $\mathbb{H}$. Recall that γ, as defined in (3.2), equals $\log(\mu_2/\lambda)$.

$$r_1 = 2\gamma(-1, -1)$$
$$r_2 = 2\gamma(-1, 0)$$
$$r_3 = (0, 0)$$
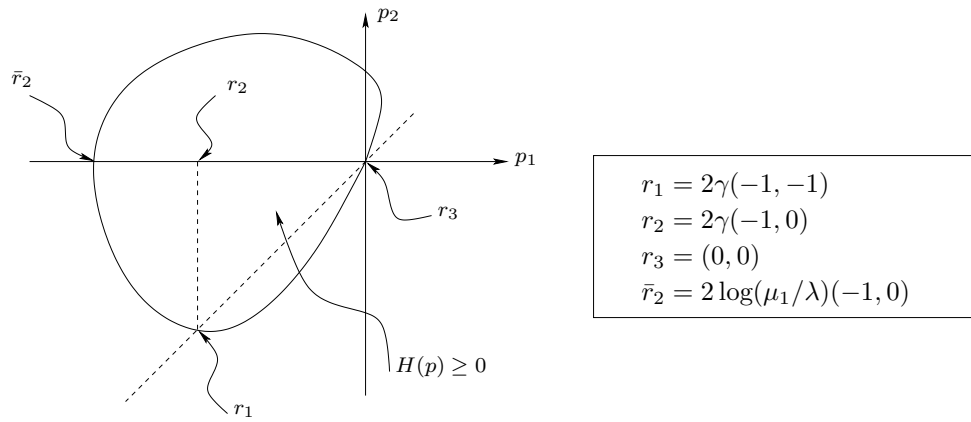$$\bar{r}_2 = 2\log(\mu_1/\lambda)(-1, 0)$$

Figure 3: Hamiltonian $\mathbb{H}$

Remark 3.6. For any $p \in \mathbb{R}^2$, we will refer to $\bar{\Theta}^*(p)$ as the (saddle point) change of measure corresponding to p.

3.7   The solution to the Isaacs equation

In [7, 8], the saddle point strategy generated by the solution to the Isaacs equation was used to construct efficient importance sampling schemes. However, the viscosity solution to the Isaacs equation (3.25) and boundary conditions (3.22), (3.23), which is only a weak-sense solution, is *not* suitable for the purpose of constructing efficient importance sampling algorithms for this tandem Jackson network.

More precisely, consider the very simple, affine function

$$W_{\mathsf{s}}(x) \doteq \langle r_1, x \rangle + 2\gamma.$$

This function is the viscosity solution to the Isaacs equation (3.25) and boundary conditions (3.22), (3.23). Even though $W_{\mathsf{s}}(0) = 2\gamma$, the optimal

decay rate, the saddle point strategy corresponding to W_s does not lead to efficient importance sampling algorithms. Indeed, thanks to Proposition 3.5 and straightforward calculation, the $\bar{\Theta}$-component in the saddle point is

$$\bar{\Theta}^*(DW_\mathsf{s}) = \bar{\Theta}^*(r_1) = (\mu_2, \mu_1, \lambda),$$

which is exactly the state-independent change of measure Θ_s based on standard heuristic; see Section 3.1.

As remarked previously, the failure of the importance sampling based on W_s is largely due to the fact that W_s is only a weak-sense viscosity solution. It is not a classical solution (or even a classical subsolution), since on the boundary ∂_2 we have

$$\langle DW_\mathsf{s}, d_2 \rangle = \langle r_1, d_2 \rangle = -2\gamma < 0.$$

In some sense, it suggests that the (artificial) large deviation player in the game, who determines the game dynamics, may be able to exploit this "bad" boundary to a degree that the importance sampling estimator based on W_s becomes inefficient. It is not coincidental, as observed by [11], that the inefficiency of Θ_s in general is because a sample path can spend a significant amount of time near boundary ∂_2 before leaving domain D and accumulate huge Radon-Nikodým derivative.

Remark 3.7. This example shows even when there is a viscosity solution to the Isaacs equation, it may not always lead to efficient importance sampling algorithms.

3.8 Subsolutions and importance sampling schemes

The idea of [9, 10] is that classical subsolutions to Isaacs equations can be used to construct efficient importance sampling schemes. It is advantageous over solution based importance sampling schemes in simplicity, greater flexibility, and general applicability. The goal of this section is to construct classical subsolutions and identify the corresponding changes of measure. The analysis of the asymptotic behaviors of the importance sampling estimator will be carried out in Appendix B.

Definition 3.8. *A function $W : \bar{D} \to \mathbb{R}$ is said to be a* classical subsolution *to the Isaacs equation (3.25) and boundary condition (3.22), (3.23) if*

1. *W is continuously differentiable,*

2. *$\mathbb{H}(DW(x)) \geq 0$ for every $x \in D$,*

3. $W(x) \leq 0$ for $x \in \partial_{\mathsf{e}}$,

4. $\langle DW(x), d_i \rangle \geq 0$ for $x \in \partial_i$, $i = 1, 2$.

As in [9, 10], the construction of classical subsolutions are divided into two steps. We first identify a subsolution as the minimum of affine functions and then mollify it to obtain a classical subsolution.

3.8.1 Construction of piecewise affine subsolutions

We would like to construct a piecewise affine subsolution $\bar{W}$ that has the following property.

1. The function $\bar{W}$ can be written as $\bar{W} = \bar{W}_1 \wedge \bar{W}_2 \wedge \bar{W}_3$ where $\bar{W}_k$ is an affine function for each $k = 1, 2, 3$.

2. $\bar{D}$ is divided into three regions R_1, R_2, and R_3 such that in each region R_k, we have $\bar{W} = \bar{W}_k$.

3. We have the subsolution property $H(D\bar{W}(x)) = H(DW_k(x)) \geq 0$ for every x in the interior of region R_k.

4. The Dirichlet boundary inequality $\bar{W}(x) \leq 0$ for $x \in \partial_{\mathsf{e}}$.

5. The Neumann boundary inequality $\langle D\bar{W}(x), d_i \rangle \geq 0$, whenever $x \in \partial_i$ and $D\bar{W}(x)$ exists.
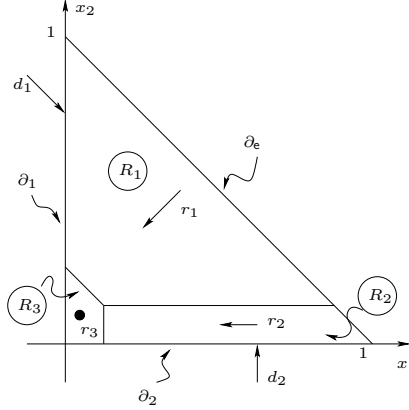

Figure 4: Piecewise affine subsolution

This can be easily achieved – indeed, fixing an arbitrary $\delta > 0$, one can let, for each k,

$$\bar{W}_k^\delta(x) \doteq \langle r_k, x \rangle + 2\gamma - k\delta, \tag{3.26}$$

where r_k is depicted in Figure 3. It is not difficult to check that

$$\bar{W}^\delta \doteq \bar{W}_1^\delta \wedge \bar{W}_2^\delta \wedge \bar{W}_3^\delta \tag{3.27}$$

satisfies all the requirements for all small $\delta > 0$.

Remark 3.9. Thanks to Remark 3.3, the value of the subsolution $\bar{W}^\delta$ at $x = 0$ is important and we would like it to be as close to the optimal decay rate as possible. But

$$\bar{W}^\delta(0) = 2\gamma - 3\delta = \text{“optimal decay rate”} - 3\delta.$$

Therefore, δ will be taken as a small positive number.

Remark 3.10. There are many different choices in the construction of piecewise affine subsolutions. For example, if one replaces r_2 by $\bar{r}_2$ (see Figure 3) in the definition of $\bar{W}_2^\delta$, the resulting functions will also have the desired properties.

3.8.2 Mollification

There are different ways to mollify the piecewise affine subsolution $\bar{W}^\delta$. We will adopt a mollification called *exponential weighting* that is specialized to the minimum of smooth functions. For future reference, we will describe this mollification in the general form.

Consider a finite collection of continuously differentiable functions $\{h_1, h_2, \ldots, h_K\}$ and let

$$h \doteq h_1 \wedge h_2 \wedge \cdots \wedge h_K.$$

Fix a small positive number ε and define

$$h^\varepsilon(x) \doteq -\varepsilon \log \sum_{k=1}^{K} \exp\left\{-\frac{1}{\varepsilon} h_k(x)\right\}.$$

We have the following result, whose proof is straightforward and can be found in [9, Section 3.3].

Lemma 3.11. *For any $\varepsilon > 0$, h^ε is continuously differentiable with*

$$Dh^\varepsilon(x) = \sum_{k=1}^{K} \rho_k^\varepsilon(x) Dh_k(x),$$

where

$$\rho_i^\varepsilon(x) \doteq \frac{\exp\{-h_i(x)/\varepsilon\}}{\sum_{k=1}^{K} \exp\{-h_k(x)/\varepsilon\}}.$$

Furthermore, we have the uniform bounds

$$-K\varepsilon \leq h^\varepsilon(x) - h(x) \leq 0$$

for every x.

Note that $\rho^\varepsilon(x) \doteq (\rho_1^\varepsilon(x), \rho_2^\varepsilon(x), \ldots, \rho_K^\varepsilon(x))$ defines a probability vector in the sense that $\rho_k^\varepsilon(x) \geq 0$ and

$$\sum_{k=1}^{K} \rho_k^\varepsilon(x) \equiv 1.$$

Remark 3.12. A well-known general mollification method, as in the classical PDE literature, is to integrate h against a smooth convolution kernel. However, we do not recommend such an approach because it involves numerical integrations, which can be computationally demanding, especially in high dimensions. In contrast, the computations involved in exponential weighting are simple and easy to implement.

3.8.3 The classical subsolution

Applying this mollification to $\bar{W}^\delta$, we define

$$W^{\varepsilon,\delta}(x) \doteq -\varepsilon \log \sum_{k=1}^{3} \exp\left\{-\frac{1}{\varepsilon}\bar{W}_k^\delta(x)\right\}. \tag{3.28}$$

Thanks to Lemma 3.11, $W^{\varepsilon,\delta}$ is continuously differentiable with

$$DW^{\varepsilon,\delta}(x) = \sum_{k=1}^{3} \rho_k^{\varepsilon,\delta}(x) r_k, \tag{3.29}$$

where

$$\rho_i^{\varepsilon,\delta}(x) \doteq \frac{\exp\{-\bar{W}_i^\delta(x)/\varepsilon\}}{\sum_{k=1}^{3} \exp\{-\bar{W}_k^\delta(x)/\varepsilon\}}. \tag{3.30}$$

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We should notice that with this mollification, the function $\bar{W}^{\varepsilon,\delta}$ is not precisely a classical subsolution, but only approximately. Indeed, Lemma B.1 states that the Neumann boundary conditions $\langle DW^{\varepsilon,\delta}, d_i \rangle \geq 0$ are not satisfied for $x \in \partial_i$. However, the lemma also indicates that they are "approximately" satisfied in the sense that, for $x \in \partial_i$,

$$\langle DW^{\varepsilon,\delta}(x), d_i \rangle \geq -\bar{\varepsilon}$$

for some small positive number $\bar{\varepsilon}$ as long as ε/δ is chosen small. The reason for this violation of the subsolution property is that the exponential weighting is not a "local" smoothing. It can be avoided if one uses integration against a convolution kernel with small support, but the advantages of the exponential weighting outweigh the minor additional complications in the analysis introduced by this error.

3.8.4 The importance sampling estimator and its asymptotics

For each k, let $\bar{\Theta}_k^*$ is the saddle point change of measure that corresponds to the affine function $\bar{W}_k$, or equivalently,

$$\bar{\Theta}_k^* \doteq \bar{\Theta}^*(D\bar{W}_k) = \bar{\Theta}^*(r_k) \in \mathcal{P}^+(\mathbb{V}), \tag{3.31}$$

where $\bar{\Theta}^*(\cdot)$ is as defined in Proposition 3.5. Straightforward calculation yields that

$$\bar{\Theta}_1^* = (\mu_2, \mu_1, \lambda), \ \ \bar{\Theta}_2^* = \frac{1}{\lambda\mu_1 + 2\mu_2^2} \left(\mu_2^2, \lambda\mu_1, \mu_2^2 \right), \ \ \bar{\Theta}_3^* = (\lambda, \mu_1, \mu_2).$$

The change of measure based on the $W^{\varepsilon,\delta}$ is just a state-dependent mixture of $\bar{\Theta}_k^*$. More precisely, define a stochastic kernel $\bar{\Theta}^{\varepsilon,\delta}[\cdot|\cdot]$ by

$$\bar{\Theta}^{\varepsilon,\delta}[\cdot|x] \doteq \sum_{k=1}^{3} \rho_k^{\varepsilon,\delta}(x)\bar{\Theta}_k^* \in \mathcal{P}^+(\mathbb{V}), \tag{3.32}$$

and for each fixed n, let

$$\bar{\Theta}^n[\cdot|\cdot] \equiv \bar{\Theta}^{\varepsilon,\delta}[\cdot|\cdot]. \tag{3.33}$$

In other words, the importance sampling algorithm simulates $Y(k+1)$, conditional on the sample history $\{Y(j) : 1 \leq j \leq k\}$, from the distribution $\bar{\Theta}^{\varepsilon,\delta}[\cdot|X^n(k)]$, where X^n is the state process as defined in (3.9). The importance sampling estimator $\hat{p}_n$ is then given by (3.12).

We have the following result regarding its asymptotic performance, whose proof is deferred to Appendix B.

Theorem 3.13. *There exist a pair of positive constants (A, B) that only depend on the system parameters (λ, μ_1, μ_2) such that, provided $\varepsilon/\delta < B$, the second moment of the importance sampling estimator $\hat{p}_n$ satisfies*

$$\liminf_n -\frac{1}{n} \log[\text{2nd moment of } \hat{p}_n] \geq 2\gamma - F(\varepsilon, \delta),$$

where

$$F(\varepsilon, \delta) \doteq 3\varepsilon + 3\delta + A \exp\{-\delta/\varepsilon\}.$$

Since 2γ is the optimal decay rate for the second moment, the theorem suggest that the importance sampling algorithm is nearly asymptotically optimal as long as $F(\varepsilon, \delta)$ is made small. This can be achieved if one chooses δ small, and ε small compared to δ.

Remark 3.14. The proof of Theorem 3.13 indeed shows that the asymptotic decay rate of the second moment of $\hat{p}_n$ is bounded from below by

$$W^{\varepsilon,\delta}(0) - A \exp\{-\delta/\varepsilon\}.$$

The presence of the term $A \exp\{-\delta/\varepsilon\}$ is due to the fact that $W^{\varepsilon,\delta}$ is only "approximately" a classical subsolution; see Section 3.8.3. It says that the performance of an importance sampling scheme based on a subsolution is largely characterized by the value of the subsolution at $x = 0$. Similar results were obtained in [9, 10].

Remark 3.15. The formula of F also provides an interesting relation between ε and δ. For each fixed small ε, it is not difficult to check that $F(\varepsilon, \cdot)$ is minimized at

$$\delta = -\varepsilon \log \varepsilon + \varepsilon \log \frac{A}{3} \approx -\varepsilon \log \varepsilon.$$

This suggests that a good strategy is to set $\delta = -\varepsilon \log \varepsilon$. Note that in this case, when ε is small, so is δ, even though δ is comparatively larger than ε.

3.8.5 Asymptotic optimality

The previous section provides a nearly asymptotically optimal importance sampling algorithm. It is good enough for many practical purposes where n is large but not exceedingly large. However, one would still like to see an algorithm that gives optimality. This actually only requires a slight modification.

Instead of using a fixed pair of parameters ε and δ for all n, we now allow them to vary depending n and denote them by ε_n and δ_n. For each n,

we use the change of measure based on $W^{\varepsilon_n,\delta_n}$. That is, for each n, define stochastic kernel

$$\bar{\Theta}^{\varepsilon_n,\delta_n}[\cdot|x] \doteq \sum_{k=1}^{3} \rho_k^{\varepsilon_n,\delta_n}(x)\bar{\Theta}_k^* \in \mathcal{P}^+(\mathbb{V}), \qquad (3.34)$$

and let

$$\bar{\Theta}^n[\cdot|\cdot] \equiv \bar{\Theta}^{\varepsilon_n,\delta_n}[\cdot|\cdot]. \qquad (3.35)$$

Abusing the notation a bit, we again denote by $\hat{p}_n$ the corresponding importance sampling estimator.

Theorem 3.16. *The importance sampling estimator $\hat{p}_n$ is asymptotically optimal, that is*

$$\lim_n -\frac{1}{n}\log[\text{2nd moment of } \hat{p}_n] = 2\gamma,$$

provided that $\delta_n \to 0$, $\varepsilon_n/\delta_n \to 0$, and $n\varepsilon_n \to \infty$.

Remark 3.15 suggests that a good choice is to set $\delta_n = -\varepsilon_n \log \varepsilon_n$. In this case, asymptotic optimality follows if $\varepsilon_n \to 0$ and $n\varepsilon_n \to \infty$.

3.8.6 Further remarks on the importance sampling algorithms

The computation of the weights $\{\rho_k^{\varepsilon,\delta}\}$ or $\{\rho_k^{\varepsilon_n,\delta_n}\}$ is very simple. As a consequence, the dynamic importance sampling algorithms based on (3.32)-(3.33) or (3.34)-(3.35) are practically as fast as the standard heuristic where a constant change of measure is used.

It is possible that one can associate other changes of measure with subsolutions. For example, one can define $\bar{\Theta}^n[\cdot|x] \equiv \bar{\Theta}^*(DW^{\varepsilon,\delta}(x))$ in lieu of (3.32)-(3.33), and the resulting algorithms will have similar asymptotic performance. However, the idea of using mixtures such as (3.32) is computationally more convenient. This is especially the case when the change of measure is not easily obtainable; e.g., for system with Markov modulated arrival and service rates, the computation of change of measure requires solving some eigenvalue/eigenvector problems. The mixtures like (3.32) only require the computation of the changes of measure that correspond to a finite collection of vectors, while $\bar{\Theta}^*(DW^{\varepsilon,\delta}(x))$-type of changes of measure will require the computation for every x in the state space.

3.9 Numerical results

In this section we present some numerical results under the setting where $\lambda = 0.1$, $\mu_1 = \mu_2 = 0.45$. The importance sampling algorithm based on the standard heuristic [14], which amounts to exchanging the arrival rate and the smallest service rate, leads to estimators with infinite variance [3].

In the simulations, we always set $\delta = -\varepsilon \log \varepsilon$. For comparison, the theoretical value of p_n is obtained using an alternative numerical iterative algorithm (we omit the details). We run simulations for $n = 20$, with $\varepsilon = 0.01, 0.02$, and 0.03, respectively. For each ε we present two estimates and each estimat consists of 20,000 replications. The theoretical is $p_n = 6.0 \times 10^{-12}$.

In all the tables, "Std. Err." stands for "Standard Error" and "C.I." for "Confidence Interval". The performance of the dynamic importance sampling shemes based on subsolution is stable across different simulations and the estimates are close to the theoretical value with small standard errors.

	$\varepsilon = 0.01$		$\varepsilon = 0.02$		$\varepsilon = 0.03$	
	No.1	No. 2	No.1	No.2	No.1	No. 2
Estimate ($\times 10^{-12}$)	5.7	5.5	5.8	6.1	6.3	5.8
Std. Err. ($\times 10^{-12}$)	0.4	0.3	0.3	0.5	0.4	0.2
95% C.I. ($\times 10^{-12}$)	[4.9, 6.4]	[4.9,6.1]	[5.2,6.4]	[5.1,7.1]	[5.5, 7.1]	[5.3,6.3]

Table 1. IS based on subsolutions, two-node tandem, total population overflow.

Below are more simulation results with $n = 30, 40, 50$, with $\varepsilon = 0.02$ and $\delta = -\varepsilon \log \varepsilon$. Each estimate consists of 20,000 replications.

	$n = 30$	$n = 40$	$n = 50$
Theoretical value	2.63×10^{-18}	1.03×10^{-24}	3.80×10^{-31}
Estimate	2.73×10^{-18}	1.05×10^{-24}	3.75×10^{-31}
Std. Err.	0.18×10^{-18}	0.03×10^{-24}	0.16×10^{-31}
95% C.I.	$[2.37, 3.09] \times 10^{-18}$	$[0.99, 1.21] \times 10^{-24}$	$[3.43, 4.07] \times 10^{-31}$

Table 2. IS based on subsolutions, two-node tandem, total population overflow.

4 Extensions to d-node tandem Jackson networks

The work on the two-node tandem Jackson network can be easily extended to d-node tandem Jackson networks and more general exit probabilities. To be more precise, consider a d-node tandem Jackson network with Poisson arrival rate λ and consecutive exponential service rates $\mu_1, \ldots, \mu_d$. The state of the network is described by the embedded Markov chain $Z = \{Z(k)\} = \{(Z_1(k), \ldots, Z_d(k))\}$, where Z_i denotes the queue length at node i. The

system is assumed to be stable, that is, $\lambda < \min\{\mu_1, \ldots, \mu_d\}$. Let Γ be a closed subset of $\mathbb{R}_+^d$ such that $0 \notin \Gamma$ and the closure of $\mathbb{R}_+^d \setminus \Gamma$ is compact. We are interested in the following rare-event probability

$$p_n \doteq \mathbb{P}\{\text{Process } Z \text{ hits set } n\Gamma \text{ before returning to } 0, \text{ starting from } 0\}.$$

Without loss of generality, we assume that $\lambda + \mu_1 + \cdots + \mu_d = 1$ and that p_n decays exponentially with

$$\lim_n -\frac{1}{n} \log p_n = \gamma.$$

4.1 Isaacs equation and the Hamiltonian

In order to write down the Isaacs equation associated with this problem, we introduce the following notation. The increments of Z take values in space $\mathbb{V} = \{v_0, v_1, \ldots, v_d\}$ where v_i's are d-dimensional vectors defined by

$$v_0 = (1, 0, \ldots, 0), \quad [v_i]_j \doteq \begin{cases} -1 & ; \quad \text{if } j = i \\ 1 & ; \quad \text{if } j = i + 1 \text{ and } j \le d \\ 0 & ; \quad \text{otherwise} \end{cases}.$$

Intuitively, v_0 corresponds to an arrival and v_i to a service at node i. Similar to (3.9), the scaled state process $X^n \doteq Z/n$ satisfies

$$X^n(k+1) = X^n(k) + \frac{1}{n}\pi[X^n(k), Y(k+1)],$$

where π is defined in a completely analogous fashion as in (3.6). The sequence $\{Y(k)\}$ is iid random variables taking values in $\mathbb{V}$ with common distribution

$$\Theta = (\lambda, \mu_1, \ldots, \mu_d) \in \mathcal{P}^+(\mathbb{V}).$$

Define regions

$$\begin{aligned} D &\doteq \{x \in \mathbb{R}_+^d : x \notin \Gamma, x_i > 0, i = 1, \ldots, d\}, \\ \partial_i &\doteq \{x \in \mathbb{R}_+^d : x \notin \Gamma, x_i = 0\}, \; i = 1, \ldots, d, \end{aligned}$$

and the directions of constraints

$$d_i = -v_i.$$

The Isaacs equation is just $\mathbb{H}(DW) = 0$, where

$$\mathbb{H}(p) = \sup_{\bar{\Theta} \in \mathcal{P}^+(\mathbb{V})} \inf_{\theta \in \mathcal{P}^+(\mathbb{V})} \left[\langle p, \mathbb{F}(\theta) \rangle + \sum_{i=0}^d \theta[v_i] \log \frac{\bar{\Theta}[v_i]}{\Theta[v_i]} + R(\theta \| \Theta) \right],$$

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with

$$\mathbb{F}(\theta) \doteq \sum_{i=0}^{d} \theta[v_i] \cdot v_i$$

for every $\theta \in \mathcal{P}^+(\mathbb{V})$. The boundary conditions are $W(x) = 0$ for $x \in \Gamma$ and $\langle DW(x), d_i \rangle = 0$ for $x \in \partial_i$.

The following result is an extension of Proposition 3.5, whose proof is very similar and thus omitted.

Proposition 4.1. *For every $p \in \mathbb{R}^d$, there exists a saddle point for the Hamiltonian $\mathbb{H}$, say $(\bar{\Theta}^*(p), \theta^*(p)) \in \mathcal{P}^+(\mathbb{V}) \times \mathcal{P}^+(\mathbb{V})$, given by*

$$\bar{\Theta}^*(p)[v_i] = \theta^*(p)[v_i] = N(p) \cdot \Theta[v_i] \exp\{-\langle p, v_i \rangle / 2\},$$

where

$$N(p) \doteq \left[\sum_{i=0}^{d} \Theta[v_i] \exp\{-\langle p, v_i \rangle / 2\} \right]^{-1}.$$

Moreover, the Hamiltonian $\mathbb{H}$ is concave and $\mathbb{H}(p) = 2 \log N(p)$.

4.2 Subsolutions and importance sampling schemes

The construction of subsolution also proceeds in a similar fashion: we start with a piecewise smooth "subsolution" and then mollify it by exponential weighting. We will discuss the general case where the subsolutions can vary depending on n. To be more specific, let $(\bar{W}_1^n, \ldots, \bar{W}_K^n)$ be smooth functions and let

$$\bar{W}^n \doteq \bar{W}_1^n \wedge \cdots \wedge \bar{W}_K^n.$$

The choice of $\{\bar{W}_k^n\}$ should have the following properties:

1. $\mathbb{H}(D\bar{W}_k^n(x)) \geq 0$ for every $x \in D$ and every k,

2. $\bar{W}^n(x) \leq 0$ for every $x \in \Gamma$,

3. for x on boundary ∂_i, $\langle D\bar{W}^n(x), d_i \rangle \geq 0$ when $D\bar{W}^n(x)$ is well defined.

Fix $\varepsilon_n > 0$. The exponential weighting produces a smooth mollification of $\bar{W}^n$ by

$$W^{\varepsilon_n, n}(x) \doteq -\varepsilon_n \log \sum_{k=1}^{K} \exp\left\{ -\frac{1}{\varepsilon_n} \bar{W}_k^n(x) \right\},$$

Similar to the proof of Lemma B.1, it is not difficult to show that, thanks to the concavity of $\mathbb{H}$ and Lemma 3.11, $W^{\varepsilon_n, n}$ satisfies $\mathbb{H}(D\bar{W}^{\varepsilon, n}(x)) \geq 0$

for $x \in D$ and $\bar{W}^{\varepsilon,n}(x) \le 0$ for every $x \in \Gamma$. However, $\langle DW^{\varepsilon_n,n}(x), d_i \rangle \ge 0$ may not hold for $x \in \partial_i$. But we should expect these boundary inequalities to be true at least approximately, thanks to the third property of $\{\bar{W}_k^n\}$.

For each n, the importance sampling change of measure based on $W^{\varepsilon_n,n}$ is as follows. Let

$$\rho_i^{\varepsilon_n,n}(x) \doteq \frac{\exp\left\{-\bar{W}_i^n(x)/\varepsilon_n\right\}}{\sum_{k=1}^K \exp\left\{-\bar{W}_k^n(x)/\varepsilon_n\right\}},$$

and

$$\bar{\Theta}_k^{*,n}(x) \doteq \bar{\Theta}^*(D\bar{W}_k^n(x)) \in \mathcal{P}^+(\mathbb{V})$$

where $\bar{\Theta}^*(\cdot)$ is as defined in Proposition 4.1. The importance sampling change of measure is determined by the stochastic kernel

$$\bar{\Theta}^n[\cdot|\cdot] \equiv \bar{\Theta}^{\varepsilon_n,n}[\cdot|x] \doteq \sum_{k=1}^K \rho_k^{\varepsilon_n,n}(x)\bar{\Theta}_k^{*,n}(x) \in \mathcal{P}^+(\mathbb{V}).$$

That is, the conditional distribution of $Y(k+1)$, given the sample history $\{Y(1), \ldots, Y(k)\}$, is just $\bar{\Theta}^n[\cdot|X^n(k)]$. We denote by $\hat{p}_n$ the corresponding importance sampling estimator.

The following result is an extension of Theorem 3.16. The proof is the same except for notational difference and thus omitted.

Theorem 4.2. *We assume that $\{\bar{W}_k^n(x)\}$ has uniformly bounded first and second derivatives for $x \in D$ and that there exists $\bar{\varepsilon}_n \ge 0$ such that for $x \in \partial_i$, $\langle DW^{\varepsilon_n,n}(x), d_i \rangle \ge -\bar{\varepsilon}_n$. We also assume that $\liminf_n \bar{W}^n(0) \ge 2\gamma$. Then the importance sampling estimator $\hat{p}_n$ is asymptotically optimal, that is*

$$\lim_n -\frac{1}{n}\log[\text{2nd moment of } \hat{p}_n] = 2\gamma,$$

provided that $\varepsilon_n \to 0$, $\bar{\varepsilon}_n \to 0$, and $n\varepsilon_n \to \infty$.

In the previous two-node tandem network we have taken $\bar{W}_k^n(x) = \langle r_k, x \rangle + 2\gamma - k\delta_n$. For this choice we can set $\bar{\varepsilon}_n = 2\gamma \exp\{-\delta_n/\varepsilon_n\}$ thanks to Lemma B.1, and Theorem 4.2 reduces to Theorem 3.16.

Remark 4.3. One can also write down a result similar to Theorem 3.13 for the case where $\bar{W}_k^n \equiv \bar{W}_k$ and $\varepsilon_n \equiv \varepsilon$, $\bar{\ }_n \varepsilon \equiv \bar{\varepsilon}$. The corresponding importance sampling estimator, still denoted by $\hat{p}_n$, will satisfy

$$\liminf_n -\frac{1}{n}\log[\text{2nd moment of } \hat{p}_n] \ge \bar{W}(0) - (K\varepsilon + C\bar{\varepsilon})$$

where C is a constant only depends on the system parameters Θ, under the condition that $\bar{\varepsilon}$ is small enough.

4.3 Example and numerical results

In this section we study two examples: the individual buffer overflow for two-node tandem Jackson network and total population overflow for d-node tandem Jackson network.

4.3.1 Two-node tandem networks with individual buffer overflow

In this section we consider the two-node tandem queueing $(d = 2)$ networks with $\Theta = (\lambda, \mu_1, \mu_2)$, and the quantity of interest is

$$p_n \doteq \{\text{size of queue 1 exceeds } B_1 n \text{ or size of queue 2 exceeds } B_2 n$$
$$\text{before the system returns to empty state, starting from } 0\}.$$

One can think of $B_i n$ as the individual buffer size for node i. In the notation we just introduced, it amounts to $\Gamma = \{x \in \mathbb{R}^2_+ : x_1 \geq B_1 \text{ or } x_2 \geq B_2\}$. Assuming $\lambda + \mu_1 + \mu_2 = 1$ and $\lambda < \min\{\mu_1, \mu_2\}$, we have (following a similar argument in [11])

$$\gamma \doteq \lim_n -\frac{1}{n} \log p_n = \min_{i=1,2} B_i \log \frac{\mu_i}{\lambda}.$$

We consider piecewise affine subsolutions that takes form $\bar{W}^n \doteq \bar{W}_1^n \wedge \bar{W}_2^n \wedge \bar{W}_3^n$ where
$$\bar{W}_k^n(x) \doteq \langle r_k, x \rangle + 2\gamma - k\delta_n,$$

for some small positive constants δ_n. The choice of $\{r_k\}$ and its corresponding change of measure $\{\bar{\Theta}^*(r_k)\}$ are given in the table below.
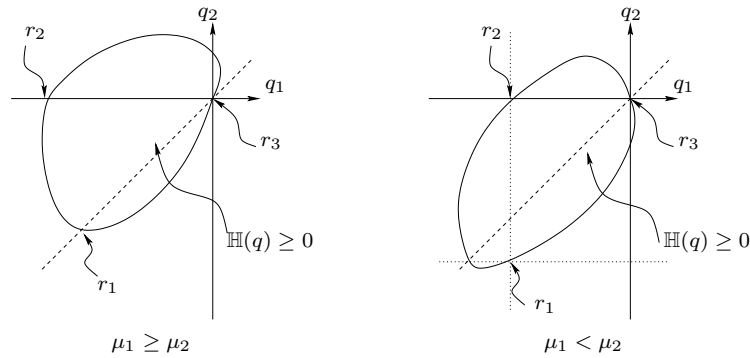

Figure 5: The choice of $\{r_k\}$

	r_k	$\bar{\Theta}^*(r_k)$
$\mu_1 \geq \mu_2$	$r_1 = 2\log(\mu_2/\lambda)(-1,-1)$ $r_2 = 2\log(\mu_1/\lambda)(-1,0)$ $r_3 = (0,0)$	(μ_2, μ_1, λ) (μ_1, λ, μ_2) (λ, μ_1, μ_2)
$\mu_1 < \mu_2$	$r_1 = (-2\log(\mu_1/\lambda), -2\log(\mu_2/\lambda))$ $r_2 = 2\log(\mu_1/\lambda)(-1,0)$ $r_3 = (0,0)$	(μ_1, μ_2, λ) (μ_1, λ, μ_2) (λ, μ_1, μ_2)

It is not difficult to check that $H(D\bar{W}^n_k(x)) = H(r_k) = 0$ and the function $\bar{W}^n$ equals $\bar{W}_k$ in region R_k (see the figure below). Furthermore, $\bar{W}^n(x) \leq 0$ for every $x \in \Gamma$ and $\langle D\bar{W}^n(x), d_i \rangle \geq 0$ whenever $x \in \partial_i$ and $D\bar{W}^n$ is well defined. As in Lemma B.1, it is also simple to show that the exponential weighting mollification $W^{\varepsilon_n, n}$ satisfies

$$\langle DW^{\varepsilon_n, n}(x), d_i \rangle \geq -\bar{\varepsilon}_n \doteq -2\log[(\mu_1 \vee \mu_2)/\lambda]\exp\{-\delta_n/\varepsilon_n\} \qquad (4.1)$$

for $x \in \partial_i$. Thanks to Theorem 4.2, the importance sampling estimator is asymptotically if $\delta_n \to 0, \varepsilon_n/\delta_n \to 0$, and $n\varepsilon_n \to \infty$.
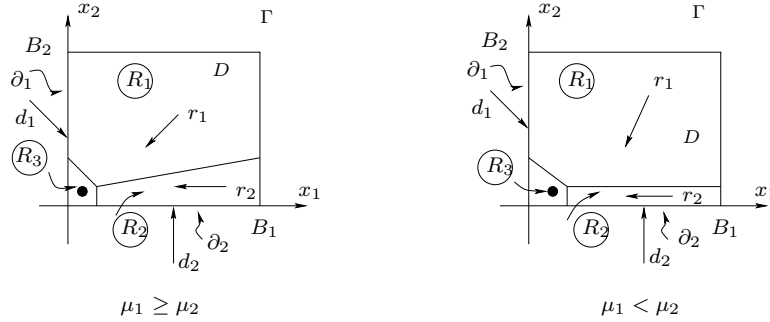

Figure 6: Piecewise affine function

4.3.2 d-node tandem networks with total population overflow

In this section we consider the total population overflow for a d-node tandem Jackson network with $d \geq 2$, that is, $\Gamma = \{x \in \mathbb{R}^+_d : x_1 + x_2 + \cdots + x_d \geq 1\}$ and

$$p_n \doteq \mathbb{P}\{\text{network total population reaches } n \text{ before returning to } 0,$$
$$\text{starting from } 0\}.$$

Specializing to the case $d = 2$ (and assuming $\mu_1 \geq \mu_2$), the results here indeed coincides with those in Section 3. Let $\bar{\mu} \doteq \mu_1 \wedge \mu_2 \ldots \wedge \mu_d$. Assuming

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$\lambda < \bar{\mu}$ and $\lambda + \mu_1 + \cdots + \mu_d = 1$, we have [11]

$$\gamma \doteq \lim_n -\frac{1}{n} \log p_n = \log \frac{\bar{\mu}}{\lambda}.$$

For any fixed n, we consider piecewise affine subsolutions of form $\bar{W}^n = \bar{W}_1^n \wedge \cdots \wedge \bar{W}_{d+1}^n$ where

$$\bar{W}_k^n(x) \doteq \langle r_k, x \rangle + 2\gamma - k\delta_n$$

for some small positive constant δ_n and

$$[r_k]_i \doteq \begin{cases} -2\gamma & ; & \text{if } 1 \le i \le d+1-k \\ 0 & ; & \text{otherwise} \end{cases}.$$

for $1 \le k \le d$ and $r_{d+1} = 0$. The change of measure corresponding to r_k is

$$\begin{aligned}
\bar{\Theta}^*(r_k) &= \left[1 - (\mu_{d+1-k} - \bar{\mu}) \frac{\bar{\mu} - \lambda}{\bar{\mu}} \right]^{-1} \\
&\quad \cdot \left(\bar{\mu}, \mu_1, \ldots, \mu_{d-k}, \frac{\lambda \mu_{d+1-k}}{\bar{\mu}}, \mu_{d+2-k}, \ldots, \mu_d \right)
\end{aligned}$$

for $1 \le k \le d$, and

$$\bar{\Theta}^*(r_{d+1}) = \Theta = (\lambda, \mu_1, \ldots, \mu_d).$$

We have the following lemma, whose proof is deferred to Appendix D.

Lemma 4.4. *We have $\mathbb{H}(r_k) \ge 0$ for every k and $\bar{W}^n(x) \le 0$ for $x \in \Gamma$. Moreover, $\langle D\bar{W}^n(x), d_i \rangle \ge 0$ for $x \in \partial_i$ such that $D\bar{W}^n(x)$ is well defined. Furthermore, let $W^{\varepsilon_n, n}$ denotes the exponential weighting of $\bar{W}^n$ with ε_n as the mollification parameter and we have*

$$\langle DW^{\varepsilon_n, n}(x), d_i \rangle \ge -\bar{\varepsilon}_n \doteq -2\gamma \exp\{-\delta_n/\varepsilon_n\}$$

for every $x \in \partial_i$.

Invoking Theorem 4.2, the importance sampling schemes corresponding to $W^{\varepsilon_n, n}$ is asymptotically optimal if $\delta_n \to 0$, $\varepsilon_n/\delta_n \to 0$, and $n\varepsilon_n \to \infty$.

4.3.3 Numerical results

For all the simulations in this section, we set $\delta = -\varepsilon \log \varepsilon$. The justification for this choice is based on an argument analogous to that of Remark 3.15, Remark 4.3, equation (4.1) and Lemma 4.4.

Consider the example of two-node tandem queue with individual buffer overflow as presented in Section 4.3.1. For the case of $\mu_1 \geq \mu_2$, we set $\lambda = 0.1$, $\mu_1 = 0.5$, $\mu_2 = 0.4$, and $B_1 = 0.9$, $B_2 = 1$. We run simulations for $n = 20, 30, 40$ with $\varepsilon = 0.01$. Below are the numerical results. Each estimate consists of 20,000 replications. Again, for comparison the theoretical value is obtained using some iterative algorithm.

	$n = 20$	$n = 30$	$n = 40$
Theoretical value	4.81×10^{-12}	3.97×10^{-18}	3.47×10^{-24}
Estimate	4.83×10^{-12}	4.04×10^{-18}	3.64×10^{-24}
Std. Err.	0.20×10^{-12}	0.15×10^{-18}	0.18×10^{-24}
95% C.I.	$[4.43, 5.23] \times 10^{-12}$	$[3.74, 4.34] \times 10^{-18}$	$[3.28, 4.00] \times 10^{-24}$

Table 3. IS based on subsolutions, two-node tandem, individual buffer overflow, $\mu_1 \geq \mu_2$

For the case of $\mu_1 < \mu_2$, we set $\lambda = 0.05$, $\mu_1 = 0.35$, $\mu_2 = 0.6$, and $B_1 = 1$, $B_2 = 0.6$. We run simulations for $n = 20, 30, 40$ with $\varepsilon = 0.1$, and each estimate consists of 20,000 replications.

	$n = 20$	$n = 30$	$n = 40$
Theoretical value	1.44×10^{-12}	4.82×10^{-19}	1.61×10^{-25}
Estimate	1.40×10^{-12}	5.01×10^{-19}	1.85×10^{-25}
Std. Err.	0.05×10^{-12}	0.29×10^{-19}	0.21×10^{-25}
95% C.I.	$[1.30, 1.50] \times 10^{-12}$	$[4.43, 5.59] \times 10^{-19}$	$[1.43, 2.27] \times 10^{-25}$

Table 4. IS based on subsolutions, two-node tandem, individual buffer overflow, $\mu_1 < \mu_2$

As for the total population overflow for general d-node tandem networks in Section 4.3.2, we run simulations for $d = 4$ and $d = 9$. For $d = 4$, we set $\lambda = 0.04$, $\mu_1 = \cdots = \mu_4 = 0.24$, and run simulations for $n = 20, 25, 30$ with $\varepsilon = 0.1$. Again, each estimate consists of $20,000$ replications, and the theoretical value is obtained using an iterative algorithm.

	$n = 20$	$n = 25$	$n = 30$
Theoretical value	2.04×10^{-12}	5.02×10^{-16}	1.10×10^{-19}
Estimate	2.05×10^{-12}	5.07×10^{-16}	1.08×10^{-19}
Std. Err.	0.04×10^{-12}	0.07×10^{-16}	0.03×10^{-19}
95% C.I.	$[1.97, 2.13] \times 10^{-12}$	$[4.93, 5.21] \times 10^{-16}$	$[1.02, 1.04] \times 10^{-19}$

Table 5. IS based on subsolutions, five-node tandem, total population overflow.

For $d = 9$, we set $\lambda = 0.01$, $\mu_1 = \cdots = \mu_9 = 0.11$, and run simulations for $n = 20, 25, 30$ with $\varepsilon = 0.12$. Each estimate consists of 100,000 replications.

In this case, the theoretical value is obtained using the same dynamic importance sampling algorithm but with 10 million replications (the iterative algorithm for computing the theoretical value in the case of $d = 4$ does not work here due to an overly large state space).

	$n = 20$	$n = 25$	$n = 30$
Theoretical value	3.18×10^{-14}	9.41×10^{-19}	2.16×10^{-23}
Estimate	2.93×10^{-14}	10.80×10^{-19}	1.98×10^{-23}
Std. Err.	0.23×10^{-14}	1.30×10^{-19}	0.30×10^{-23}
95% C.I.	$[2.47, 3.39] \times 10^{-14}$	$[8.20, 13.10] \times 10^{-19}$	$[1.38, 2.58] \times 10^{-23}$

Table 6. IS based on subsolutions, nine-node tandem, total population overflow.

5 Remarks on general queueing networks

It is possible to write down a result, analogous to Theorem 4.2, that applies to general open Jackson networks. However, due to the space limitation, we will only present a simple example to illustrate the main idea and refer the interested readers to [16] for such a general theorem. The major difference from the theory developed in Section 3 and Section 4, in short, is that the Neumann-type boundary conditions for the Isaacs equation are not sufficient anymore, and one has to consider the more elaborate boundary Hamiltonians.

To be more concrete, consider the following two-node tandem Jackson network with feedback. Again assume Poisson arrivals with rate λ and consecutive exponentially services with rate μ_i at node i. However, after being served at node 2, a job has probability β to be returned to node 1.
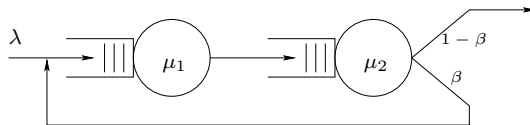

Figure 7: Two-node tandem network with feedback

Suppose that the quantity of interest is the probability of total population overflow,

$$p_n \doteq \mathbb{P}\{\text{network total population reaches } n \text{ before returning to } 0,$$
$$\text{starting from } 0\}.$$

Let $\bar{\mu} \doteq \mu_1 \wedge \mu_2$. Assuming the stability condition $\lambda < \bar{\mu}(1-\beta)$, and without

loss of generality, $\lambda + \mu_1 + \mu_2 = 1$, we have [11]

$$\gamma \doteq \lim_n -\frac{1}{n}\log p_n = \log\frac{(1-\beta)\bar{\mu}}{\lambda}.$$

The goal is to find an efficient importance sampling scheme for the estimation of p_n.

5.1 System dynamics

Let $Z = \{Z(k)\}$ be the embedded discrete time Markov chain that represents the queue lengths at the transition epochs of the network. Then the dynamics of Z can be modeled by

$$Z(k+1) = Z(k) + \pi[Z(k), Y(k+1)]$$

where $\{Y(k)\}$ are iid random variables taking values in the space

$$\mathbb{V} \doteq \{v_0 = (1,0), v_1 = (-1,1), v_2 = (0,-1), v_3 = (1,-1)\},$$

and the mapping π is defined as

$$\pi[z,y] \doteq \begin{cases} 0 & ; \quad \text{if } z_1 = 0 \text{ and } y = v_1 \\ 0 & ; \quad \text{if } z_2 = 0 \text{ and } y = v_2 \text{ or } v_3 \\ y & ; \quad \text{otherwise} \end{cases}$$

Under the original probability measure $\mathbb{P}$, the distribution of $Y(k)$ is just

$$\Theta \doteq (\lambda, \mu_1, (1-\beta)\mu_2, \beta\mu_2) \in \mathcal{P}^+(\mathbb{V}).$$

See Figure 8 for illustration.

5.2 The Isaacs equation and boundary Hamiltonian

We should use the same notation introduced in Section 3.4. Following the recipe of Section 3, define the scaled state process $X^n(k) \doteq Z(k)/n$ and dynamic importance sampling schemes are characterized by stochastic kernels $\bar{\Theta}^n[\cdot|\cdot]$ on space $\mathbb{V}$ such that the conditional distribution of $Y(k+1)$, given $\{Y(1), \ldots, Y(k)\}$, is just $\bar{\Theta}^n[\cdot|X^n(k)] \in \mathcal{P}^+(V)$.

Following the argument in Section 3.5, one can write down the Isaacs equation $\mathbb{H}(DW(x)) = 0$ for $x \in D$, where

$$\mathbb{H}(p) = \sup_{\bar{\Theta} \in \mathcal{P}^+(\mathbb{V})} \inf_{\theta \in \mathcal{P}^+(\mathbb{V})} \left[\langle p, \mathbb{F}(\theta) \rangle + \sum_{i=0}^{3} \theta[v_i] \log\frac{\bar{\Theta}[v_i]}{\Theta[v_i]} + R(\theta\|\Theta) \right]$$
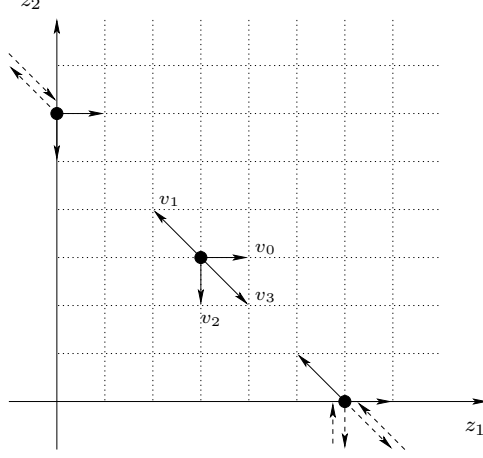
Figure 8: State dynamics

with

$$\mathbb{F}(\theta) \doteq \sum_{i=0}^{3} \theta[v_i] \cdot v_i,$$

and the Dirichlet boundary condition $W(x) = 0$ for $x \in \partial_{\mathsf{e}}$.

However, as far as the two boundaries ∂_1 and ∂_2 are concerned, the Neumann-type boundary condition $\langle DW(x), d_i \rangle = 0$ is not sufficient (more precisely, it is not sufficient for ∂_2, since the direction of constraint is not well defined on ∂_2). Instead one has to resort to *boundary Hamiltonians*, which, loosely speaking, is the Hamiltonian that one obtains using the state dynamics on the boundary [6]. Consequently, the boundary conditions become

$$\mathbb{H}_{\partial_i}(DW(x)) = 0, \quad \text{for } x \in \partial_i,\, i = 1, 2,$$

where the boundary Hamiltonian $\mathbb{H}_{\partial_i}$ is defined exactly as $\mathbb{H}$ except $\mathbb{F}(\theta)$ is replaced by $\mathbb{F}_i(\theta)$ with

$$\mathbb{F}_1(\theta) = \sum_{i \neq 1} \theta[v_i] \cdot v_i, \quad \mathbb{F}_2(\theta) = \sum_{i \neq 2,3} \theta[v_i] \cdot v_i$$

Remark 5.1. Proposition 4.1 can be easily applied to the interior Hamiltonian $\mathbb{H}$ and the boundary Hamiltonian $\mathbb{H}_{\partial_i}$ to show the existence of saddle points and the concavity of these Hamiltonians. The formulae for the saddle points are as follows. Let $(\bar{\Theta}^*(\cdot), \theta^*(\cdot))$ be the saddle point for $\mathbb{H}$, and

$\left(\bar{\Theta}^*_{\partial_i}(\cdot), \theta^*_{\partial_i}(\cdot)\right)$ be the saddle point for $\mathbb{H}_{\partial_i}$. Then

$$\bar{\Theta}^*(p) = \theta^*(p) = N(p) \cdot \left(\lambda e^{-\frac{p_1}{2}}, \mu_1 e^{\frac{p_1-p_2}{2}}, (1-\beta)\mu_2 e^{\frac{p_2}{2}}, \beta\mu_2 e^{\frac{p_2-p_1}{2}}\right),$$

$$\bar{\Theta}^*_{\partial_1}(p) = \theta^*_{\partial_1}(p) = N_1(p) \cdot \left(\lambda e^{-\frac{p_1}{2}}, \mu_1, (1-\beta)\mu_2 e^{\frac{p_2}{2}}, \beta\mu_2 e^{\frac{p_2-p_1}{2}}\right),$$

$$\bar{\Theta}^*_{\partial_2}(p) = \theta^*_{\partial_2}(p) = N_2(p) \cdot \left(\lambda e^{-\frac{p_1}{2}}, \mu_1 e^{\frac{p_1-p_2}{2}}, (1-\beta)\mu_2, \beta\mu_2\right),$$

where $N(p), N_i(p)$ are normalizing constants so that all these vectors are probability vectors (i.e., elements in $\mathcal{P}^+(\mathbb{V})$). Moreover, $\mathbb{H}(p) = 2\log N(p)$ and $\mathbb{H}_{\partial_i}(p) = 2\log N_i(p)$.

5.3   Piecewise affine subsolutions and mollification

The definition of a classical subsolution is the same as Definition 3.8, except that Neumann boundary inequality $\langle DW(x), d_i \rangle \geq 0$ is replaced by $\mathbb{H}_{\partial_i}(DW(x)) \geq 0$ for $x \in \partial_i$, $i = 1, 2$.

The construction of piecewise affine subsolution is very similar to that in Section 3.8.1. Define

$$r_1 \doteq 2\gamma(-1, -1), \quad r_2 \doteq 2\gamma(-1, 0) + 2(\gamma - a)(0, -1), \quad r_3 \doteq (0, 0),$$

where the choice of a is given by

$$a \doteq \begin{cases} \log[\mu_1/(\mu_1 + \lambda - (1-\beta)\mu_2)] & ; \quad \text{if } \mu_1 \geq \mu_2 \\ \log[\mu_1/(\lambda + \beta\mu_1)] & ; \quad \text{if } \mu_1 < \mu_2 \end{cases}.$$
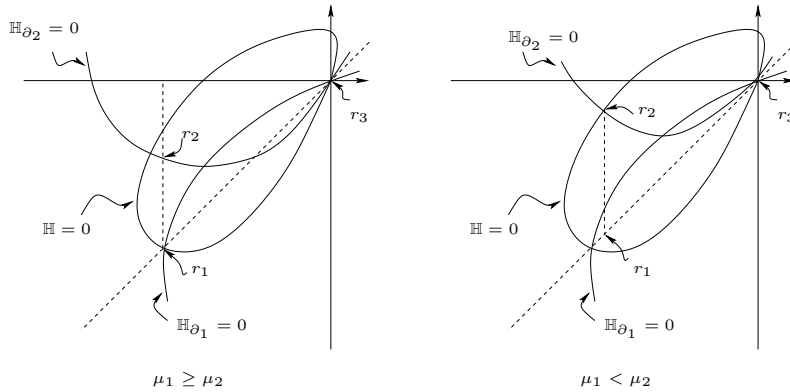

Figure 9: The Hamiltonians and the choice of $\{r_k\}$

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It is not difficult to check that $0 < a \leq \gamma$. Now let $\bar{W}^\delta = \bar{W}_1^\delta \wedge \bar{W}_2^\delta \wedge \bar{W}_3^\delta$ where

$$
\begin{aligned}
\bar{W}_1^\delta(x) &\doteq \langle r_1, x \rangle + 2\gamma - \delta \\
\bar{W}_2^\delta(x) &\doteq \langle r_2, x \rangle + 2\gamma - 2\delta \\
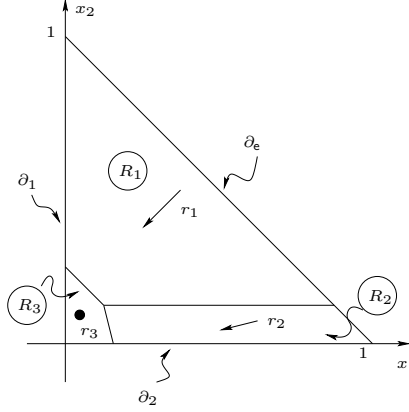\bar{W}_3^\delta(x) &\doteq \langle r_3, x \rangle + 2\gamma - (1 + 2\gamma/a)\delta.
\end{aligned}
$$

Figure 10: The subsolution

The exponential weighting of $\bar{W}^\delta$ with parameter ε yields a smooth function

$$
W^{\varepsilon,\delta}(x) \doteq -\varepsilon \log \sum_{k=1}^{3} \exp\left\{-\frac{1}{\varepsilon}\bar{W}_k^\delta(x)\right\}.
$$

that satisfies

$$
DW^{\varepsilon,\delta}(x) = \sum_{k=1}^{3} \rho_k^{\varepsilon,\delta}(x) r_k, \quad \rho_i^{\varepsilon,\delta}(x) \doteq \frac{\exp\left\{-\bar{W}_i^\delta(x)/\varepsilon\right\}}{\sum_{k=1}^{3} \exp\left\{-\bar{W}_k^\delta(x)/\varepsilon\right\}}.
$$

We have the following result, whose proof is deferred to Appendix D.

Lemma 5.2. *For each k, we have $\mathbb{H}(r_k) \geq 0$, and the function $W^{\varepsilon,\delta}$ satisfies*

1. *$\mathbb{H}(DW^{\varepsilon,\delta}(x)) \geq 0$ for $x \in D$,*

2. *$W^{\varepsilon,\delta}(x) \leq 0$ for $x \in \partial_{\mathsf{e}}$,*

3. *For each $i = 1, 2$, and $x \in \partial_i$, we have*

$$
\mathbb{H}_{\partial_i}(DW^{\varepsilon,\delta}(x)) \geq \sum_{k=1}^{3} \rho_k^{\varepsilon,\delta}(x)\mathbb{H}_{\partial_i}(r_k) \geq -\bar{C}\exp\{-\delta/\varepsilon\}
$$

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for some constant $\bar{C}$ that only depends on the system parameter Θ.

5.4 The importance sampling scheme and its asymptotics

The importance sampling scheme based on $\bar{W}^{\varepsilon,\delta}$ is as follows. Define stochastic kernel $\bar{\Theta}^{\varepsilon,\delta}[\cdot|\cdot]$ on $\mathbb{V}$ by

$$\bar{\Theta}^{\varepsilon,\delta}[\cdot|x] \doteq \sum_{k=1}^{3} \rho_k^{\varepsilon,\delta}(x)\bar{\Theta}^*(r_k), \qquad \text{if } x \in D$$

and

$$\bar{\Theta}^{\varepsilon,\delta}[\cdot|x] \doteq \sum_{k=1}^{3} \rho_k^{\varepsilon,\delta}(x)\bar{\Theta}_{\partial_i}^*(r_k), \qquad \text{if } x \in \partial_i.$$

Here the formulae for $\bar{\Theta}^*$ and $\bar{\Theta}_{\partial_i}^*$ can be found in Remark 5.1.

We will allow ε and δ to be n-dependent, denoted by ε_n, δ_n, and let $\bar{\Theta}^n[\cdot|\cdot] \equiv \bar{\Theta}^{\varepsilon_n,\delta_n}[\cdot|\cdot]$. Denote by $\hat{p}_n$ the corresponding importance sampling estimator. We have the following result, whose proof is very similar to that of Theorem 3.16. Indeed, in the proof of Theorem 3.16, the Neumann boundary condition is used to derive (implicitly) certain inequalities associated with boundary Hamiltonians. Such inequalities can now be obtained using Lemma 5.2. We omit the details.

Theorem 5.3. *The importance sampling estimator $\hat{p}_n$ is asymptotically optimal if $\delta_n \to 0$, $\varepsilon_n/\delta_n \to 0$, and $n\varepsilon_n \to \infty$.*

One can also use a fixed pair of parameters ε and δ for all n, which leads to a result similar to Theorem 3.13 and suggests a good choice may be to take $\delta_n = -\varepsilon_n \log \varepsilon_n$.

5.5 Numerical results

For all the simulation in this section, we set $\varepsilon = 0.02$ and $\delta = -\varepsilon \log \varepsilon$. For the case of $\mu_1 \geq \mu_2$, we choose $\lambda = 0.1, \mu_1 = 0.5, \mu_2 = 0.4$, and $\beta = 0.1$. We run simulations for $n = 20, 30, 40$ and each estimate consists of 20,000 replications. The theoretical value is obtained using a numerical iterative algorithm.

	$n = 20$	$n = 30$	$n = 40$
Theoretical value	9.60×10^{-11}	2.66×10^{-16}	7.27×10^{-22}
Estimate	9.31×10^{-11}	$2.60 \times 10-16$	7.33×10^{-22}
Std. Err.	0.17×10^{-11}	0.07×10^{-16}	0.33×10^{-22}
95% C.I.	$[8.97, 9.65] \times 10^{-11}$	$[2.46, 2.74] \times 10^{-16}$	$[6.67, 7.99] \times 10^{-22}$

Table 7. IS based on subsolutions, two-node tandem with feedback, $\mu_1 \geq \mu_2$

For the case of $\mu_1 < \mu_2$, we choose $\lambda = 0.1$, $\mu_1 = 0.43$, $\mu_2 = 0.47$, and $\beta = 0.2$. We run simulations for $n = 20, 30, 40$ and each estimate consists of 20,000 replications.

	$n = 20$	$n = 30$	$n = 40$
Theoretical value	4.39×10^{-10}	2.13×10^{-15}	9.60×10^{-21}
Estimate	4.62×10^{-10}	$1.91 \times 10{-15}$	9.88×10^{-21}
Std. Err.	0.46×10^{-10}	0.13×10^{-15}	0.87×10^{-21}
95% C.I.	$[3.70, 5.54] \times 10^{-10}$	$[1.65, 2.17] \times 10^{-15}$	$[8.14, 11.64] \times 10^{-21}$

Table 8. IS based on subsolutions, two-node tandem with feedback, $\mu_1 < \mu_2$

A   Appendix. A large deviation result

In this appendix we prove a useful large deviation result. The proof is of its own interest. For completeness, recall the definition of process Z by (3.5):

$$Z(k+1) = Z(k) + \pi[Z(k), Y(k+1)]$$

where $\{Y(k)\}$ is a sequence of iid random variables taking values in $\mathbb{V} = \{v_0, v_1, v_2\}$ with distribution $\Theta = (\lambda, \mu_1, \mu_2)$. Define the hitting times

$$\sigma_n \;\dot=\; \inf\{k \geq 0 : Z_1(k) + Z_2(k) = n\},$$
$$\sigma_0 \;\dot=\; \inf\{k \geq 0 : Z_1(k) + Z_2(k) = 0\}.$$

These quantities differ slightly from T_n and T_0, which are defined only for the initial condition 0 and which use $k > 0$ in the definition of T_0. To ease notation, let
$$\mathbb{Z}_n \;\dot=\; \left\{ (z_1, z_2) \in \mathbb{Z}_+^2 : z_1 + z_2 \leq n \right\}.$$

We have the following result.

Proposition A.1. *There exists a positive constant $c > 0$, which only depends on the system parameter (λ, μ_1, μ_2), such that*

$$\limsup_n \sup_{z \in \mathbb{Z}_n} \frac{1}{n} \log E_z \left[e^{c(\sigma_n \wedge \sigma_0)} \right] < \infty.$$

Here E_z denotes the expectation conditional on initial condition $Z(0) = z$.

The main difficulty to prove such a result is the discontinuity of the dynamics on the boundaries. The key idea to overcome this is to study a closely related one-dimensional process. More precisely, let $S(z) \;\dot=\; E_z[\sigma_0]$

for every $z \in \mathbb{Z}_+^2$. S is finite, thanks to the stability assumption. Define the process

$$Q(k) \doteq \begin{cases} S(Z(k)) & ; \quad \text{if } k \leq \sigma_0 \\ \sigma_0 - k & ; \quad \text{if } k > \sigma_0 \end{cases} . \qquad (1.1)$$

In other words, the process Q is random until the process Z hits the origin, after which Q becomes deterministic and decreases by 1 each step. The scaled continuous-time piecewise affine interpolation process is just

$$Q_n(t) \doteq \frac{1}{n} Q(\lfloor nt \rfloor) + \frac{nt - \lfloor nt \rfloor}{n} \left[Q(\lfloor nt \rfloor + 1) - Q(\lfloor nt \rfloor) \right], \qquad (1.2)$$

for $t \geq 0$.

In order to give a large deviation upper bound for the processes $\{Q_n\}$, we need the following definitions. Fix any $\alpha \in \mathbb{R}$. For each $z \in \mathbb{Z}_+^2$, define

$$h(z; \alpha) \doteq \log E_z \exp\{\alpha(Q(1) - Q(0))\} \qquad (1.3)$$

and

$$H(\alpha) \doteq \sup_{z \in \mathbb{Z}_+^2} h(z; \alpha). \qquad (1.4)$$

Clearly, H is convex since $h(z; \cdot)$ is convex for each z. The convex conjugate of H is denoted by L, or,

$$L(\beta) \doteq \sup_{\alpha \in \mathbb{R}} \left[\alpha\beta - H(\alpha) \right]. \qquad (1.5)$$

The function L is non-negative since $H(0) = 0$, and it will serve as a local rate function. For any fixed time $T \geq 0$, let $C([0, T]; \mathbb{R})$ be the Polish space of continuous functions on interval $[0, T]$ equipped with the supremum metric ρ. Define a mapping $I_T : C([0, T]; \mathbb{R}) \to \mathbb{R}_+ \cup \{\infty\}$ by

$$I_T(\phi) \doteq \begin{cases} \int_0^T L(\dot{\phi}(t)) \, dt & ; \quad \text{if } \phi \text{ is absolutely continuous} \\ \infty & ; \quad \text{otherwise} \end{cases} ,$$

and its level set

$$\Phi_x(s) \doteq \{\phi \in C([0, T]; \mathbb{R}) : \phi(0) = x, I_T(\phi) \leq s\}$$

for every $x \in \mathbb{R}$ and $s \geq 0$.

We have the following results, whose proofs are deferred to the end of this appendix. Proposition A.1 is a consequence of these lemmas.

Lemma A.2. *There exists a constant $M > 0$ such that $S(z) \leq M(z_1 + z_2)$ for every $z \in \mathbb{Z}_+^2$, and the increments of $\{Q(k)\}$ is uniformly bounded by M.*

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Lemma A.3. *Let $T > 0$ be given.*

1. *$I_T(\phi) \geq 0$ for every ϕ, and $I_T(\phi) = 0$ if and only if $\dot\phi(t) \equiv -1$.*

2. *There exists a constant K such that $I_T(\phi)$ is finite only if ϕ is Lipschitz continuous with Lipschitz constant K.*

3. *Given any compact set $F \subset \mathbb{R}$, the union of level sets, $\bigcup_{x \in F} \Phi_x(s)$, is compact for any $s \geq 0$. In particular, I_T is lower semicontinuous.*

4. *For any $h > 0$ and $s \geq 0$, we have*

$$\limsup_n \sup_{z \in \mathbb{Z}_n} \frac{1}{n} \log \mathbb{P}_z \left\{ \rho(Q_n, \Phi_{S(z)/n}(s)) > h \right\} \leq -s,$$

Proof of Proposition A.1. Let M be the constant in Lemma A.2, and K be the Lipschitz constant in Lemma A.3. For any $\delta > 0$ and $T > 0$, define

$$F_T^\delta \doteq \{ \phi \in C([0,T]; \mathbb{R}) : \phi(0) \in [0, M], -\delta \leq \phi \leq M + \delta,$$
$$\phi \text{ is absolutely continuous}, |\dot\phi| \leq K \vee M \},$$

which is a compact subset of $C([0,T]; \mathbb{R})$. Since I_T is lower semicontinuous by Lemma A.3, it attains its minimum on F_T^δ. However, it is not difficult to see that $I_T(\phi) > 0$ for any $\phi \in F_T^\delta$ if $T > M + \delta$. Indeed, suppose $I_T(\phi) = 0$. Then by Lemma A.3 we have $\phi(t) = \phi(0) - t$. If $\phi(0) \in [0, M]$ then for any $M + \delta < t \leq T$, $\phi(t) = \phi(0) - t < -\delta$. Thus $\phi \notin F_T^\delta$. It follows that, as long as $T > M + \delta$, $\min\{I_T(\phi) : \phi \in F_T^\delta\} > 0$, thanks to the lower-semicontinuity of I_T and the compactness of F_T^δ.

Now fix an arbitrary δ (the specific value of δ is not important), and let $t_0 = M + 4\delta$. Define

$$s \doteq \frac{1}{2} \min\{I_{t_0}(\phi) : \phi \in F_{t_0}^{2\delta}\} > 0.$$

For any x and $\phi \in \Phi_x(s)$, by Lemma A.3 again, ϕ is Lipschitz continuous with $|\dot\phi| \leq K$. However, it is trivial that $\Phi_x(s) \cap F_{t_0}^{2\delta} = \emptyset$ by definition. Therefore, for any $x \in [0, M]$ and $\phi \in \Phi_x(s)$, we must have

$$\inf\{t \geq 0 : \phi(t) \notin [-2\delta, M + 2\delta]\} \leq t_0. \tag{1.6}$$

Define the following stopping time

$$\tau_n^\delta \doteq \inf\{t \geq 0 : Q_n(t) \notin [-\delta, M + \delta]\}.$$

Thanks to Lemma A.2, the increment of $\{Q(k)\}$ is uniformly bounded by M, which in turn implies that Q_n has Lipschitz continuous sample paths and $|\dot{Q}_n| \leq M$. Moreover, for any initial condition $Z(0) = z \in \mathbb{Z}_n$, Lemma A.2 implies $Q_n(0) = S(z)/n \in [0, M]$. It follows that

$$\mathbb{P}_z(\tau_n^\delta > t_0) = \mathbb{P}_z(Q_n \in F_{t_0}^\delta).$$

Thanks to equation (1.6), it is trivial that for every $Q_n \in F_{t_0}^\delta$, we have

$$\rho(Q_n, \Phi_{S(z)/n}(s)) > \delta,$$

Therefore,

$$\mathbb{P}_z(\tau_n^\delta > t_0) \leq \mathbb{P}_z\left(\rho(Q_n, \Phi_{S(z)/n}(s)) > \delta\right)$$

However, it follows from Lemma A.2 that $\{\sigma_n \wedge \sigma_0 > nt_0\} \subset \{\tau_n^\delta > t_0\}$ for $n \geq M/\delta$. As a consequence,

$$\limsup_n \sup_{z \in \mathbb{Z}_n} \frac{1}{n} \log \mathbb{P}_z(\sigma_n \wedge \sigma_0 > nt_0)$$

$$\leq \quad \limsup_n \sup_{z \in \mathbb{Z}_n} \frac{1}{n} \log \mathbb{P}_z(\tau_n^\delta > t_0)$$

$$\leq \quad \limsup_n \sup_{z \in \mathbb{Z}_n} \frac{1}{n} \log \mathbb{P}_z\left(\rho(Q_n, \Phi_{S(z)/n}(s)) > \delta\right)$$

$$\leq \quad -s,$$

here the last inequality is by Lemma A.3. In particular,

$$\sup_{z \in \mathbb{Z}_n} \mathbb{P}_z(\sigma_n \wedge \sigma_0 \geq \lfloor nt_0 \rfloor + 1) \leq \sup_{z \in \mathbb{Z}_n} \mathbb{P}_z(\sigma_n \wedge \sigma_0 > nt_0) \leq e^{-ns/2}$$

for n big enough. Let $k_n \doteq \lfloor nt_0 \rfloor + 1$. Thanks to the Markov property, we have for n big enough,

$$\sup_{z \in \mathbb{Z}_n} \mathbb{P}_z(\sigma_n \wedge \sigma_0 \geq jk_n) \leq e^{-jns/2}, \quad \forall j \geq 0.$$

Let c be any constant such that $0 < c < s/(4t_0)$. We have, for n big enough,

$ck_n \le ns/4$, which implies that

$$
\begin{aligned}
E_z\left[e^{c(\sigma_n \wedge \sigma_0)}\right] &= \sum_{j=0}^{\infty} \sum_{i=jk_n}^{(j+1)k_n-1} e^{ci} \mathbb{P}_z(\sigma_n \wedge \sigma_0 = i) \\
&\le e^{ck_n} \sum_{j=0}^{\infty} e^{cjk_n} \mathbb{P}_z(jk_n \le \sigma_n \wedge \sigma_0 \le (j+1)k_n - 1) \\
&\le e^{ck_n} \sum_{j=0}^{\infty} e^{-j(ns/2 - ck_n)} \\
&\le e^{ck_n} \sum_{j=0}^{\infty} e^{-jns/4} \\
&= e^{ck_n} \frac{1}{1 - e^{-ns/4}}.
\end{aligned}
$$

Therefore,

$$
\limsup_n \sup_{z \in \mathbb{Z}_n} \frac{1}{n} \log E_z\left[e^{c(\sigma_n \wedge \sigma_0)}\right] \le \lim_n \frac{ck_n}{n} + \lim_n \frac{1}{n} \log \frac{1}{1 - e^{-ns/4}} = ct_0.
$$

We complete the proof. ∎

It remains to show Lemma A.2 and A.3. The proof is technical and we need to investigate the processes in great detail. We will begin with the following result, whose proof is trivial and thus omitted.

Lemma A.4. *Let $\{\mathcal{F}_k = \sigma(Z(0), Y(1), \ldots, Y(k))\}$ be the filtration. Then*

$$
E_z[Q(k+1) - Q(k)|\mathcal{F}_k] = -1,
$$

for every $z \in \mathbb{Z}_+^2$ and every $k \ge 0$.

The next lemma is concerned with the monotonicity of the sample path with respect to the initial conditions. To be more precise, for $\bar{z}, z \in \mathbb{Z}_+^2$, we say $\bar{z} \le z$ if the inequality holds component-wise. Also for $z \in \mathbb{Z}_+^2$, denote by Q^z the sample path corresponding with initial condition z, that is,

$$
Q^z(0) = z, \quad Q^z(k+1) = Q^z(k) + \pi[Q^z(k), Y(k+1)].
$$

Lemma A.5. *Define $g : \mathbb{Z}_+^2 \to \mathbb{Z}_+$ by $g(z) = z_1 + z_2$. Given any $\bar{z}, z \in \mathbb{Z}_+^2$ such that $\bar{z} \le z$, we have*

$$
\begin{aligned}
Q^{\bar{z}}(k) &\le Q^z(k) \\
g\left(Q^z(k)\right) - g\left(Q^{\bar{z}}(k)\right) &\le g(z) - g(\bar{z})
\end{aligned}
$$

for every $k \ge 0$.

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Proof. We will use induction. The claim is trivial for $k = 0$. Assume for now that it holds for some $k \geq 0$. Introduce the following notation.

$$
\begin{aligned}
\Gamma &\doteq \{z \in \mathbb{Z}_+^2 : z_1 > 0, z_2 > 0\}, \\
\Gamma_1 &\doteq \{z \in \mathbb{Z}_+^2 : z_1 = 0, z_2 > 0\}, \\
\Gamma_2 &\doteq \{z \in \mathbb{Z}_+^2 : z_1 > 0, z_2 = 0\}.
\end{aligned}
$$

We should consider the following possible scenarios separately: (i) $Q^{\bar{z}}(k) \in \Gamma$; (ii) $Q^{\bar{z}}(k) \in \Gamma_1$; (iii) $Q^{\bar{z}}(k) \in \Gamma_2$; (iv) $Q^{\bar{z}}(k) = 0$. Since the proofs for these cases are essentially the same, we choose to only present case (ii). Assume that $Q^{\bar{z}}(k) \in \Gamma_1$. Thanks to induction hypothesis $Q^{\bar{z}}(k) \leq Q^{z}(k)$, we must have $Q^{z}(k) \in \Gamma_1$ or $Q^{z}(k) \in \Gamma$. If $Q^{z}(k) \in \Gamma_1$, or $Q^{z}(k) \in \Gamma$ but $Y(k+1) \neq v_1$, then it is trivial to check that $\pi[Q^{\bar{z}}(k), Y(k+1)] = \pi[Q^{z}(k), Y(k+1)]$ and the claim holds for $k+1$. It only remains to show for the case where $Q^{z}(k) \in \Gamma$ and $Y(k+1) = v_1$. In this case $Q^{\bar{z}}(k+1) = Q^{\bar{z}}(k)$ and $Q^{z}(k+1) = Q^{z}(k) + v_1 = Q^{z}(k) + (-1, 1)$. But since $Q_1^z(k) > 0$ and $Q_1^{\bar{z}}(k) = 0$, it is not difficult to see that $Q^{\bar{z}}(k+1) \leq Q^{z}(k+1)$. Furthermore, note that $g(Q^{z}(k+1)) = g(Q^{z}(k))$, $g(Q^{\bar{z}}(k+1)) = g(Q^{\bar{z}}(k))$. We complete the proof. ∎

Proof of Lemma A.2. Let $\bar{M} \doteq 2S((1,0)) + 2S((0,1))$. We would like to show that for any $z \in \mathbb{Z}_+^2$ and any $i = 0, 1, 2$,

$$
|S(z + \pi[z, v_i]) - S(z)| \leq \bar{M}. \tag{1.7}
$$

We can assume that $\pi[z, v_i] = v_i$, since otherwise there is nothing to prove. First we consider the case $i = 2$, and let $\bar{z} \doteq z + v_2 = (z_1, z_2 - 1) \leq z$. Define stopping times

$$
T^z \doteq \inf\{k \geq 0 : Q^z(k) = 0\}, \quad T^{\bar{z}} \doteq \inf\{k \geq 0 : Q^{\bar{z}}(k) = 0\}.
$$

Thanks to Lemma A.5, we have $Q^{\bar{z}}(k) \leq Q^z(k)$ for any $k \geq 0$, which implies $T^{\bar{z}} \leq T^z$. By the same lemma, for every $k \geq 0$, $g(Q^z(k)) - g(Q^{\bar{z}}(k)) \leq g(z) - g(\bar{z}) = 1$. In particular, for $k = T^{\bar{z}}$, we have $g(Q^z(T^{\bar{z}})) \leq 1$. It follows that

$$
Q^z(T^{\bar{z}}) \in \{(0,0), (1,0), (0,1)\}.
$$

Now the strong Markov property yields

$$
S(z) = S(\bar{z}) + \mathbb{P}\{Q^z(T^{\bar{z}}) = (1,0)\}S((1,0)) + \mathbb{P}\{Q^z(T^{\bar{z}}) = (0,1)\}S((0,1)).
$$

Thus $|S(z) - S(\bar{z})| \leq S((1,0)) + S((0,1)) \leq \bar{M}/2$. The proof for the case $i = 0$ is almost a verbatim. For $i = 1$, $z + v_i = z + (-1, 1)$. One can

use the same argument to prove that $|S(z) - S(z + (-1, 0))| \leq \bar{M}/2$ and $|S(z + (-1, 0)) - S(z + (-1, 1))| \leq \bar{M}/2$, and then use the triangle inequality to show $|S(z + v_1) - S(z)| \leq \bar{M}$. We omit the details.

It follows from (1.7) that the increment of $\{Q(k)\}$ is uniformly bounded by $\bar{M}$ (note that $\bar{M} \geq 1$ trivially since $S(z) \geq 1$ for every $z \neq 0$). Now for every $z \in \mathbb{Z}_+^2$, we can write $S(z)$ as

$$
\begin{aligned}
S(z) &= [S(z) - S((0, z_1 + z_2))] + [S((0, z_1 + z_2)) - S((0, 0))] \\
&= \sum_{i=0}^{z_1 - 1} [S(z + iv_1) - S(z + (i+1)v_1)] \\
&\quad + \sum_{i=0}^{z_1 + z_2 - 1} [S(z + z_1 v_1 - iv_2)) - S(z + z_1 v_1 - (i+1)v_2))].
\end{aligned}
$$

Thanks to (1.7) again, the absolutely value of each summand is bounded by $\bar{M}$. Thus $S(z) \leq \bar{M}(2z_1 + z_2)$. Take $M \doteq 2\bar{M}$ and we complete the proof. ∎

Proof of Lemma A.3. Recall the definition of $h(z; \alpha)$ and $H(\alpha)$ by (1.3)-(1.4). We first show that H is convex and Lipschitz continuous with $H(0) = 0$. To this end, note that $h(z; \cdot)$ is convex and satisfies $h(z; 0) = 0$ for each fixed $z \in \mathbb{Z}_+^2$. Therefore H is convex with $H(0) = 0$. Let M be the uniform bound on the increments of $\{Q(k)\}$ given by Lemma A.2. It follows easily

$$|h(z; \alpha)| \leq M|\alpha|.$$

Therefore $|H(\alpha)| \leq M|\alpha|$ for every α, whence H is Lipschitz continuous (thanks to its convexity).

We claim that H is differentiable at $\alpha = 0$ and $H'(0) = -1$. Indeed, since $h(z; \alpha)$ is differentiable with respect to α and $h(z; 0) = 0$, we have

$$
\frac{H(\alpha)}{\alpha} = \sup_{z \in \mathbb{Z}_+^2} \frac{h(z; \alpha)}{\alpha} = \sup_{z \in \mathbb{Z}_+^2} D_\alpha h(z; \alpha[z]),
$$

where $\alpha[z]$ is some number between 0 and α. But thanks to Lemma A.4, $D_\alpha h(z; 0) = E_z[Q(1) - Q(0)] = -1$. Moreover, Lemma A.2 and simple algebra yield that $|D_{\alpha\alpha} h(z; \alpha)| \leq \bar{K}$ for some constant $\bar{K}$ and for every $z \in \mathbb{Z}_+^2$ and $\alpha \in \mathbb{R}$. It follows that

$$
\left| \frac{H(\alpha)}{\alpha} + 1 \right| \leq \bar{K}|\alpha|,
$$

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which converges to 0 as $\alpha \to 0$, or H is differentiable at $\alpha = 0$ with $H'(0) = -1$.

The convexity of H and $H(0) = 0$ imply that L, defined by (1.5), is convex and non-negative. The Lipschitz continuity of H implies that L takes value infinity outside a compact set. Lastly, the differentiability of H at $\alpha = 0$ with $H'(0) = -1$ imply that $L(\beta) = 0$ if and only if $\beta = -1$. Now Parts 1 and 2 of Lemma A.3 are trivial from these properties of L. The rest of the Lemma follows from Theorem 4.1 of [5] and that

$$L(\beta) \leq l(z; \beta) \doteq \sup_{\alpha}[\alpha\beta - h(z; \alpha)].$$

We complete the proof. ■

B   Appendix. Proof of main theorems

We put the proofs of the main results, Theorem 3.13 and Theorem 3.16, together in this appendix. These proofs are, in essence, verification type arguments. We start with a few useful technical results.

Lemma B.1. *The function $W^{\varepsilon,\delta}$ as defined in (3.28) satisfies*

1. $\mathbb{H}(DW^{\varepsilon,\delta}(x)) \geq 0$ *for all $x \in D$,*

2. $W^{\varepsilon,\delta}(x) \leq 0$ *for all $x \in \partial_{\mathsf{e}}$,*

3. $\langle DW^{\varepsilon,\delta}(x), d_i \rangle \geq -2\gamma \exp\{-\delta/\varepsilon\}$ *for every $x \in \partial_i$.*

4. *There exists a constant C which only depends on the system parameter (λ, μ_1, μ_2) such that*
$$\left| \frac{\partial^2 W^{\varepsilon,\delta}(x)}{\partial x_i \partial x_j} \right| \leq \frac{C}{\varepsilon}$$
   for every $x \in \bar{D}$ and every i, j.

Proof. Thanks to (3.29), the concavity of H (Proposition 3.5), and that $\mathbb{H}(r_k) \geq 0$, it follows that

$$\mathbb{H}(DW^{\varepsilon,\delta}(x)) = \mathbb{H}\left(\sum_{k=1}^{3} \rho_k^{\varepsilon,\delta}(x) r_k \right) \geq \sum_{k=1}^{3} \rho_k^{\varepsilon,\delta}(x) \mathbb{H}(r_k) \geq 0.$$

By Lemma 3.11 we have $\bar{W}^{\varepsilon,\delta}(x) \leq \bar{W}^{\delta}(x)$. But $\bar{W}^{\delta}(x) \leq 0$ for $x \in \partial_{\mathsf{e}}$ by definition, the second claim follows.

Since $\langle r_1, d_1 \rangle = \langle r_3, d_1 \rangle = 0$ and $\langle r_2, d_1 \rangle = -2\gamma$, we have

$$\langle DW^{\varepsilon,\delta}(x), d_1 \rangle = -2\gamma \rho_2^{\varepsilon,\delta}(x).$$

For $x \in \partial_1$, thanks to (3.30) and (3.26), we have

$$\rho_2^{\varepsilon,\delta}(x) \leq \frac{\exp\left\{-\bar{W}_2^{\delta}(x)/\varepsilon\right\}}{\exp\left\{-\bar{W}_3^{\delta}(x)/\varepsilon\right\}} = \exp\{-\delta/\varepsilon\}.$$

Similarly, for $x \in \partial_2$, we have

$$\langle DW^{\varepsilon,\delta}(x), d_2 \rangle = -2\gamma \rho_1^{\varepsilon,\delta}(x) \geq -2\gamma \frac{\exp\left\{-\bar{W}_1^{\delta}(x)/\varepsilon\right\}}{\exp\left\{-\bar{W}_2^{\delta}(x)/\varepsilon\right\}} = -2\gamma \exp\{-\delta/\varepsilon\}.$$

This ends the proof of the third claim.

As for the last claim, it follows easily from (3.29) that

$$\frac{\partial^2 W^{\varepsilon,\delta}(x)}{\partial x_i \partial x_j} = \sum_{j=1}^{3} \frac{\partial \rho_k^{\varepsilon,\delta}(x)}{\partial x_j} \langle r_k, e_i \rangle,$$

where e_i is the standard i-th unit vector. However,

$$\begin{aligned}
\frac{\partial \rho_k^{\varepsilon,\delta}(x)}{\partial x_j} &= \frac{1}{\varepsilon} \cdot \rho_k^{\varepsilon,\delta}(x) \left[-\frac{\partial \bar{W}_k^{\delta}(x)}{\partial x_j} + \sum_{m=1}^{3} \rho_j^{\varepsilon,\delta}(x) \frac{\partial \bar{W}_m^{\delta}(x)}{\partial x_j} \right] \\
&= \frac{1}{\varepsilon} \cdot \rho_k^{\varepsilon,\delta}(x) \left[-\langle r_k, e_j \rangle + \sum_{m=1}^{3} \rho_j^{\varepsilon,\delta}(x) \langle r_m, e_j \rangle \right].
\end{aligned}$$

The last claim follows readily from the definition of $\{r_k\}$ and that $\rho_k^{\varepsilon,\delta}(x)$ is bounded between 0 and 1. ∎

We now define a few functions that are closely related to the interior Hamiltonian H and the boundary Hamiltonians. For each $\alpha \geq 0$, let

$$\bar{L}(\alpha, p; \bar{\Theta}, \theta) \doteq (1+\alpha)\langle p, \mathbb{F}(\theta) \rangle + (1+2\alpha) \sum_{i=0}^{2} \theta[v_i] \log \frac{\bar{\Theta}[v_i]}{\Theta[v_i]} + R(\theta \| \Theta),$$

for every $\bar{\Theta}, \theta \in \mathcal{P}^+(\mathbb{V})$. Similarly, for each $j = 1, 2$, let

$$\mathbb{F}_j(\theta) = \sum_{i \neq j} \theta[v_i] \cdot v_i,$$

and define

$$\bar{L}_j(\alpha, p; \bar{\Theta}, \theta) \doteq (1+\alpha)\langle p, \mathbb{F}_j(\theta) \rangle + (1+2\alpha) \sum_{i=0}^{2} \theta[v_i] \log \frac{\bar{\Theta}[v_i]}{\Theta[v_i]} + R(\theta \| \Theta).$$

Lemma B.2. *Let $p \in \mathbb{R}^2$ such that $\mathbb{H}(p) \geq 0$. Then for an arbitrarily given $\alpha \geq 0$, we have*

$$\inf_{\theta \in \mathcal{P}^+(\mathbb{V})} \bar{L}(\alpha, p; \bar{\Theta}^*(p), \theta) \geq 0,$$

where $\bar{\Theta}^*(p)$ is as defined in Proposition 3.5.

Proof. By definition of $\bar{L}$, (3.21), and Proposition 3.5, it is not difficult to check that

$$\begin{aligned}
\bar{L}(\alpha, p; \bar{\Theta}^*(p), \theta) &= \bar{L}(0, p; \bar{\Theta}^*(p), \theta) + 2\alpha \log N(p) \\
&= \bar{L}(0, p; \bar{\Theta}^*(p), \theta) + \alpha \mathbb{H}(p).
\end{aligned}$$

However, thanks to Proposition 3.5 again, we have

$$\inf_{\theta \in \mathcal{P}^+(\mathbb{V})} \bar{L}(0, p; \bar{\Theta}^*(p), \theta) = \bar{L}(0, p; \bar{\Theta}^*(p), \theta^*(p)) = \mathbb{H}(p).$$

This completes the proof. ∎

Proof of Theorem 3.13. To ease exposition, we will use notation $W = W^{\varepsilon, \delta}$, $\rho_k = \rho_k^{\varepsilon, \delta}$, and set $\bar{\varepsilon} \doteq 2\gamma \exp\{-\delta/\varepsilon\}$. Fix any $\alpha > 0$. We claim that, for every $x \in D_n$,

$$\inf_{\theta \in \mathcal{P}^+(\mathbb{V})} \bar{L}(\alpha, DW(x); \bar{\Theta}^n[\cdot|x], \theta) \geq 0. \tag{2.1}$$

Indeed, thanks to the definition of $\bar{L}$, the concavity of logarithmic function, and that $DW(x) = \sum \rho_k(x) r_k$, $\Theta^n[\cdot|x] = \sum \rho_k(x) \bar{\Theta}^*(r_k)$, one can check that

$$\bar{L}(\alpha, DW(x); \bar{\Theta}^n[\cdot|x], \theta) \geq \sum_{k=0}^{2} \rho_k(x) \bar{L}(\alpha, r_k; \bar{\Theta}^*(r_k), \theta).$$

Since $\mathbb{H}(r_k) \geq 0$, it follows from Lemma B.2 that

$$\inf_{\theta \in \mathcal{P}^+(\mathbb{V})} \bar{L}(\alpha, r_k; \bar{\Theta}^*(r_k), \theta) \geq 0,$$

which in turn implies (2.1). For every $x \in \partial_j \cap \bar{D}_n$, we have an analogous inequality

$$\inf_{\theta \in \mathcal{P}^+(\mathbb{V})} \bar{L}_j(\alpha, DW(x); \bar{\Theta}^n[\cdot|x], \theta) \geq -(1 + \alpha)\bar{\varepsilon}. \tag{2.2}$$

Actually, thanks to (2.1),

$$\begin{aligned}
\bar{L}_j&(\alpha, DW(x); \bar{\Theta}^n[\cdot|x], \theta) \\
&= \bar{L}(\alpha, DW(x); \bar{\Theta}^n[\cdot|x], \theta) - (1 + \alpha)\theta[v_j] \cdot \langle DW(x), v_j \rangle \\
&\geq -(1 + \alpha)\theta[v_j] \cdot \langle DW(x), v_j \rangle.
\end{aligned}$$

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Recall that $d_j = -v_j$ and (2.2) follows readily from Lemma B.1.

We now show that inequalities (2.1) and (2.2) imply

$$\inf_{\theta \in \mathcal{P}^+(\mathbb{V})} \left\{ \sum_{i=0}^{2} (1+\alpha)n \left[W\left(x + \frac{1}{n}\pi(x, v_i)\right) - W(x) \right] \cdot \theta[v_i] \right. \tag{2.3}$$

$$\left. +(1+2\alpha) \sum_{i=0}^{2} \theta[v_i] \log \frac{\bar{\Theta}^n[v_i|x]}{\Theta[v_i]} + R(\theta\|\Theta) \right\} \geq -(1+\alpha)\left[\frac{C}{n\varepsilon} + \bar{\varepsilon} \right]$$

for every $x \in \bar{D}_n$, where C is a constant that only depends on the system parameter (λ, μ_1, μ_2). To this end, we consider separately the cases $x \in D_n$ (interior) and $x \in \partial_j \cap \bar{D}_n$ (boundary). For $x \in D_n$, $\pi(x, v_i) \equiv v_i$. Therefore, by Taylor expansion,

$$n \left[W\left(x + \frac{1}{n}v_i\right) - W(x) \right] \cdot \theta[v_i] = \langle DW(x), v_i \rangle \cdot \theta[v_i] + \frac{1}{2n}v_i^T D^2 W(\bar{x}_i) v_i \cdot \theta[v_i],$$

where $\bar{x}_i$ is some point on the line connecting x and $x + v_i$. Thanks to Lemma B.1, the definition of $\mathbb{F}$ (3.21), and that $\|v_i\|^2 \leq 2$, we have

$$\sum_{i=0}^{2} n \left[W\left(x + \frac{1}{n}v_i\right) - W(x) \right] \cdot \theta[v_i] \geq \langle DW(x), \mathbb{F}(\theta) \rangle - \frac{C}{n\varepsilon}.$$

This and inequality (2.1) immediately leads to (2.3). The case of $x \in \partial_j \cap \bar{D}_n$ is similar, except now that $\pi(x, v_i) = v_i$ if $i \neq j$ and $\pi(x, v_j) = 0$. We omit the details.

Applying the relative entropy representation (Remark 3.4) to the left-hand-side of (2.3) and adopting notation $\beta_n \doteq C/(n\varepsilon) + \bar{\varepsilon}$, we have, for every $x \in \bar{D}_n$,

$$e^{-(1+\alpha)\beta_n} \cdot \sum_{i=0}^{2} e^{-(1+\alpha)n[W(x+\pi(x,v_i)/n)-W(x)]} \left(\frac{\Theta[v_i]}{\bar{\Theta}^n[v_i|x]} \right)^{1+2\alpha} \cdot \Theta[v_i] \leq 1.$$

Recalling the definition of X^n in (3.9), the above display implies that the process $M = \{M(k) : k \geq 0\}$, where

$$M(k) \doteq e^{-(1+\alpha)\beta_n k} e^{-(1+\alpha)nW(X^n(k))} \left(\prod_{j=0}^{k-1} \frac{\Theta[Y(j+1)]}{\bar{\Theta}^n[Y(j+1)|X^n(j)]} \right)^{1+2\alpha},$$

is indeed a supermartingale under the original probability measure $\mathbb{P}$. Thanks to the Optional Sampling Theorem and the non-negativity of M, we have

$$E^{\mathbb{P}} M(T_n \wedge T_0) \leq E^{\mathbb{P}} M(0) = e^{-(1+\alpha)nW(0)}.$$

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Recalling that $\hat{p}_n = \hat{p}_n \cdot 1_{\{T_n < T_0\}}$ and $W(x) \leq 0$ for $x \in \partial_{\mathsf{e}}$, we have

$$
\begin{aligned}
M(T_n \wedge T_0) &\geq M(T_n) \cdot 1_{\{T_n < T_0\}} \\
&= e^{-(1+\alpha)\beta_n T_n} e^{-(1+\alpha)nW(X^n(T_n))} \hat{p}_n^{1+2\alpha} \\
&\geq e^{-(1+\alpha)\beta_n T_n} \hat{p}_n^{1+2\alpha}.
\end{aligned}
$$

It follows that

$$
E^{\mathbb{P}}\left[e^{-(1+\alpha)\beta_n T_n} \hat{p}_n^{1+2\alpha} \right] \leq e^{-(1+\alpha)nW(0)}.
$$

By Hölder's inequality,

$$
\begin{aligned}
[\text{2nd moment of } \hat{p}_n] &= E^{\mathbb{P}}[\hat{p}_n] \qquad\qquad\qquad\qquad\qquad\qquad\qquad (2.4) \\
&\leq E^{\mathbb{P}}\left[e^{-(1+\alpha)\beta_n T_n} \hat{p}_n^{1+2\alpha} \right]^{\frac{1}{1+2\alpha}} \cdot E^{\mathbb{P}}\left[e^{\frac{1+\alpha}{2\alpha}\beta_n T_n} \cdot 1_{\{T_n < T_0\}} \right]^{\frac{2\alpha}{1+2\alpha}} \\
&\leq e^{-\frac{1+\alpha}{1+2\alpha}nW(0)} \cdot E^{\mathbb{P}}\left[e^{\frac{1+\alpha}{2\alpha}\beta_n(T_n \wedge T_0)} \right]^{\frac{2\alpha}{1+2\alpha}},
\end{aligned}
$$

which yields

$$
\begin{aligned}
&\liminf_n -\frac{1}{n} \log [\text{2nd moment of } \hat{p}_n] \qquad\qquad\qquad\qquad\qquad (2.5) \\
&\geq \frac{1+\alpha}{1+2\alpha}W(0) - \frac{2\alpha}{1+2\alpha} \limsup_n \frac{1}{n} \log E^{\mathbb{P}}\left[e^{\frac{1+\alpha}{2\alpha}\beta_n(T_n \wedge T_0)} \right].
\end{aligned}
$$

Let c be the constant in Proposition A.1, and let

$$
\bar{C} \doteq \limsup_n \sup_{x \in \bar{D}_n} \frac{1}{n} \log E_x^{\mathbb{P}}\left[e^{c(T_n \wedge T_0)} \right],
$$

It follows immediately from Proposition A.1 that $\bar{C}$ is finite. Note that (2.5) holds for any $\alpha > 0$. In particular, it holds for $\alpha \doteq \bar{\varepsilon}/c$. With this choice of α, we have

$$
\frac{1+\alpha}{2\alpha}\beta_n = \frac{1+\alpha}{2\alpha}\frac{C}{n\varepsilon} + \frac{\bar{\varepsilon}}{2} + \frac{c}{2}.
$$

Therefore, if $\bar{\varepsilon} < c$, then for n big enough,

$$
\frac{1+\alpha}{2\alpha}\beta_n < c,
$$

and (2.5) yields

$$
\liminf_n -\frac{1}{n} \log [\text{2nd moment of } \hat{p}_n] \geq \frac{1+\alpha}{1+2\alpha}W(0) - \frac{2\alpha}{1+2\alpha}\bar{C}.
$$

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But the right-hand-side equals

$$W(0) - \frac{\alpha}{1+2\alpha}[W(0) + 2\bar{C}] = W(0) - \bar{\varepsilon}\frac{1}{c+2\bar{\varepsilon}}[W(0) + 2\bar{C}].$$

However, since $W(0) \leq 2\gamma$, we have

$$\liminf_n -\frac{1}{n}\log\left[\text{2nd moment of } \hat{p}_n\right] \geq W(0) - \bar{\varepsilon}\frac{1}{c}[2\gamma + 2\bar{C}].$$

It follows from Lemma 3.11 that

$$W(0) = W^{\varepsilon,\delta}(0) \geq \bar{W}^{\delta}(0) - 3\varepsilon = 2\gamma - 3\delta - 3\varepsilon. \qquad (2.6)$$

Recall that $\bar{\varepsilon} = 2\gamma\exp\{-\delta/\varepsilon\}$. We conclude the proof by setting $A = 2\gamma[2\gamma + 2\bar{C}]/c$ and setting $B = 1/\log(2\gamma/c)$ if $c < 2\gamma$ and ∞ if $c \geq 1$. ∎

Proof of Theorem 3.16. It suffices to show that

$$\liminf_n -\frac{1}{n}\log\left[\text{2nd moment of } \hat{p}_n\right] \geq 2\gamma,$$

since the other direction is automatic by Jensen's inequality (see Section 2). We should use the notation $W^n = W^{\varepsilon_n,\delta_n}$, $\rho_k^n \doteq \rho_k^{\varepsilon_n,\delta_n}$, and $\bar{\varepsilon}_n = \exp\{-\delta_n/\varepsilon_n\}$. The same argument leading to inequality (2.5) gives that, for any strictly positive sequence $\{\alpha_n\}$,

$$\liminf_n -\frac{1}{n}\log\left[\text{2nd moment of } \hat{p}_n\right]$$
$$\geq \liminf_n \frac{1+\alpha_n}{1+2\alpha_n}W^n(0) - \limsup_n \frac{2\alpha_n}{1+2\alpha_n}\frac{1}{n}\log E^{\mathbb{P}}\left[e^{\frac{1+\alpha_n}{2\alpha_n}\beta_n(T_n \wedge T_0)}\right],$$

where

$$\beta_n \doteq \frac{C}{n\varepsilon_n} + \bar{\varepsilon}_n.$$

In particular, we should choose α_n so that

$$\frac{1+\alpha_n}{2\alpha_n}\beta_n = c, \text{ or, } \alpha_n = \frac{\beta_n}{2c - \beta_n}.$$

Note α_n is strictly positive (at least for n big enough) and $\alpha_n \to 0$ since $\beta_n \to 0$ by assumption. It follows that

$$\liminf_n -\frac{1}{n}\log\left[\text{2nd moment of } \hat{p}_n\right] \geq \liminf_n W^n(0).$$

But thanks to (2.6), $W^n(0) \geq 2\gamma - 3\delta_n - 3\varepsilon_n$. We complete the proof. ∎

C   Appendix. Proof of Proposition 3.5

Fix arbitrarily $p \in \mathbb{R}^2$. For $p \in \mathbb{R}^2$ and $\bar{\Theta}, \theta \in \mathcal{P}^+(\mathbb{V})$, define

$$L(p; \bar{\Theta}, \theta) \doteq \langle p, \mathbb{F}(\theta) \rangle + \sum_{i=0}^{2} \theta[v_i] \log \frac{\bar{\Theta}[v_i]}{\Theta[v_i]} + R(\theta \| \Theta).$$

It follows from the definition of relative entropy that

$$
\begin{aligned}
L(p; \bar{\Theta}, \theta) &= \langle p, \mathbb{F}(\theta) \rangle - \sum_{i=0}^{2} \theta[v_i] \log \frac{\theta[v_i]}{\bar{\Theta}[v_i]} + \sum_{i=0}^{2} \theta[v_i] \log \frac{\theta[v_i]}{\Theta[v_i]} + R(\theta \| \Theta) \\
&= \langle p, \mathbb{F}(\theta) \rangle - R(\theta \| \bar{\Theta}) + 2R(\theta \| \Theta).
\end{aligned}
$$

We first show that $(\bar{\Theta}^*(p), \theta^*(p))$ is a saddle point, or

$$L(p; \bar{\Theta}, \theta^*(p)) \leq L(p; \bar{\Theta}^*(p), \theta^*(p)) \leq L(p; \bar{\Theta}^*(p), \theta)$$

for every $\bar{\Theta}, \theta \in \mathcal{P}^+(\mathbb{V})$. The first inequality is rather trivial, thanks to the non-negativity of relative entropy, that $R(\gamma \| \theta) = 0$ if and only if $\gamma = \theta$, and that $\bar{\Theta}^*(p) = \theta^*(p)$. Now we consider the second inequality. It is easy to verify that

$$\log \frac{\bar{\Theta}^*(p)[v_i]}{\Theta[v_i]} = \log N(p) - \frac{1}{2} \langle p, v_i \rangle,$$

and

$$\langle p, \mathbb{F}(\theta) \rangle = \sum_{i=0}^{2} \theta[v_i] \langle p, v_i \rangle.$$

Therefore,

$$L(p; \bar{\Theta}^*(p), \theta) = \log N(p) + \frac{1}{2} \sum_{i=0}^{2} \theta[v_i] \langle p, v_i \rangle + R(\theta \| \Theta).$$

It is now straightforward computation by calculus that $L(p; \bar{\Theta}^*(p), \theta)$, as a function of θ, attains its minimum at $\theta = \theta^*(p)$. The second inequality follows readily.

The existence of saddle point $(\bar{\Theta}^*(p), \theta^*(p))$ implies that

$$\mathbb{H}(p) = L(p; \bar{\Theta}^*(p), \theta^*(p)) = 2 \log N(p),$$

and that the order of sup and ind can be exchanged, or

$$\mathbb{H}(p) = \inf_{\theta \in \mathcal{P}^+(\mathbb{V})} \sup_{\bar{\Theta} \in \mathcal{P}^+(\mathbb{V})} L(p; \bar{\Theta}, \theta).$$

But

$$\sup_{\bar{\Theta} \in \mathcal{P}^+(\mathbb{V})} L(p; \bar{\Theta}, \theta) = \sup_{\bar{\Theta} \in \mathcal{P}^+(\mathbb{V})} \left[\langle p, \mathbb{F}(\theta) \rangle - R(\theta \| \bar{\Theta}) + 2R(\theta \| \Theta) \right]$$

$$= \langle p, \mathbb{F}(\theta) \rangle + 2R(\theta \| \Theta) - \inf_{\bar{\Theta} \in \mathcal{P}^+(\mathbb{V})} R(\theta \| \bar{\Theta})$$

$$= \langle p, \mathbb{F}(\theta) \rangle + 2R(\theta \| \Theta).$$

Since $\mathbb{H}$ is the infimum of affine functions (of p), it is concave. We complete the proof. ∎

D   Collection of miscellaneous proofs

Proof of Lemma 4.4. Clearly, $\mathbb{H}(r_{d+1}) = \mathbb{H}(0) = 0$. For $1 \le k \le d$, Proposition 4.1 implies that $\mathbb{H}(r_k) = 2 \log N(r_k)$ where

$$\frac{1}{N(r_k)} = \bar{\mu} + \mu_1 + \cdots + \mu_{d-k} + \frac{\lambda \mu_{d+1-k}}{\bar{\mu}} + \mu_{d+2-k} + \cdots + \mu_d.$$

In order to show $N(r_k) \ge 1$, it suffices to show that

$$\bar{\mu} + \frac{\lambda \mu_{d+1-k}}{\bar{\mu}} \le \lambda + \mu_{d+1-k},$$

or equivalently

$$(\mu_{d+1-k} - \bar{\mu})(\bar{\mu} - \lambda) \ge 0,$$

which is trivial from assumption. Therefore we have $\mathbb{H}(r_k) \ge 0$. Furthermore, for $x \in \Gamma$, we have

$$\bar{W}^n(x) \le \bar{W}_1^n(x) = -2\gamma(x_1 + x_2 + \cdots + x_d) + 2\gamma - \delta \le -\delta < 0.$$

Now assume $x \in \partial_i$ for some $1 \le i \le d$. Suppose $D\bar{W}^n(x)$ is well defined, or equivalently, the $\bar{W}_1^n(x) \wedge \cdots \wedge \bar{W}_{d+1}^n(x)$ attains the minimum uniquely at some $k^* \in \{1, 2, \ldots, d+1\}$. In this case, $D\bar{W}^n(x) = r_{k^*}$, and we wish to show $\langle r_{k^*}, d_i \rangle \ge 0$. However, for every k we have

$$\langle r_k, d_i \rangle = \begin{cases} -2\gamma & ; \quad \text{if } k + i = d + 1 \\ 0 & ; \quad \text{otherwise} \end{cases}. \tag{4.1}$$

Thus it suffices to show that $k^* + i \ne d+1$. However, thanks to the definition of $\{r_k\}$ and that $x_i = 0$, this is trivial since

$$\bar{W}_{d+2-i}^n(x) = \langle r_{d+2-i}, x \rangle + \gamma - (d+2-i)\delta_n$$

$$= \langle r_{d+1-i}, x \rangle + \gamma - (d+2-i)\delta_n$$

$$= \bar{W}_{d+1-i}^n(x) - \delta$$

$$< \bar{W}_{d+1-i}^n(x).$$

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It remains to show that $\langle DW^{\varepsilon_n,n}(x), d_i \rangle - 2\gamma \exp\{-\delta_n/\varepsilon\}$ for $x \in \partial_i$. But

$$DW^{\varepsilon_n,n}(x) = \sum_{k=1}^{d+1} \rho_k^{\varepsilon_n,n}(x) r_k.$$

Thanks to (4.1), we only need to show $\rho_{d+1-i}^{\varepsilon_n,n}(x) \leq \exp\{-\delta_n/\varepsilon_n\}$ for $x \in \partial_i$. To this end, observe that

$$\rho_{d+1-i}^{\varepsilon_n,n}(x) \leq \frac{\exp\{-\bar{W}_{d+1-i}^n(x)/\varepsilon_n\}}{\exp\{-\bar{W}_{d+2-i}^n(x)/\varepsilon_n\}} = \exp\{-\delta_n/\varepsilon_n\}.$$

We complete the proof. ∎

Proof of Lemma 5.2. We will only present the proof for the case $\mu_1 < \mu_2$, and omit the analogous proof for $\mu_1 \geq \mu_2$.

Assume $\mu_1 < \mu_2$ hereafter, and use the notation $W \equiv W^{\varepsilon,\delta}$ and $\rho_k = \rho_k^{\varepsilon,\delta}$. The formulae in Remark 5.1 yield

$$\mathbb{H}(r_1) = 2\log N(r_1) = -2\log\left[(1-\beta)\mu_1 + \mu_1 + \beta\mu_2 + \frac{\lambda\mu_2}{\mu_1}\right].$$

By assumption $\lambda \leq (1-\beta)\mu_1$ and $\mu_1 < \mu_2$. Therefore

$$\left(\frac{\mu_2}{\mu_1} - 1\right)((1-\beta)\mu_1 - \lambda) \geq 0 \Rightarrow (1-\beta)\mu_1 + \frac{\lambda\mu_2}{\mu_1} \leq \lambda + (1-\beta)\mu_2.$$

Since $\lambda + \mu_1 + \mu_2 = 1$, we have $\mathbb{H}(r_1) \geq 0$. Similarly, we have $\mathbb{H}(r_2) = 0$ and $\mathbb{H}(r_3) = 0$. Thanks to the concavity of $\mathbb{H}$, $DW(x) = \sum_k \rho_k(x) r_k$, and $\sum_k \rho_k(x) = 1$, $\rho_k(x) \geq 0$, we have $\mathbb{H}(DW(x)) \geq 0$. As for $x \in \partial_e$, we have $W(x) \leq \langle r_1, x \rangle + 2\gamma - \delta = -\delta \leq 0$.

It remains to show the Claim 3. Thanks to the concavity of $\mathbb{H}_{\partial_i}$, we have, for $x \in \partial_i$,

$$\mathbb{H}_{\partial_i}(DW(x)) \geq \sum_{k=1}^{3} \rho_k(x) \mathbb{H}_{\partial_i}(r_k) = \sum_{k=1}^{2} \rho_k(x) \mathbb{H}_{\partial_i}(r_k).$$

However, it is not difficult to show that

$$\mathbb{H}_{\partial_1}(r_1) \geq 0, \quad \mathbb{H}_{\partial_2}(r_2) = 0.$$

Therefore, we only need to show $\rho_2(x) \leq \exp\{-\delta/\varepsilon\}$ for $x \in \partial_1$ and $\rho_1(x) \leq \exp\{-\delta/\varepsilon\}$ for $x \in \partial_2$. For $x \in \partial_2$, we have $x_2 = 0$ and

$$\rho_1(x) \leq \frac{\exp\{-W_1^\delta(x)/\varepsilon\}}{\exp\{-W_2^\delta(x)/\varepsilon\}} = \exp\{-\delta/\varepsilon\}.$$

For $x = (0, x_2) \in \partial_1$, we consider two cases: $x_2 \leq x_2^*$ and $x_2 > x_2^*$ separately, where $x_2^* \doteq \delta/a$. For $x_2 \leq x_2^*$, we have

$$\rho_2(x) \leq \frac{\exp\{-W_2^\delta(x)/\varepsilon\}}{\exp\{-W_3^\delta(x)/\varepsilon\}} = \exp\left\{\frac{2(\gamma - a)}{\varepsilon}x_2 + \left(1 - \frac{2\gamma}{a}\right)\frac{\delta}{\varepsilon}\right\} \leq \exp\{-\delta/\varepsilon\}.$$

Similarly, for $x \geq x_2^*$, we have

$$\rho_2(x) \leq \frac{\exp\{-W_2^\delta(x)/\varepsilon\}}{\exp\{-W_1^\delta(x)/\varepsilon\}} = \exp\left\{-\frac{2a}{\varepsilon}x_2 + \frac{\delta}{\varepsilon}\right\} \leq \exp\{-\delta/\varepsilon\}.$$

We complete the proof. ■

References

[1] S. Asmussen. Conditioned limit theorems relating a random walk to its associates, with applications to risk reverse process and GI/G/1 queue. *Adv. Appl. Prob.*, pages 143–170, 1982.

[2] S. Asmussen. *Applied Probability and Queues*. Spring-Verlag, New York, 2003.

[3] P.J. De Boer. Analysis of state-independent importance sampling measures for the two-node tandem queue. *Preprint.*

[4] P. Dupuis and R. S. Ellis. *A Weak Convergence Approach to the Theory of Large Deviations*. John Wiley & Sons, New York, 1997.

[5] P. Dupuis, R.S. Ellis, and A. Weiss. Large deviations for Markov processes with discontinuous statistics, I: General upper bounds. *Annals of Probability*, 19:1280–1297, 1991.

[6] P. Dupuis, H. Ishii, and H. M. Soner. A viscosity solution approach to the asymptotic analysis of queueing systems. *The Annals of Probab.*, 18:226–255, 1990.

[7] P. Dupuis and H. Wang. Importance sampling, large deviations, and differential games. *Stoch. and Stoch. Reports.*, 76:481–508, 2004.

[8] P. Dupuis and H. Wang. Dynamic importance sampling for uniformly recurrent Markov chains. *Annals of Applied Probab.*, 15:1–38, 2005.

[9] P. Dupuis and H. Wang. Subsolutions of an Isaacs equation and efficient schemes for importance sampling: Convergence analysis. *Preprint*, 2005.

[10] P. Dupuis and H. Wang. Subsolutions of an Isaacs equation and efficient schemes for importance sampling: Examples and numerics. *Preprint*, 2005.

[11] P. Glasserman and S. Kou. Analysis of an importance sampling estimator for tandem queues. *ACM Trans. Modeling Comp. Simulation*, 4:22–42, 1995.

[12] P. Heidelberger. Fast simulation of rare events in queueing and reliability models. *ACM Trans. Modeling Comp. Simulation*, 4:43–85, 1995.

[13] P.-L. Lions. Neumann type boundary conditions for Hamilton-Jacobi-Bellman equations. *Duke Mathmatical Journal.*, 52:793–820, 1985.

[14] S. Parekh and J. Walrand. A quick simulation method for excessive backlogs in networks of queues. *IEEE. Trans. Autom. Control*, 34:54–66, 1989.

[15] J.S. Sadowsky. Large deviations and efficient simulation of excessive backlogs in a $GI/G/m$ queue. *IEEE. Trans. Automat. Control*, 36:1383–1394, 1991.

[16] A.D. Sezer. *Dynamic Importance sampling for queueing networks.* PhD thesis, Brown University, Rhode Island, 2005.

[17] R.R. Weber. The interchangeability in M/M/1 queues in series. *J. Appl. Prob.*, 16:690–695, 1979.